An Accurate Five Off-Step Points Implicit Block Method for Direct Solution of Fourth Order Differential Equations

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Abstract

In this article, my focus is the derivation, analysis and implementation of a new modified one-step implicit hybrid block method with five off-step points. The derived method is to solve directly initial value problems of fourth order ordinary differential equations. The approach for the derivation of the method is to interpolate the approximate power series solution to the problem and to collocate its fourth derivative at the grid and off-grid points to generate systems of linear equations for the determination of the unknown parameters. The derived method is tested for consistency, zero stability, convergence and absolute stability. Accuracy and usability of the method are determined with some test problems and the results obtained are found to be better in accuracy than some existing methods.

Keywords

Interpolation, Continuous Coefficients, Block Method, Numerical Integration, Fourth Order Ordinary Differential Equations

Subject Areas: **Ordinary Differential Equation**

1. Introduction

In sciences and engineering, mathematical models are developed to understand as well as to interpret physical phenomena, many of such phenomena, when modeled, often result into higher order ordinary differential equations of the form:

$$y^{(iv)} = f\left(x, y, y', y'', y'''\right), \quad y\left(a\right) = \eta_1, y'\left(a\right) = \eta_2, y''\left(a\right) = \eta_3, y'''\left(a\right) = \eta_4 \tag{1}$$

An old conventional way to solve (1) is the method of first reducing (1) to system of first order differential

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equation of the form:

$$y' = f(x,y), y(a) = \eta_0, f \in C[a,b], x, y \in \Re \tag{2}$$

and to solve the resulting system of equations by any of the existing methods of solving first order ordinary differential equations. Literatures abounded in this old conventional method of solving problems of type (1) numerically are [1]-[3]. The drawbacks of this method include computational cumbersomeness and longer computer tine and space. In addition, [4] observes that these methods do not utilize additional information associated with a specific ordinary differential equation, such as oscillatory nature of the solution. To circumvent these drawbacks, many researchers have solved (1) directly; amongst these are [5]-[7] who develop blocked methods for numerical solution of fourth order ordinary differential equations. [8] develops linear multistep method for solution of fourth order ordinary differential equations whose implementation is Predictor-Corrector mode. Consequently, my motivation in this work is the success story of the adoption of single step method with five off-step points for direct numerical solution of fourth order ordinary differential equations which eliminate the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivatives.

2. Derivation of the Method

We take our basis function to be a power series of the form:

$$y(x) = \sum_{j=0}^{(r+s)-1} a_j \psi^j(x) \tag{3}$$

The fourth derivative of (3) gives

$$y^{(iv)}(x) = \sum_{j=0}^{(r+s)-1} j(j-1)(j-2)(j-3) a_j \psi^{j-4}(x) \tag{4}$$

By putting (4) into (1) we have the differential system:

$$\sum_{j=0}^{(r+s)-1} j(j-1)(j-2)(j-3) a_j \Psi^{j-4}(x) = f(x, y, y', y'', y''') \tag{5}$$

where a_j 's are the parameters to be determining while $r+s$ denotes number of collation and interpolation points. By collocating (5) at the mesh points $x = x_{n+j}, j = 0\left(\frac{1}{6}\right)$ and interpolating (3) at $x = x_{n+j}, j = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$ yields a system of equations:

$$\sum_{j=4}^{(r+s)-1} j(j-1)(j-2)(j-3) a_j \psi_{j-4}(x_{n+r}) = f_{n+r}, r = o, v_i, k \quad i = 1, 2, \cdots, m \tag{6}$$

$$\sum_{j=0}^{(r+s)-1} a_j \psi_j(x_{n+s}) = y_{n+s}, s = o, v_i, k \quad i = 0, 1, 2, \cdots, m \tag{7}$$

By putting these system of equations in the matrix form and then solved to obtain values of parameters a_j 's, $j = 0, \frac{1}{6}, \cdots$ which when substituted in (3), yields, after some manipulation, a hybrid linear method with continuous coefficients of the form:

$$y(t) = \sum_{j=0}^{1} \alpha_j(t) y_{n+j}(t) + h^4 \sum_{j=0}^{1} \beta_j(t) f_{n+j}(t) \tag{8}$$

The coefficient of $\alpha_j(t)$ and $\beta_j(t)$ are

$$\alpha_0(t) = \frac{1}{8}\left(2t^3 + t^2 + 5t\right), \quad \alpha_{\frac{1}{3}}(t) = -\frac{1}{4}\left(8t^3 + 22t^2 + 10t\right), \quad \alpha_{\frac{1}{2}}(t) = \frac{1}{2}\left(10t^3 + 24t^2 + 6t\right)$$

$$\alpha_{\frac{2}{3}}(t) = -\frac{1}{4}\left(32t^3 + 18t^2 + 7t + 3\right), \quad \alpha_{\frac{5}{6}}(t) = \frac{1}{2}\left(3t^3 + 4t^2 + t\right)$$

$$\beta_0(t) = \frac{1}{19595520}\left(12t^8 + t^7 - 4t^6 + 2t^5 - 4t^4 + t^3 + 4t - 6\right)$$

$$\beta_{\frac{1}{6}}(t) = \frac{1}{19595520}\left(258t^8 + 496t^7 - 315t^6 + 998t^5 + 945t^4 - 190t^3 + 178t\right)$$

