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On Starlike Functions Using the Generalized Salagean Differential Operator

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Abstract

In this paper we investigate the new subclass of starlike functions in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ via the generalized salagean differential operator. Basic properties of this new subclass are also discussed.

Subject Areas

Mathematical Analysis

Keywords

Salagean Differential Operator, Starlike Functions, Unit Disk, Univalent Functions, Analytic Functions and Subordination

1. Introduction

Let $\mathbb{A}$ denote the class of functions:

$$f(z)=z+a_2z^2+\cdots \tag{1}$$

which are analytic in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Denote by

$S^*=\left\{f\in\mathbb{A}:Re\dfrac{zf'(z)}{f(z)}>0,z\in U\right\}$ the class of normalized univalent functions in *U*.

Let $g(z)=z+b_2z^2+\cdots\in\mathbb{A}$. We say that $f(z)$ is subordinate to $g(z)$ (written as $f\prec g$) if there is a function *w* analytic in *U*, with $w(0)=0,|w(z)|<1$, for all $z\in U$. If *g* is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$ [1].

Definition 1 ([2]). *Let* $f\in\mathbb{A},\lambda\in(0,1]$ *and* $n\in\mathbb{N}$. *The operator* D_λ^n *is defined by*

$$
\begin{aligned}
&D_\lambda^0 f(z) = f(z),\\
&D_\lambda^1 = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z),\\
&\qquad\vdots\\
&D_\lambda^{n+1} f(z) = (1-\lambda) D_\lambda^n f(z) + \lambda z \left(D_\lambda^n f(z)\right)' = D_\lambda \left(D_\lambda^n f(z)\right), z \in U.
\end{aligned} \tag{2}
$$

Remark 1. *If* $f \in \mathbb{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then* $D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^n a_k z^k, z \in U$.

Remark 2. *For* $\lambda = 1$ *in* (2), *we obtain the Salagean differential operator.*

From (2), the following relations holds:

$$\left(D_\lambda^{n+1} f(z)\right)' = \left(D_\lambda^n f(z)\right)' + z\lambda \left(D_\lambda^n f(z)\right)'' \tag{3}$$

and from which, we get

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} = (1-\lambda) + z\lambda \frac{\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} \tag{4}$$

Definition 2 ([3]). *Let* $f \in \mathbb{A}$, *and* $n \in \mathbb{N}_0$. *Then*

$$
\begin{aligned}
I_\lambda^n f(z) = I_\lambda \left(I_\lambda^{n-1} f(z)\right) &= \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{1-\frac{1}{\lambda}} I_\lambda^{n-1} f(t) \mathrm{d}t\\
&= z + \sum_{k=2}^{\infty} \frac{1}{\left[1 + (k-1)\lambda\right]^n} a_k z^k,
\end{aligned}
$$

with $I_\lambda^0 f(z) = f(z)$.

This operator is a particular case of the operator defined in [3] and it is easy to see that for any $f \in \mathbb{A}$, $I_\lambda^n \left(D_\lambda^n f(z)\right) = D_\lambda^n \left(I_\lambda^n f(z)\right) = f(z)$.

Next, we define the new subclasses of S^*.

Definition 3. *A function* $f \in \mathbb{A}$ *belongs to the class* S_λ^n *if and only if*

$$Re \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} > (1-\lambda), \quad \lambda \in (0,1]. \tag{5}$$

Remark 3. $S_\lambda^0 \equiv S^*$.

Remark 4. $f \in S_\lambda^n$ *if and only if* $D_\lambda^n f(z) \in S^*$.

Definition 4. *Let* $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ *and* Ψ , *the set of functions* $\psi(u,v): \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *satisfying*:

i) $\psi(u,v)$ is continuous in a domain Ω of $\mathbb{C} \times \mathbb{C}$,

ii) $(1,0) \in \Omega$ and $Re\psi(1,0) > 0$,

iii) $Re\psi(u_2 i, v_1) \le 0$ when $(u_2 i, v_1) \in \Omega$ and $v_1 \le -\frac{1}{2}\left(1 + u_2^2\right)$ for $z \in U$.

