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3D Matrix Ring with a "Common" Multiplication

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Abstract

In this article, starting from geometrical considerations, he was born with the idea of 3D matrices, which have developed in this article. A problem here was the definition of multiplication, which we have given in analogy with the usual 2D matrices. The goal here is 3D matrices to be a generalization of 2D matrices. Work initially we started with $3\times3\times3$ matrix, and then we extended to $m\times n\times p$ matrices. In this article, we give the meaning of 3D matrices. We also defined two actions in this set. As a result, in this article, we have reached to present 3-dimensional unitary ring matrices with elements from a field F.

Subject Areas

Algebra, Applied Statistical Mathematics, Geometry

Keywords

Linear Algebra, Matrices, Ring Theory

1. Introduction

Based on the meaning of the addition and the multiplication of 2D matrices [1]-[6], this article stretches this sense, the idea, the addition and the multiplication of 3D matrices. Starting from geometrical considerations, concretely taking into account the cube, he was born with the idea of 3D matrices, which have developed in this article. A problem here was the definition of multiplication, for which we have acted pages, analogously acted as the columns, which we have given in analogy with the usual 2D matrices [6] [9]. The goal here is 3D matrices to be a generalization of 2D matrices. We proved that this set of two actions together in forming the "unitary ring" [7] [8] [10] [11]. In literature and in various mathematical forums, we noticed an interest in the 3D matrices, but on the other hand are missing results associated with them; this was a sufficient reason to explore. We introduced the meaning of the scalar multiplication, and finally we have shown that we have an F-module connected to this ring 3D matrices or vector spaces [8] [10]. As indications for this paper were simply geometric imaginations. Everything presented in this article are my results.

2. Addition of 3 × 3 × 3, 3-D Matrices over Field *F*, and the Addition Abelian Group of Their 3-D Matrices

Imagining a parallelepiped, with born idea of 3D matrices, which are define as follows

Definition 2.1 3-dimensional 3 × 3 × 3 matrice will call, a matrix which has: *three horizontal layers* (*analogous to three rows*), *three vertical page* (analogue with three columns in the usual matrices) and *three vertical layers* two of which are hidden.

The set of these matrices the write how:

$$\mathcal{M}_{3\times3\times3}(F)=\left\{\left(a_{ijk}\right)\mid a_{ijk}\in F \text{ and } i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$

The appearance of these matrices will be as in Figure 1.

Definition 2.2 The addition of two matrices $\boldsymbol{A}_{3\times3\times3},\boldsymbol{B}_{3\times3\times3}\in\mathcal{M}_{3\times3\times3}(F)$ we will call the matrix:

$$\boldsymbol{C}_{3\times3\times3}=\left\{\left(c_{ijk}\right)\mid c_{ijk}=a_{ijk}+b_{ijk},\forall i,j,k\in\{1,2,3\}\right\}$$

The appearance of the addition of 3 × 3 × 3, 3D matrices, will be as in Figure 2, where matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the following appearance,

$$\boldsymbol{A}_{3\times3\times3}=\left\{\left(a_{ijk}\right)\mid a_{ijk}\in F \text{ for } i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$
$$\boldsymbol{B}_{3\times3\times3}=\left\{\left(b_{ijk}\right)\mid b_{ijk}\in F \text{ for } i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$

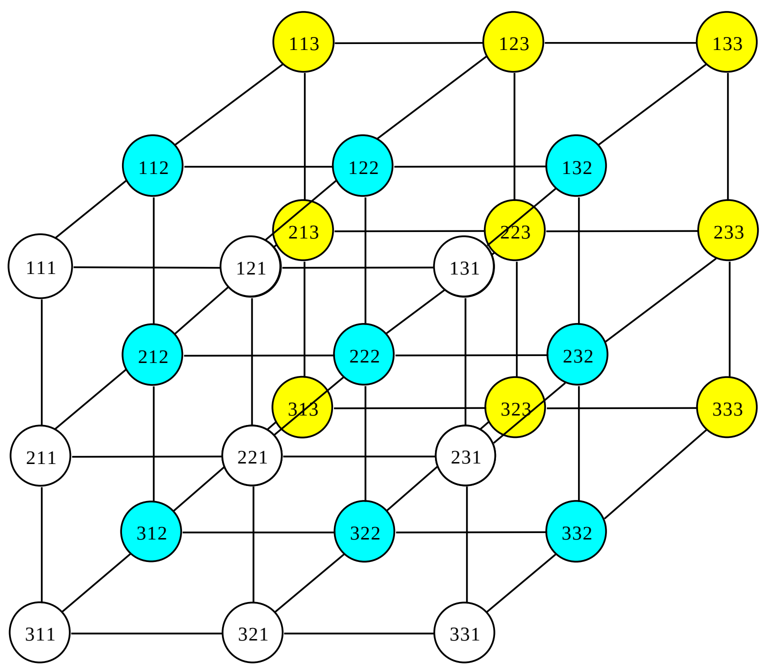


Figure 1. 3-D Matrice.