Several examples of members of the set Ψ have been mentioned in [4] [5] and ([6], p. 27).

2. Preliminary Lemmas

Let P denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which are analytic in U and satisfy $Re\, p(z) > 0, z \in U$.

Lemma 1 ([5] [7]) *Let* $\psi \in \Psi$ *with corresponding domain* Ω. *If* $P(\Psi)$ *is defined as the set of functions* $p(z)$ *given as* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *which are regular in U and satisfy*:

i) $(p(z), zp'(z)) \in \Omega$

ii) $Re\psi(p(z), zp'(z)) > 0$ when $z \in U$. Then $Re\, p(z) > 0$ in *U*.

More general concepts were discussed in [4]-[6].

Lemma 2 ([8]). *Let* η *and* μ *be complex constants and* $h(z)$ *a convex univalent function in U satisfying* $h(0) = 1$, *and* $Re(\eta h(z) + \mu)$. *Suppose* $p \in P$ *satisfies the differential subordination*:

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec h(z), \quad z \in U. \tag{6}$$

If the differential subordination:

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = h(z), \quad q(0) = 1. \tag{7}$$

has univalent solution $q(z)$ in *U*. Then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant in (6).

The formal solution of (6) is given as

$$q(z) = \frac{zF'(z)}{F(z)} \tag{8}$$

where

$$F(z)^{\eta} = \frac{\eta + \mu}{z^{\mu}} \int_0^z t^{\mu - 1} H(t)^{\mu}\, \mathrm{d}t$$

and

$$H(z) = z \exp\left(\int_0^z \frac{h(t) - 1}{t} \mathrm{d}t \right)$$

see [9] [10].

Lemma 3 ([9]). *Let* $\eta \neq 0$ *and* μ *be complex constants and* $h(z)$ *regular in U with* $h'(0) \neq 0$, *then the solution* $q(z)$ *of* (7) *given by* (8) *is univalent in U if* (i) *Re* $\{G(z) = \eta h(z) + \mu\} > 0$, (ii) $Q(z) = z\frac{G'(z)}{G(z)} \in S^*$ (iii) $R(z) = \frac{Q(z)}{G(z)} \in S^*$.

3 Main Results

Theorem 1. *Let* $\lambda \in (0,1]$ *and* $h(z)$ *a convex univalent function in U satisfying* $h(0) = 1$, *and* $Re\left(\frac{1-\lambda}{\lambda} + h(z)\right)$, $z \in U$. *Let* $f \in \mathbb{A}$. *If* $\frac{D_{\lambda}^{n+2} f(z)}{D_{\lambda}^{n+1} f(z)} \prec h(z)$, *then* $\frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} \prec h(z)$.

Proof. From (4), we have

$$\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} = (1-\lambda) + z\lambda \frac{\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^{n+1} f(z)}.$$

If we suppose $\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \prec h(z)$, we need to show that $\frac{D_\lambda^{n+1} f(z)}{D_\lambda^{n} f(z)} \prec h(z)$. Using the above equation and (4) and Remark 4, it suffices to show that if $\frac{z\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^{n+1} f(z)} \prec h(z)$, then $\frac{z\left(D_\lambda^{n} f(z)\right)'}{D_\lambda^{n} f(z)} \prec h(z)$.

Now, let

$$p = \frac{z\left(D_\lambda^{n} f(z)\right)'}{D_\lambda^{n} f(z)}.$$

Then

$$z\left(D_\lambda^{n} f(z)\right)'' + \left(D_\lambda^{n} f(z)\right)' = p'(z) D_\lambda^{n} f(z) + \left(D_\lambda^{n} f(z)\right)'.$$

By (2) and (3) we have

$$\begin{aligned} \frac{z\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^{n+1} f(z)} &= \frac{\lambda z p'(z) + (1-\lambda) p(z) + \lambda p(z)^2}{(1-\lambda) + \lambda p(z)} \\ &= p(z) + \frac{z p'(z)}{\frac{1-\lambda}{\lambda} + p(z)}. \end{aligned} \tag{9}$$