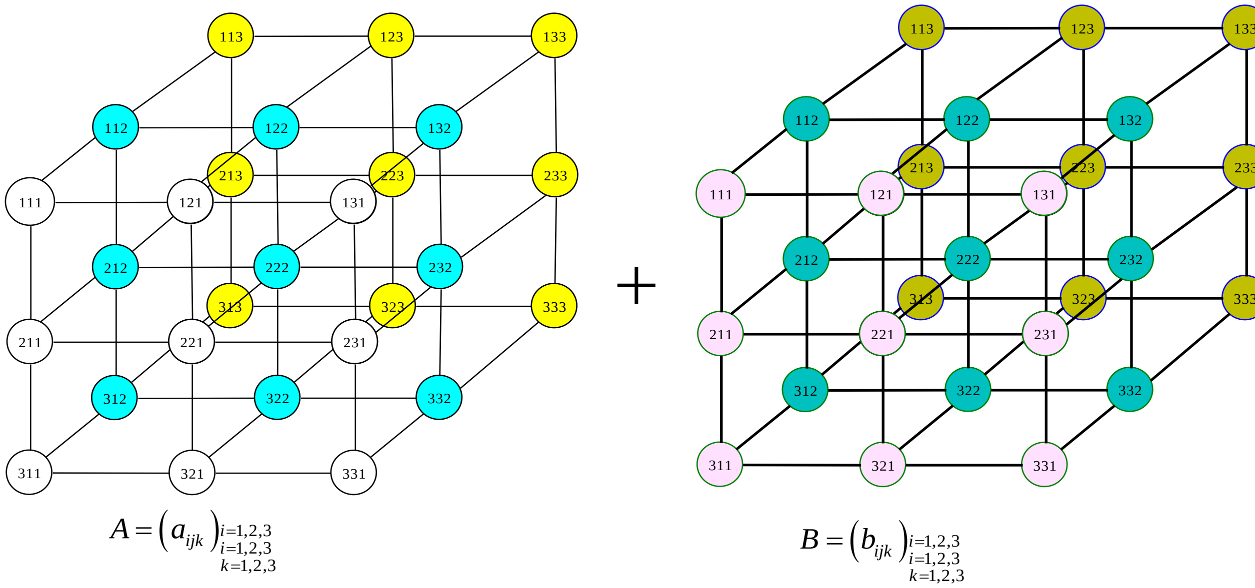


Figure 2. The addition of 3 × 3 × 3, 3D matrices.

Definition 2.3 Zero matrix 3 × 3, 3D we will called the matrix that has all its elements zero.

$$\boldsymbol{O}_{3\times3\times3}=\left\{\left(0_F\right)_{ijk} \mid i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$

Definition 2.4. *The opposite matric of anmatrice*

$$\boldsymbol{A}_{3\times3\times3}=\left\{\left(a_{ijk}\right) \mid i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$

will, called matrix

$$-\boldsymbol{A}_{3\times3\times3}=\left\{\left(-a_{ijk}\right) \mid i=1,2,3;\ j=1,2,3;k=1,2,3\right\}$$

(where $-a_{ijk}$ is a opposite element of element $a_{ijk}\in F$, so $a_{ijk}+\left(-a_{ijk}\right)=0_F$ and $(F,+,\cdot)$ is field [8] [10] [11]), which satisfies the condition

$$\begin{aligned}\boldsymbol{A}_{3\times3\times3}+\left(-\boldsymbol{A}_{3\times3\times3}\right)&=\left\{\left(a_{ijk}+\left(-a_{ijk}\right)\right) \mid i=1,2,3;\ j=1,2,3;k=1,2,3\right\}\\&=\left\{(0)_{ijk} \mid i=1,2,3;\ j=1,2,3;k=1,2,3\right\}=\boldsymbol{O}_{3\times3\times3}\end{aligned}$$

Theorem 2.1 $\left(\mathcal{M}_{3\times3\times3}(F),+\right)$ *is a beliangrup.*

Proof. Truly from the definition 2.2, of addition the 3-Dmatrices, we see that addition is the sustainable in $\mathcal{M}_{3\times3\times3}(F)$, because

$$a_{ijk}\in F, b_{ijk}\in F\Rightarrow c_{ijk}=a_{ijk}+b_{ijk}\in F, \forall i,j,k\in\{1,2,3\}$$

1) *Associative property,*

$$\forall\boldsymbol{A}=\left(a_{ijk}\right),\boldsymbol{B}=\left(b_{ijk}\right),\boldsymbol{C}=\left(c_{ijk}\right)\in\mathcal{M}_{3\times3\times3}(F)\Rightarrow(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$$

truly

$$\begin{aligned}(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}&=\left[\left(a_{ijk}\right)+\left(b_{ijk}\right)\right]+\left(c_{ijk}\right)=\left(a_{ijk}+b_{ijk}\right)+\left(c_{ijk}\right)=\left(\left(a_{ijk}+b_{ijk}\right)+c_{ijk}\right)\\&=\left(a_{ijk}+b_{ijk}+c_{ijk}\right)=\left(a_{ijk}+\left(b_{ijk}+c_{ijk}\right)\right)=\left(a_{ijk}\right)+\left(b_{ijk}+c_{ijk}\right)=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})\end{aligned}$$

2) $\forall \boldsymbol{A}=\left(a_{ijk}\right)\in \mathcal{M}_{3\times 3\times 3}\left(F\right), \exists \boldsymbol{O}=\left(0_{ijk}\right) / \boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}.$
truly, $\forall \boldsymbol{A}=\left(a_{ijk}\right)\in \mathcal{M}_{3\times 3\times 3}\left(F\right), \exists \boldsymbol{O}=\left(0_{ijk}\right) / \boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}.$

$$\begin{aligned}\boldsymbol{A}+\boldsymbol{O}&=\left\{\left(a_{ijk}\right)+\left(0\right)_{ijk} \mid i=1,2,3;\ j=1,2,3; k=1,2,3\right\}\\&=\left\{\left(a_{ijk}+0\right)_{ijk} \mid i=1,2,3;\ j=1,2,3; k=1,2,3\right\}\\&=\left\{\left(a_{ijk}\right) \mid i=1,2,3;\ j=1,2,3; k=1,2,3\right\}=\boldsymbol{A}\end{aligned}$$

3) $\forall \boldsymbol{A}=\left(a_{ijk}\right)\in \mathcal{M}_{3\times 3\times 3}\left(F\right), \exists -\boldsymbol{A}=\left(-a_{ijk}\right)\in \mathcal{M}_{3\times 3\times 3}\left(F\right) / \boldsymbol{A}+\left(-\boldsymbol{A}\right)=\boldsymbol{O}.$
truly, from Definition 2.4, we have

$$\begin{aligned}\boldsymbol{A}+\left(-\boldsymbol{A}\right)&=\left\{\left(a_{ijk}+\left(-a_{ijk}\right)\right) \mid i=1,2,3;\ j=1,2,3; k=1,2,3\right\}\\&=\left\{\left(0\right)_{ijk} \mid i=1,2,3;\ j=1,2,3; k=1,2,3\right\}=\boldsymbol{O}\end{aligned}$$

4) Addition is commutative.

$$\forall \boldsymbol{A}=\left(a_{ijk}\right), \boldsymbol{B}=\left(b_{ijk}\right)\in \mathcal{M}_{3\times 3\times 3}\left(F\right), \boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}.$$

truly

$$\boldsymbol{A}+\boldsymbol{B}=\left(a_{ijk}\right)+\left(b_{ijk}\right)=\left(a_{ijk}+b_{ijk}\right)\overset{(\mathbb{R},+)\text{ is abelian}}{=}\left(b_{ijk}+a_{ijk}\right)=\left(b_{ijk}\right)+\left(a_{ijk}\right)=\boldsymbol{B}+\boldsymbol{A}$$

3. Addition of $m\times n\times p$, 3-D Matrices over Any Field *F* and the Addition Abelian Group of Their 3-D Matrices

Definition 3.1 *3-dimensional$m$xnxp* matrix will call, a matrix which has: *m-horizontal layers* (*analogous to m-rows*), *n-vertical page* (analogue with n-columns in the usualmatrices) and *p-vertical layers* ($p-1$ of which are hidden).

The set of these matrixes the write how:

$$\mathcal{M}_{m\times n\times p}\left(F\right)=\left\{\left(a_{ijk}\right) \mid a_{ijk}\in F\text{-field and } i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}$$

Definition 3.2 The addition of two matrices $\boldsymbol{A},\boldsymbol{B}\in \mathcal{M}_{m\times n\times p}\left(F\right)$ we will call the matrix:

$$\boldsymbol{C}_{m\times n\times p}=\left\{\left(c_{ijk}\right) \mid c_{ijk}=a_{ijk}+b_{ijk}, \forall i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}$$

The appearance of the addition of mxnxp, 3D matrices will be as in **Figure 3**, where matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the following appearance,

$$\boldsymbol{A}_{m\times n\times p}=\left\{\left(a_{ijk}\right) \mid i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}$$
$$\boldsymbol{B}_{m\times n\times p}=\left\{\left(b_{ijk}\right) \mid i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}$$

Definition 3.3 3-D, Zero matrix $m\times n\times p$, we will called the matrix that has all its elements zero.

$$\boldsymbol{O}=\boldsymbol{O}_{m\times n\times p}=\left\{\left(0\right)_{ijk} \mid i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}$$

Definition 3.4 *The opposite matric of anmatrice*

$$\boldsymbol{A}_{m\times n\times p}=\left\{\left(a_{ijk}\right) \mid i=\overline{1,m};\ j=\overline{1,n}; k=\overline{1,p}\right\}\in \mathcal{M}_{m\times n\times p}\left(F\right)$$

will, called matrix

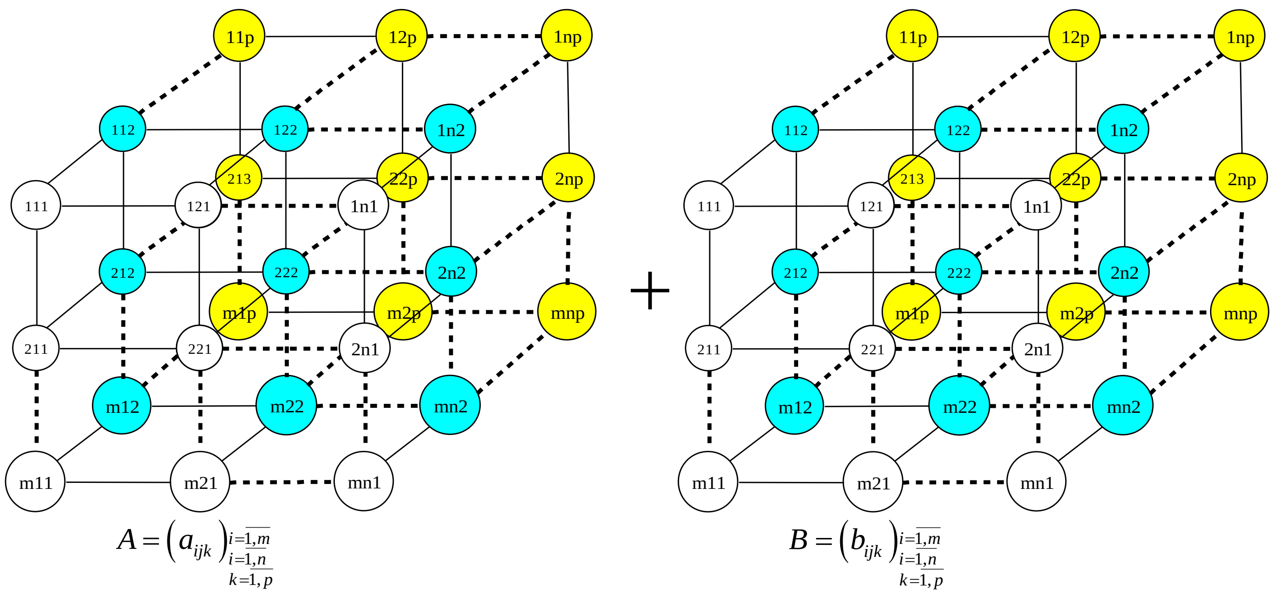


Figure 3. The addition of mxnxp, 3D matrices.

$$-\boldsymbol{A}_{m\times n\times p}=\left\{\left(-a_{ijk}\right)\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}\in\mathcal{M}_{m\times n\times p}\left(F\right)$$

(where $-a_{ijk}$ is a opposite element of element $a_{ijk}\in F$, so $a_{ijk}+\left(-a_{ijk}\right)=0_F$ and $\left(F,+,\cdot\right)$ is field), which satisfies the condition

$$\begin{aligned}\boldsymbol{A}_{m\times n\times p}+\left(-\boldsymbol{A}_{m\times n\times p}\right)&=\left\{\left(a_{ijk}+\left(-a_{ijk}\right)\right)\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}\\&=\left\{\left(0\right)_{ijk}\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}=\boldsymbol{O}\end{aligned}$$

Theorem 3.1 $\left(\mathcal{M}_{m\times n\times p}\left(F\right),+\right)$ *is abeliangrup.*