Applying Lemma 2 with $\eta = 1$ and $\mu = \frac{1-\lambda}{\lambda}$, the proof is complete.□

Theorem 2. *Let* $\lambda \in (0, 1/2]$ *and* $h(z)$ *a convex univalent function in* U *satisfying* $h(0) = 1$, *and* $Re\left(\frac{1-\lambda}{\lambda} + h(z)\right) > 0, z \in U$. *Let* $f \in \mathbb{A}$. *If* $f \in S_\lambda^n$, *then*

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^{n} f(z)} \prec q(z)$$

where

$$q(z) = \frac{1 + \sum_{k=0}^{\infty} \frac{(1+k)^2}{(1+\lambda k)} z^k}{1 + \sum_{k=0}^{\infty} \frac{(1+k)}{(1+\lambda k)} z^k}$$

is the best dominant.

Proof. Let $f \in S_\lambda^{n+1}$, then by Remark 4,

$$\frac{z\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^{n+1} f(z)} \prec \frac{1+z}{1-z}.$$

By (9), we have

$$p(z)+\frac{zp'(z)}{\frac{1-\lambda}{\lambda}+p(z)}\prec\frac{1+z}{1-z},$$

where

$$p(z)=\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}.$$

To show that $\frac{z\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^n f(z)}\prec q(z)$, by Remark 4, it suffices to show that $p(z)\prec q(z)$.

Now, considering the differential equation

$$q(z)+\frac{zq'(z)}{\frac{1-\lambda}{\lambda}+q(z)}=\frac{1+z}{1-z}$$

whose solution is obtained from (8). If we proof that $q(z)$ is univalent in *U*, our result follows trivially from Lemma 2. Setting $\mu=\frac{1-\lambda}{\lambda},\eta$ and $h(z)=\frac{1+z}{1-z}$ in Lemma 3, we have

i) $ReG(z)=Re\left(\mu+h(z)\right)>0$,

ii) $Q(z)=\frac{zG'(z)}{G(z)}=2\lambda\frac{z}{(1+\beta z)(1-z)}$

where $\beta=2\lambda-1$, so that by logarithmic differentiation, we have

$$\frac{zQ'(z)}{Q(z)}=\frac{1}{1-z}+\frac{1}{1+\beta z}-1.$$

Therefore, $Re\frac{zQ'(z)}{Q(z)}>\frac{(1-2\lambda)(1+\lambda)}{2\lambda^2}>0$,

iii) $R(z)=\frac{Q(z)}{G(z)}=2\lambda^2\frac{z}{(1+\beta z)^2}$

so that

$$Re\frac{zR'(z)}{R(z)}>\frac{1-\beta}{1+\beta}=\mu>0.$$

Hence, $q(z)$ is univalent in U since it satisfies all the conditions of Lemma 3. This completes the proof.□

Theorem 3. $S_\lambda^{n+1}\subset S_\lambda^n$.

Proof. Let $f\in S_\lambda^{n+1}$. By Remark 4

$$Re\frac{z\left(D_\lambda^{n+1} f(z)\right)'}{D_\lambda^{n+1} f(z)}>0.$$

From (9), let $\psi\left(p(z), zp'(z)\right) := p(z) + \dfrac{zp'(z)}{\frac{1-\lambda}{\lambda} + p(z)}$ with $p(z) = \dfrac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}$ for $\Omega = \left[\mathbb{C} - \left\{-\frac{1-\lambda}{\lambda}\right\}\right] \times \mathbb{C}$. Conditions (i) and (ii) of Lemma 1 are clearly satisfied by ψ.