Proof. Truly from the definition 3.2, of additions the 3-D matrices, we see that addition is the sustainable in $\mathcal{M}_{m\times n\times p}\left(F\right)$, because

$$a_{ijk}\in F,b_{ijk}\in F\Rightarrow c_{ijk}=a_{ijk}+b_{ijk}\in F,\forall i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}$$

1) *Associative property,*

$$\forall\boldsymbol{A}=\left(a_{ijk}\right),\boldsymbol{B}=\left(b_{ijk}\right),\boldsymbol{C}=\left(c_{ijk}\right)\in\mathcal{M}_{m\times n\times p}\left(F\right)\Rightarrow\left(\boldsymbol{A}+\boldsymbol{B}\right)+\boldsymbol{C}=\boldsymbol{A}+\left(\boldsymbol{B}+\boldsymbol{C}\right)$$

truly

$$\begin{aligned}\left(\boldsymbol{A}+\boldsymbol{B}\right)+\boldsymbol{C}&=\left[\left(a_{ijk}\right)+\left(b_{ijk}\right)\right]+\left(c_{ijk}\right)=\left(a_{ijk}+b_{ijk}\right)+\left(c_{ijk}\right)=\left(\left(a_{ijk}+b_{ijk}\right)+c_{ijk}\right)\\&=\left(a_{ijk}+b_{ijk}+c_{ijk}\right)=\left(a_{ijk}+\left(b_{ijk}+c_{ijk}\right)\right)=\left(a_{ijk}\right)+\left(b_{ijk}+c_{ijk}\right)=\boldsymbol{A}+\left(\boldsymbol{B}+\boldsymbol{C}\right)\end{aligned}$$

2) $\forall\boldsymbol{A}=\left(a_{ijk}\right)\in\mathcal{M}_{m\times n\times p}\left(F\right),\exists\boldsymbol{O}=\left(0_{ijk}\right)\in\mathcal{M}_{m\times n\times p}\left(F\right)/\ \boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}.$

truly,

$$\forall\boldsymbol{A}=\left(a_{ijk}\right)\in\mathcal{M}_{m\times n\times p}\left(F\right),\exists\boldsymbol{O}=\left(0_{ijk}\right)\in\mathcal{M}_{m\times n\times p}\left(F\right)/\ \boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}.$$

$$\begin{aligned}\boldsymbol{A}+\boldsymbol{O}&=\left\{\left(a_{ijk}\right)+\left(0\right)_{ijk}\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}\\&=\left\{\left(a_{ijk}+0_F\right)_{ijk}\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}\\&=\left\{\left(a_{ijk}\right)\mid i=\overline{1,m};\ j=\overline{1,n};k=\overline{1,p}\right\}=\boldsymbol{A}\end{aligned}$$

3) $\forall \boldsymbol{A} = (a_{ijk}) \in \mathcal{M}_{m\times n\times p}(F), \exists - \boldsymbol{A} = (-a_{ijk}) \in \mathcal{M}_{m\times n\times p}(F) / \boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{O}.$

truly, from Definition 2.4, we have

$$\begin{aligned}\boldsymbol{A} + (-\boldsymbol{A}) &= \left\{\left(a_{ijk} + (-a_{ijk})\right) \mid i = \overline{1,m};\ j = \overline{1,n}; k = \overline{1,p}\right\} \\ &= \left\{(0_F)_{ijk} \mid i = \overline{1,m};\ j = \overline{1,n}; k = \overline{1,p}\right\} = \boldsymbol{O}\end{aligned}$$

4) Addition is commutative.

$$\forall \boldsymbol{A} = (a_{ijk}), \boldsymbol{B} = (b_{ijk}) \in \mathcal{M}_{m\times n\times p}(F) / \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}.$$

truly

$$\boldsymbol{A} + \boldsymbol{B} = (a_{ijk}) + (b_{ijk}) = (a_{ijk} + b_{ijk}) \overset{(F,+)\text{ is abelian}}{=} (b_{ijk} + a_{ijk}) = (b_{ijk}) + (a_{ijk}) = \boldsymbol{B} + \boldsymbol{A}$$

4. The "Common" Multiplication of $3\times3\times3$, 3-D Matrices with Elements Froman Field F

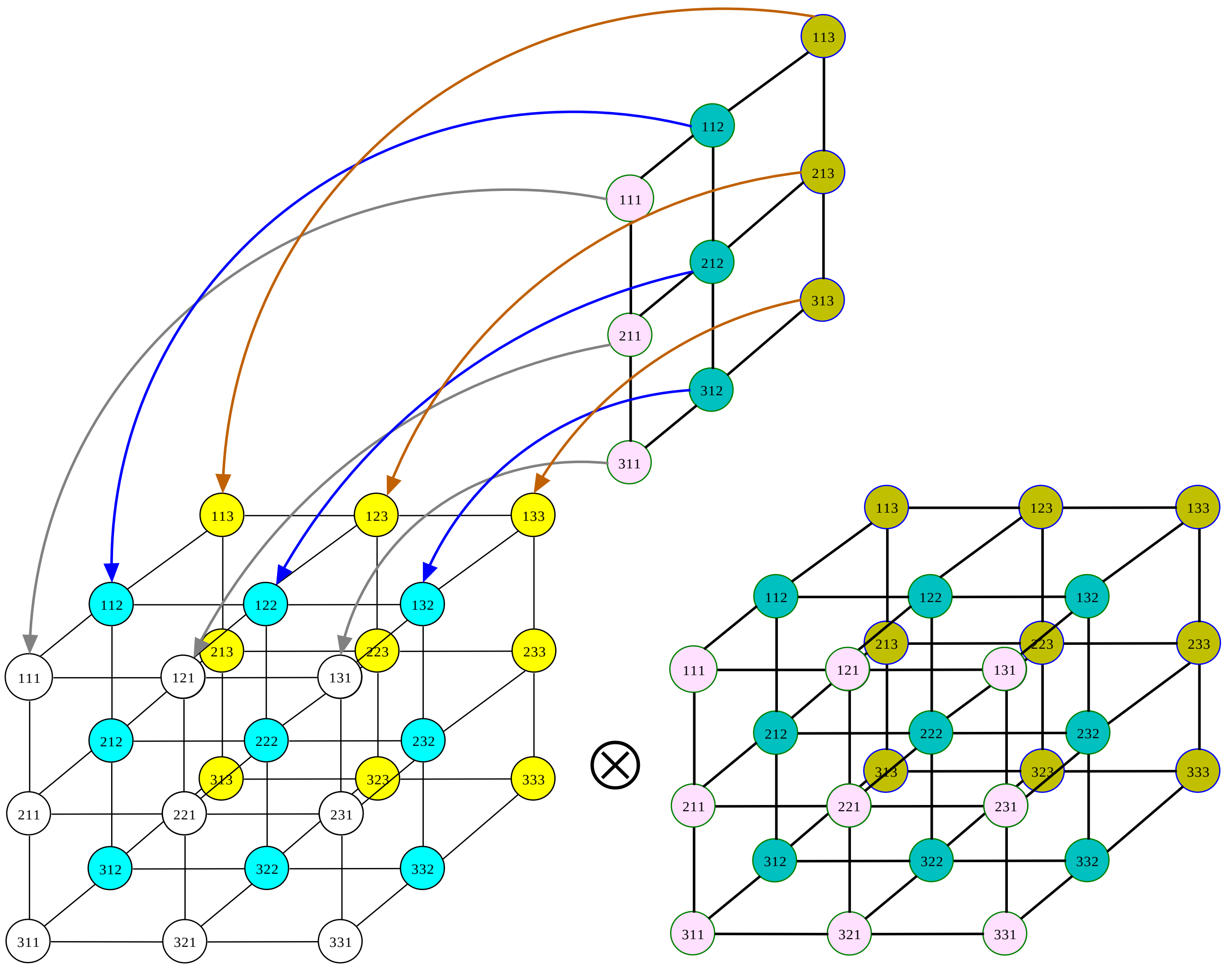
Definition 4.1: *The multiplication of two matrices* $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}_{3\times3\times3}(F)$ *we will call the matrix* $\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B} \in \mathcal{M}_{3\times3\times3}(F)$ *calculated as follows*:

$$\forall \left| \begin{array}{l} \text{3-vertical layer} \begin{pmatrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{pmatrix} \\ \\ \text{2-vertical layer} \begin{pmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{pmatrix} \\ \\ \text{1-vertical layer} \begin{pmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{pmatrix} \end{array} \right. , \begin{array}{l} \text{3-vertical layer} \begin{pmatrix} b_{113} & b_{123} & b_{133} \\ b_{213} & b_{223} & b_{233} \\ b_{313} & b_{323} & b_{333} \end{pmatrix} \\ \\ \text{2-vertical layer} \begin{pmatrix} b_{112} & b_{122} & b_{132} \\ b_{212} & b_{222} & b_{232} \\ b_{312} & b_{322} & b_{332} \end{pmatrix} \\ \\ \text{1-vertical layer} \begin{pmatrix} b_{111} & b_{121} & b_{131} \\ b_{211} & b_{221} & b_{231} \\ b_{311} & b_{321} & b_{331} \end{pmatrix} \end{array} \in \mathcal{M}_{3\times3\times3}(F)$$

The appearance of the multiplication of 3 × 3 × 3, 3D matrices will be as in **Figure 4**.

$$\boldsymbol{C} = \begin{array}{r} (\text{3-vertical layer}) \\ \\ \\ (\text{2-vertical layer}) \\ \\ \\ (\text{1-vertical layer}) \end{array} \begin{pmatrix} c_{113} & c_{123} & c_{133} \\ c_{213} & c_{223} & c_{233} \\ c_{313} & c_{323} & c_{333} \\ \\ c_{112} & c_{122} & c_{132} \\ c_{212} & c_{222} & c_{232} \\ c_{312} & c_{322} & c_{332} \\ \\ c_{111} & c_{121} & c_{131} \\ c_{211} & c_{221} & c_{231} \\ c_{311} & c_{321} & c_{331} \end{pmatrix} = \begin{pmatrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \\ \\ a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \\ \\ a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{pmatrix} \otimes \begin{pmatrix} b_{113} & b_{123} & b_{133} \\ b_{213} & b_{223} & b_{233} \\ b_{313} & b_{323} & b_{333} \\ \\ b_{112} & b_{122} & b_{132} \\ b_{212} & b_{222} & b_{232} \\ b_{312} & b_{322} & b_{332} \\ \\ b_{111} & b_{121} & b_{131} \\ b_{211} & b_{221} & b_{231} \\ b_{311} & b_{321} & b_{331} \end{pmatrix}$$

where, the first vertical *page* is:



$A = \left(a_{ijk}\right)_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}}$ $\otimes$ $B = \left(b_{ijk}\right)_{\substack{i=1,2,3\\ j=1,2,3\\ k=1,2,3}}$

Figure 4. The multiplication of 3 × 3 × 3, 3D matrices.

$$c_{111} = a_{111} \cdot b_{111} + a_{121} \cdot b_{211} + a_{131} \cdot b_{311};$$
$$c_{112} = a_{112} \cdot b_{112} + a_{122} \cdot b_{211} + a_{132} \cdot b_{312};$$
$$c_{113} = a_{113} \cdot b_{113} + a_{123} \cdot b_{213} + a_{133} \cdot b_{313};$$

$$c_{211} = a_{211} \cdot b_{111} + a_{221} \cdot b_{211} + a_{231} \cdot b_{311};$$
$$c_{212} = a_{212} \cdot b_{112} + a_{222} \cdot b_{211} + a_{232} \cdot b_{312};$$
$$c_{213} = a_{213} \cdot b_{113} + a_{223} \cdot b_{213} + a_{233} \cdot b_{313};$$

$$c_{311} = a_{311} \cdot b_{111} + a_{321} \cdot b_{211} + a_{331} \cdot b_{311};$$
$$c_{312} = a_{312} \cdot b_{112} + a_{322} \cdot b_{211} + a_{332} \cdot b_{312};$$
$$c_{313} = a_{313} \cdot b_{113} + a_{323} \cdot b_{213} + a_{333} \cdot b_{313};$$

the second vertical *page* is:

$$c_{121} = a_{111} \cdot b_{121} + a_{121} \cdot b_{221} + a_{131} \cdot b_{321};$$
$$c_{122} = a_{112} \cdot b_{122} + a_{122} \cdot b_{222} + a_{132} \cdot b_{322};$$
$$c_{123} = a_{113} \cdot b_{123} + a_{123} \cdot b_{223} + a_{133} \cdot b_{323};$$

$$c_{221} = a_{211} \cdot b_{121} + a_{221} \cdot b_{221} + a_{231} \cdot b_{321};$$
$$c_{222} = a_{212} \cdot b_{122} + a_{222} \cdot b_{222} + a_{232} \cdot b_{322};$$
$$c_{223} = a_{213} \cdot b_{123} + a_{223} \cdot b_{223} + a_{233} \cdot b_{323};$$

$$c_{321} = a_{311} \cdot b_{121} + a_{321} \cdot b_{221} + a_{331} \cdot b_{321};$$
$$c_{322} = a_{312} \cdot b_{122} + a_{322} \cdot b_{222} + a_{332} \cdot b_{322};$$
$$c_{323} = a_{313} \cdot b_{123} + a_{323} \cdot b_{223} + a_{333} \cdot b_{323};$$

and third vertical *page* is:

$$c_{131} = a_{111} \cdot b_{131} + a_{121} \cdot b_{231} + a_{131} \cdot b_{331};$$
$$c_{132} = a_{112} \cdot b_{132} + a_{122} \cdot b_{232} + a_{132} \cdot b_{332};$$
$$c_{133} = a_{113} \cdot b_{133} + a_{123} \cdot b_{233} + a_{133} \cdot b_{333};$$

$$c_{231} = a_{211} \cdot b_{121} + a_{221} \cdot b_{221} + a_{231} \cdot b_{321};$$
$$c_{232} = a_{212} \cdot b_{122} + a_{222} \cdot b_{222} + a_{232} \cdot b_{322};$$
$$c_{233} = a_{213} \cdot b_{123} + a_{223} \cdot b_{223} + a_{233} \cdot b_{323};$$

$$c_{331} = a_{311} \cdot b_{121} + a_{321} \cdot b_{221} + a_{331} \cdot b_{321};$$
$$c_{332} = a_{312} \cdot b_{122} + a_{322} \cdot b_{222} + a_{332} \cdot b_{322};$$
$$c_{333} = a_{313} \cdot b_{123} + a_{323} \cdot b_{223} + a_{333} \cdot b_{323};$$

It is reduce the above notes through matrix blocks

$$\begin{pmatrix} \boldsymbol{C}_3 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{B}_3 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \times \boldsymbol{B}_3 \\ \boldsymbol{A}_2 \times \boldsymbol{B}_2 \\ \boldsymbol{A}_1 \times \boldsymbol{B}_1 \end{pmatrix}$$

where

$$\boldsymbol{A}_1 = \begin{pmatrix} a_{111} & a_{121} & a_{131} \\ a_{211} & a_{221} & a_{231} \\ a_{311} & a_{321} & a_{331} \end{pmatrix}; \boldsymbol{A}_2 = \begin{pmatrix} a_{112} & a_{122} & a_{132} \\ a_{212} & a_{222} & a_{232} \\ a_{312} & a_{322} & a_{332} \end{pmatrix}; \boldsymbol{A}_3 = \begin{pmatrix} a_{113} & a_{123} & a_{133} \\ a_{213} & a_{223} & a_{233} \\ a_{313} & a_{323} & a_{333} \end{pmatrix};$$

$$\boldsymbol{B}_1 = \begin{pmatrix} b_{111} & b_{121} & b_{131} \\ b_{211} & b_{221} & b_{231} \\ b_{311} & b_{321} & b_{331} \end{pmatrix}; \boldsymbol{B}_2 = \begin{pmatrix} b_{112} & b_{122} & b_{132} \\ b_{212} & b_{222} & b_{232} \\ b_{312} & b_{322} & b_{332} \end{pmatrix}; \boldsymbol{B}_3 = \begin{pmatrix} b_{113} & b_{123} & b_{133} \\ b_{213} & b_{223} & b_{233} \\ b_{313} & b_{323} & b_{333} \end{pmatrix};$$

$$\begin{pmatrix} \boldsymbol{C}_3 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{B}_3 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \times \boldsymbol{B}_3 \\ \boldsymbol{A}_2 \times \boldsymbol{B}_2 \\ \boldsymbol{A}_1 \times \boldsymbol{B}_1 \end{pmatrix}$$

and

$$\boldsymbol{C}_1 = \begin{pmatrix} c_{111} & c_{121} & c_{131} \\ c_{211} & c_{221} & c_{231} \\ c_{311} & c_{321} & c_{331} \end{pmatrix}; \boldsymbol{C}_2 = \begin{pmatrix} c_{112} & c_{122} & c_{132} \\ c_{212} & c_{222} & c_{232} \\ c_{312} & c_{322} & c_{332} \end{pmatrix}; \boldsymbol{C}_3 = \begin{pmatrix} c_{113} & c_{123} & c_{133} \\ c_{213} & c_{223} & c_{233} \\ c_{313} & c_{323} & c_{333} \end{pmatrix}$$

$$\boldsymbol{C}_1 = \boldsymbol{A}_1 \times \boldsymbol{B}_1; \boldsymbol{C}_2 = \boldsymbol{A}_2 \times \boldsymbol{B}_2; \boldsymbol{C}_3 = \boldsymbol{A}_3 \times \boldsymbol{B}_3$$

Remark 4.1 Two dimensional matrices can think like matrix with size $m \times n \times 1$

Easy seen from the definition 1, *above it that, if* $a_{ij2} = 0, a_{ij3} = 0$ *and* $b_{ij2} = 0, b_{ij3} = 0, \forall i, j \in (1,2,3)$ *we get, the usual* 3×3*-matrix multiplication, then will take only the first vertical layer is* (*or, in the language of matrix blocks* *would say that*: $\boldsymbol{A}_2 = 0$; $\boldsymbol{A}_3 = 0$; $\boldsymbol{B}_2 = 0$; $\boldsymbol{B}_3 = 0$):

$$\begin{pmatrix} a_{111}\cdot b_{111}+a_{121}\cdot b_{211}+a_{131}\cdot b_{311} & a_{111}\cdot b_{121}+a_{121}\cdot b_{221}+a_{131}\cdot b_{321} & a_{111}\cdot b_{131}+a_{121}\cdot b_{231}+a_{131}\cdot b_{331} \\ a_{211}\cdot b_{111}+a_{221}\cdot b_{211}+a_{231}\cdot b_{311} & a_{211}\cdot b_{121}+a_{221}\cdot b_{221}+a_{231}\cdot b_{321} & a_{211}\cdot b_{121}+a_{221}\cdot b_{221}+a_{231}\cdot b_{321} \\ a_{311}\cdot b_{111}+a_{321}\cdot b_{211}+a_{331}\cdot b_{311} & a_{311}\cdot b_{121}+a_{321}\cdot b_{221}+a_{331}\cdot b_{321} & a_{311}\cdot b_{121}+a_{321}\cdot b_{221}+a_{331}\cdot b_{321} \end{pmatrix}$$

Definition 4.2. The3-D,unit matrix, associated with the "common" multiplication, must be:

$$\boldsymbol{I}_{3\times3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ \text{third vertical layer} \\ \\ \\ \\ \text{the second vertical layer} \\ \\ \\ \\ \text{the first vertical layer} \\ \\ \end{array}$$

or, in the language of matrix blocks:

$$\boldsymbol{I}_{3\times3\times3} = \begin{pmatrix} \boldsymbol{I}_{3\times3} \\ \boldsymbol{I}_{3\times3} \\ \boldsymbol{I}_{3\times3} \end{pmatrix}$$

Easy distinguish that, $\forall \boldsymbol{A} \in \mathcal{M}_{3\times3\times3}(F) / \boldsymbol{A} \otimes \boldsymbol{I}_{3\times3\times3} = \boldsymbol{A}$.

Theorem 4.1 $\left(\mathcal{M}_{3\times3\times3}(F), \otimes\right)$ is a unitary semi-Group with regard to this ordinary multiplication

Proof: 1) associative property. $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{M}_{3\times3\times3}(F)$

$$\left[\begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{B}_3 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{pmatrix}\right] \otimes \begin{pmatrix} \boldsymbol{C}_3 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \times \boldsymbol{B}_3 \\ \boldsymbol{A}_2 \times \boldsymbol{B}_2 \\ \boldsymbol{A}_1 \times \boldsymbol{B}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{C}_3 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_1 \end{pmatrix} = \begin{pmatrix} (A_3 \times B_3) \times C_3 \\ (A_2 \times B_2) \times C_2 \\ (A_1 \times B_1) \times C_1 \end{pmatrix}$$

$$\overset{(\mathcal{M}_{3\times3}(F),\times)\text{ is a semigroup}}{=} \begin{pmatrix} \boldsymbol{A}_3 \times (\boldsymbol{B}_3 \times \boldsymbol{C}_3) \\ \boldsymbol{A}_2 \times (\boldsymbol{B}_2 \times \boldsymbol{C}_2) \\ \boldsymbol{A}_1 \times (\boldsymbol{B}_1 \times \boldsymbol{C}_1) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix} \otimes \left[\begin{pmatrix} \boldsymbol{B}_3 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{C}_3 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_1 \end{pmatrix}\right].$$

2) $\exists \boldsymbol{I}_{3\times3\times3} \in \mathcal{M}_{3\times3\times3}(F) / \forall \boldsymbol{A} \in \mathcal{M}_{3\times3\times3}(F) \Rightarrow \boldsymbol{A} \times \boldsymbol{I}_{3\times3\times3} = \boldsymbol{A}$.

$$\begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{I}_{3\times3} \\ \boldsymbol{I}_{3\times3} \\ \boldsymbol{I}_{3\times3} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_3 \times \boldsymbol{I}_{3\times3} \\ \boldsymbol{A}_2 \times \boldsymbol{I}_{3\times3} \\ \boldsymbol{A}_1 \times \boldsymbol{I}_{3\times3} \end{pmatrix} \overset{(\mathcal{M}_{3\times3}(F),\times)\text{ is a unitary semigroup}}{=} \begin{pmatrix} \boldsymbol{A}_3 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_1 \end{pmatrix}$$

Theorem 4.2 $\left(\mathcal{M}_{3\times3\times3}(F), +, \otimes\right)$ *is a unitary Ring*.

Proof: 1) From Theorem 2.1. $\left(\mathcal{M}_{3\times3\times3}(F), +\right)$ is abeliangrup.

2) From Theorem 4.1. $\left(\mathcal{M}_{3\times3\times3}(F), \otimes\right)$ is a unitary semi-Group, and consequently also, $\left(\mathcal{M}_{3\times3\times3}(F), \otimes\right)$ is a unitary semi-Group

3) $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{M}_{3\times3\times3}(F)$,

a) $\boldsymbol{A} \otimes (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \otimes \boldsymbol{B} + \boldsymbol{A} \otimes \boldsymbol{C}$. b) $(\boldsymbol{A} + \boldsymbol{B}) \otimes \boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{C} + \boldsymbol{B} \otimes \boldsymbol{C}$.

truly

$$\boldsymbol{A}\otimes(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix}\boldsymbol{A}_3\\\boldsymbol{A}_2\\\boldsymbol{A}_1\end{pmatrix}\otimes\left[\begin{pmatrix}\boldsymbol{B}_3\\\boldsymbol{B}_2\\\boldsymbol{B}_1\end{pmatrix}+\begin{pmatrix}\boldsymbol{C}_3\\\boldsymbol{C}_2\\\boldsymbol{C}_1\end{pmatrix}\right]=\begin{pmatrix}\boldsymbol{A}_3\\\boldsymbol{A}_2\\\boldsymbol{A}_1\end{pmatrix}\otimes\begin{pmatrix}\boldsymbol{B}_3+\boldsymbol{C}_3\\\boldsymbol{B}_2+\boldsymbol{C}_2\\\boldsymbol{B}_1+\boldsymbol{C}_1\end{pmatrix}=\begin{pmatrix}\boldsymbol{A}_3\times(\boldsymbol{B}_3+\boldsymbol{C}_3)\\\boldsymbol{A}_2\times(\boldsymbol{B}_2+\boldsymbol{C}_2)\\\boldsymbol{A}_1\times(\boldsymbol{B}_1+\boldsymbol{C}_1)\end{pmatrix}$$