Next, $\psi(u_2 i, v_1) = u_2 i + \dfrac{v_1}{\frac{1-\lambda}{\lambda} + u_2 i}$. Then $Re\psi(u_2 i, v_1) = \dfrac{\frac{1-\lambda}{\lambda} v_1}{\left(\frac{1-\lambda}{\lambda}\right)^2 + u_2^2} \le 0$ if $v_1 \le -\frac{1}{2}\left(1 + u_2^2\right)$. Hence, $Re\, p(z) > 0$. Using Remark 4, $Re \dfrac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} > 1 - \lambda$ which complete the proof.□

Corollary 1. *All functions in* S_λ^n *are starlike univalent in U.*

Proof. The proof follows directly from Theorem 3 and Remark 4.□

Corollary 2. *The class* S_1^n "*clone*" *the analytic representation of convex functions.*

Proof. The proof is obvious from the above corollary and Definition 4.□

The functions $f(z) = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ and $g(z) = z - \frac{z^2}{2 \times 2!} + \frac{z^3}{3 \times 3!} + \cdots$ are examples of functions in S_1^n.

Theorem 4. *The class* S_λ^n *is preserve under the Bernardi integral transformation*:

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,\mathrm{d}t, \quad c > -1. \tag{10}$$

Proof. let $f \in S_\lambda^n$, then by Remark 4 $D_\lambda^n f(z) \in S^*$. From (10) we get

$$(c+1) f(z) = cF(z) + zF'(z). \tag{11}$$

Applying D_λ^n on (10) and noting from Remark 1 that $D_\lambda^n\left(zF'(z)\right) = z\left(D_\lambda^n F(z)\right)'$, we have

$$\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} = \frac{(c+1) z\left(D_\lambda^n F(z)\right)' + z^2\left(D_\lambda^n F(z)\right)''}{cD_\lambda^n F(z) + z\left(D_\lambda^n F(z)\right)'}.$$

Let $p(z) = \dfrac{z\left(D_\lambda^n F(z)\right)'}{D_\lambda^n F(z)}$ and noting that $\dfrac{z^2\left(D_\lambda^n F(z)\right)''}{D_\lambda^n f(z)} = zp'(z) + p(z)^2 - p(z)$, we get

$$\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} = p(z) + \frac{zp'(z)}{c + p(z)}$$

Let $\psi\left(p(z), zp'(z)\right) =: p(z) + \dfrac{zp'(z)}{c + p(z)}$ for $\Omega = \left[\mathbb{C} - \{-c\}\right] \times \mathbb{C}$. Then ψ satisfies all the conditions of Lemma 1 and so $Re \dfrac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} > 0 \Rightarrow Re \dfrac{z\left(D_\lambda^n F(z)\right)'}{D_\lambda^n F(z)} > 0.$ By

Remark 4 $F \in S_\lambda^n$.□

Theorem 5. *Let* $f \in S_\lambda^n$. *Then f has integral representation*:

$$f(z) = I_\lambda^n \left\{ z \exp\left(\int_0^z \frac{p(t)-1}{t} \mathrm{d}t \right) \right\}$$

for some $p \in P$.

Proof. Let $f \in S_\lambda^n$. Then by Remark 4, $D_\lambda^n F(z) \in S^*$ and so for some $p \in P$

$$\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)} = p(z).$$

But $\frac{\mathrm{d}}{\mathrm{d}z}\left(\log \frac{D_\lambda^n f(z)}{z}\right) = \frac{p(z)-1}{z}$, so that

$$D_\lambda^n f(z) = z \exp\left(\int_0^z \frac{p(t)-1}{t} \mathrm{d}t \right).$$

Applying the operator in Definition 2, we have the result.□

With $p(z) = \frac{1+z}{1-z}$, we have the extremal function for this new subclass of S^* which is

$$f_\lambda^n(z) = z + \sum_{k=2}^{\infty} \frac{k}{\left[1+(k-1)\lambda\right]^n} z^k.$$

Theorem 6. *Let* $f \in S_\lambda^n$. *Then*

$$|a_k| \le \frac{k}{\left(1+(k-1)\lambda\right)^n}, \quad k \ge 2.$$

The function $f_\lambda^n(z)$ given by (13) shows that the result is sharp.

Proof. Let $f \in S_\lambda^n$, then by Remark 4, $D_\lambda^n f(z) \in S^*$. Since it is well known that for any $f \in S^*$, $|a_k| \le k, k \ge 2$, then from Remark 1 we get the result.□

Theorem 7. *Let* $f \in S_\lambda^n$. *Then*

$$r\left(1-\mathcal{R}_\lambda^n\right) < |f(z)| < r\left(1+\mathcal{R}_\lambda^n\right)$$

and

$$1 - rR_\lambda^n \le |f'(z)| \le 1 + rR_\lambda^n,$$

where

$$\mathcal{R}_\lambda^n = \sum_{k=2}^{\infty} \frac{k}{\left[1+(k-1)\lambda\right]^n} \quad \text{and} \quad R_\lambda^n = \sum_{k=2}^{\infty} \frac{k^2}{\left[1+(k-1)\lambda\right]^n}.$$

Proof. Let $f \in \mathbb{A}$. Then by Theorem 6, we have

$$|f(z)| \le |z| + \sum_{k=2}^{\infty} |a_k||z^k| = r\left(1 + \sum_{k=2}^{\infty} \frac{k}{\left[1+(k-1)\lambda\right]^n}\right)$$

and

$$\left|f(z)\right| \geq |z| - \sum_{k=2}^{\infty}|a_k||z^k| = r\left(1 - \sum_{k=2}^{\infty}\frac{k}{\left[1+(k-1)\lambda\right]^n}\right)$$

for $|z| = r < 1$.

Also, upon differentiating $f \in \mathbb{A}$, we get

$$\left|f'(z)\right| \leq 1 + \sum_{k=2}^{\infty}k|a_k||z^{k-1}| \leq 1 + r\sum_{k=2}^{\infty}\frac{k^2}{\left[1+(k-1)\lambda\right]^n}$$

and

$$\left|f'(z)\right| \geq 1 - \sum_{k=2}^{\infty}k|a_k||z^{k-1}| \geq 1 - r\sum_{k=2}^{\infty}\frac{k^2}{\left[1+(k-1)\lambda\right]^n}$$

for $|z| = r < 1$. This complete the proof.□

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References

[1] Duren, P.L. (1983) Univalent Functions. Springer Verlag, New York Inc.

[2] Al-Oboudi, F.M. (2004) On Univalent Functions Defined by a Generalized Salagean Operator. *International Journal of Mathematics and Mathematical Sciences*, **27**, 1429-1436. http://dx.doi.org/10.1155/S0161171204108090

[3] Faisal, I. and Darus, M. (2011) Application of a New Family of Functions on the Space of Analytic Functions. *Revista Notas de Matemtica*, **7**, 144-151.

[4] Babalola, K.O. and Opoola, T.O. (2006) Iterated Integral Transforms of Caratheodory Functions and Their Applications to Analytic and Univalent Functions. *Tamkang Journal of Mathematics*, **37**, 355-366.

[5] Miller, S.S. and Mocanu, P.T. (1978) Second Order Differential Inequalities in the Complex Plane. *Journal of Mathematical Analysis and Applications*, **65**, 289-305. http://dx.doi.org/10.1016/0022-247X(78)90181-6

[6] Miller, S.S. and Mocanu, P.T. (2000) Differential Subordination, Theory and Applications. Marcel Dekker, 2000.

[7] Babalola, K.O. and Opoola, T.O. (2008) On the Coefficients of Certain Analytic and Univalent Functions. In: Dragomir, S.S. and Sofo, A., Eds., *Advances in Inequalities for Series*, Nova Science Publishers, 5-17. http://www.novapublishers.com

[8] Eenigenburg, P., Miller, S.S., Mocanu, P.T. and Read, M.O. (1984) On Briot-Bouquet Differential Surbordination. *Revue Roumaine de Mathématiques pures et Appliquées*, **29**, 567-573.

[9] Miller, S.S. and Mocanu, P.T. (1983) Univalent Solution of Briot-Bouquet Differential Equations. *Lecture Notes in Mathematics*, **1013**, Springer Berlin/Heidelberg, 292-310.

[10] Srivastava, H.M. and Lashin A.Y. (1936) Some Applications of the Briot-Bouquet Differential Surbordination. *Journal of Inequalities in Pure and Applied Mathematics*, **37**, 374-408.