$$\overset{(\mathcal{M}_{3\times3}(F),+,\times)\text{ is a unitary Ring}}{=}\begin{pmatrix}\boldsymbol{A}_3\times\boldsymbol{B}_3+\boldsymbol{A}_3\times\boldsymbol{C}_3\\\boldsymbol{A}_2\times\boldsymbol{B}_2+\boldsymbol{A}_2\times\boldsymbol{C}_2\\\boldsymbol{A}_1\times\boldsymbol{B}_1+\boldsymbol{A}_1\times\boldsymbol{C}_1\end{pmatrix}=\begin{pmatrix}\boldsymbol{A}_3\times\boldsymbol{B}_3\\\boldsymbol{A}_2\times\boldsymbol{B}_2\\\boldsymbol{A}_1\times\boldsymbol{B}_1\end{pmatrix}+\begin{pmatrix}\boldsymbol{A}_3\times\boldsymbol{C}_3\\\boldsymbol{A}_2\times\boldsymbol{C}_2\\\boldsymbol{A}_1\times\boldsymbol{C}_1\end{pmatrix}$$

$$=\left(\begin{pmatrix}\boldsymbol{A}_3\\\boldsymbol{A}_2\\\boldsymbol{A}_1\end{pmatrix}\otimes\begin{pmatrix}\boldsymbol{B}_3\\\boldsymbol{B}_2\\\boldsymbol{B}_1\end{pmatrix}\right)+\left(\begin{pmatrix}\boldsymbol{A}_3\\\boldsymbol{A}_2\\\boldsymbol{A}_1\end{pmatrix}\otimes\begin{pmatrix}\boldsymbol{C}_3\\\boldsymbol{C}_2\\\boldsymbol{C}_1\end{pmatrix}\right)=\boldsymbol{A}\otimes\boldsymbol{B}+\boldsymbol{A}\otimes\boldsymbol{C}.$$

In a similar manner proved the point (b).

5. Multiplication of a 3-D, $3\times3\times3$-Matrix by a Scalar

Definition 5.1 *The multiplication of matrix* $\mathbf{A}\in\mathcal{M}_{3\times3\times3}(F)$ *with scalar* $\lambda\in F$, *is matrix* $\boldsymbol{C}=\lambda\circ\mathbf{A}\in\mathcal{M}_{3\times3\times3}(F)$:

$$\boldsymbol{C}=\lambda\circ\begin{pmatrix}a_{113}&a_{123}&a_{133}\\a_{213}&a_{223}&a_{233}\\a_{313}&a_{323}&a_{333}\\\\a_{112}&a_{122}&a_{132}\\a_{212}&a_{222}&a_{232}\\a_{312}&a_{322}&a_{332}\\\\a_{111}&a_{121}&a_{131}\\a_{211}&a_{221}&a_{231}\\a_{311}&a_{321}&a_{331}\end{pmatrix}=\begin{pmatrix}\lambda\cdot a_{113}&\lambda\cdot a_{123}&\lambda\cdot a_{133}\\\lambda\cdot a_{213}&\lambda\cdot a_{223}&\lambda\cdot a_{233}\\\lambda\cdot a_{313}&\lambda\cdot a_{323}&\lambda\cdot a_{333}\\\\\lambda\cdot a_{112}&\lambda\cdot a_{122}&\lambda\cdot a_{132}\\\lambda\cdot a_{212}&\lambda\cdot a_{222}&\lambda\cdot a_{232}\\\lambda\cdot a_{312}&\lambda\cdot a_{322}&\lambda\cdot a_{332}\\\\\lambda\cdot a_{111}&\lambda\cdot a_{121}&\lambda\cdot a_{131}\\\lambda\cdot a_{211}&\lambda\cdot a_{221}&\lambda\cdot a_{231}\\\lambda\cdot a_{311}&\lambda\cdot a_{321}&\lambda\cdot a_{331}\end{pmatrix}$$

So

$$\circ: F\times\mathcal{M}_{3\times3\times3}(F)\to\mathcal{M}_{3\times3\times3}(F)$$
$$(\lambda,\mathbf{A})\mapsto\lambda\circ\mathbf{A}$$

Theorem 5.1 $(\mathcal{M}_{3\times3\times3}(F),+,\circ_F)$ *is a vector space*

Proof. is evident because, $(\mathcal{M}_{3\times3}(F),+,\circ_F)$ it is the vector space, see [6] [8] [9] [10].

Definition 5.2 *The multiplication of matrix* $\mathbf{A}\in\mathcal{M}_{m\times n\times p}(F)$ *with scalar* $\lambda\in F$, *is matrix* $\boldsymbol{C}=\lambda\circ\mathbf{A}\in\mathcal{M}_{m\times n\times p}(F)$*:*
wherein each element of the matrix is multiplied (by multiplication of the field *F*) with the element $\lambda\in F$. Well, so we have

$$\circ: F\times\mathcal{M}_{m\times n\times p}(F)\to\mathcal{M}_{m\times n\times p}(F)$$
$$(\lambda,\mathbf{A})\mapsto\lambda\circ\mathbf{A}$$

Theorem 5.2 $(\mathcal{M}_{m\times n\times p}(F),+,\circ_F)$ *is a vector space*

Proof. Is evident because, $(\mathcal{M}_{m\times n}(F),+,\circ_F)$ it is the vector space, see [6] [8] [9] [10]

6. Conclusion

In this article, based on geometric considerations, and mostly considering the cube, we managed to develop the idea of the 3D matrix doing so a generalization of the 2D matrices, step by step. Furthermore, we gave a unitary ring with the elements of a field F. Initially we gave the ring $3\times 3\times 3$, 3D matrices and then generalized this concept for $m\times n\times p$, 3D matrices. At the end of this article, we present the scalar multiplication with the 3D matrices and we show that the set of $3\times 3\times 3$, 3D matrix, forms a vector space over the field *F*.

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