$$\beta_{\frac{1}{3}}(t) = \frac{1}{19595520}\left(870t^8 + 9236t^7 - 7812t^6 + 7924t^5 + 7938t^4 + 7157t^3 - 5404t + 836\right)$$

$$\beta_{\frac{1}{2}}(t) = \frac{1}{19595520}\left(1840t^8 + 12846t^7 + 14410t^6 - 10450t^5 + 14224t^4 - 7102t^3 + 13368t + 2784\right)$$

$$\beta_{\frac{2}{3}}(t) = \frac{1}{19595520}\left(746t^8 + 3876t^7 + 2860t^6 - 2656t^5 + 4624t^4 - 3102t^3 + 3708t + 714\right)$$

$$\beta_{\frac{5}{6}}(t) = \frac{1}{19595520}\left(22t^8 - 96t^7 + 40t^6 - 136t^5 - 64t^4 + 74t^3 - 88t + 14\right)$$

$$\beta_1(t) = \frac{1}{19595520}\left(5t^8 + 12t^7 - 10t^6 + 8t^5 - 6t^4 - 8t^3 + 14t + 10\right) \tag{9}$$

where $t = \dfrac{x - x_{n+v_i}}{h}$.

We evaluate (9) at $t = 0, \frac{1}{6}, 1$ to obtain the discrete one step formula

$$y_n - 10y_{n+\frac{1}{3}} + 20y_{n+\frac{1}{2}} - 15y_{n+\frac{2}{3}} + 4y_{n+\frac{5}{6}} = \frac{h^4}{19595520}\left[4f_n + 2370f_{n+\frac{1}{6}} + 20745f_{n+\frac{1}{3}} + 41920f_{n+\frac{1}{2}} + 10770f_{n+\frac{2}{3}} - 234f_{n+\frac{5}{6}} + 25f_{n+1}\right] \tag{10a}$$

$$y_{n+\frac{1}{6}} - 4y_{n+\frac{1}{3}} + 6y_{n+\frac{1}{2}} - 4y_{n+\frac{2}{3}} + y_{n+\frac{5}{6}} = \frac{h^4}{19595520}\left[5f_n - 51f_{n+\frac{1}{6}} + 2679f_{n+\frac{1}{3}} + 9854f_{n+\frac{1}{2}} + 2679f_{n+\frac{2}{3}} - 51f_{n+\frac{5}{6}} + 5f_{n+1}\right] \tag{10b}$$

$$y_{n+1} - 4y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} - 4y_{n+\frac{1}{2}} + y_{n+\frac{1}{3}} = \frac{h^4}{19595520}\left[5f_n - 30f_{n+\frac{1}{6}} + 54f_{n+\frac{1}{3}} + 2504f_{n+\frac{1}{2}} + 10029f_{n+\frac{2}{3}} + 2574f_{n+\frac{5}{6}} - 16f_{n+1}\right] \tag{10c}$$

The first derivative of $\alpha(t)$ and $\beta(t)$ in (9) gives:

$$\alpha_0'(t) = \frac{1}{8}\left(6t^2 + 2t + 5\right) \quad \alpha_{\frac{1}{3}}'(t) = -\frac{1}{4}\left(24t^2 + 44t + 10\right)$$

$$\alpha_{\frac{1}{2}}'(t) = \frac{1}{2}\left(30t^2 + 48t + 6\right) \quad \alpha_{\frac{2}{3}}'(t) = -\frac{1}{4}\left(96t^2 + 36t + 7\right)$$

$$\alpha_{\frac{5}{6}}'(t) = \frac{1}{2}\left(9t^2 + 8t + 1\right) \quad \beta_{\frac{1}{6}}'(t) = \frac{1}{19595520}\left(2064t^7 + 3472t^6 - 1890t^5 + 4990t^4 + 3780t^3 - 570t^2 + 178\right)$$

$$\beta'_{\frac{1}{3}}(t)=\frac{1}{19595520}\left(6960t^7+64652t^6-46872t^5+39620t^4+31752t^3+21471t^2-5404\right)$$

$$\beta'_{\frac{1}{2}}(t)=\frac{1}{19595520}\left(14880t^7+12922t^6+86460t^5-52250t^4+56896t^3-21306t^2+13368\right)$$

$$\beta'_{\frac{2}{3}}(t)=\frac{1}{19595520}\left(5968t^7+27132t^6+17160t^5-13280t^4+18496t^3-9306t^2+3708\right)$$

$$\beta'_{\frac{5}{6}}(t)=\frac{1}{19595520}\left(176t^7-672t^6+240t^5-680t^4-256t^3+222t^2-88\right)$$

$$\beta'_{1}(t)=\frac{1}{19595520}\left(40t^7+84t^6-60t^5+40t^4-24t^3-24t^2+14\right) \tag{11}$$

Similarly, the second derivative of $\alpha(t)$ and $\beta(t)$ in (9) gives

$$\alpha''_0(t)=\frac{1}{8}(12t+2)\ \alpha''_{\frac{1}{3}}(t)=-\frac{1}{4}(48t+44)$$

$$\alpha''_{\frac{1}{2}}(t)=\frac{1}{2}(60t+48)\ \alpha''_{\frac{2}{3}}(t)=-\frac{1}{4}(192t+36)$$

$$\alpha''_{\frac{5}{6}}(t)=\frac{1}{2}(18t+8)$$

$$\beta'_0(t)=\frac{1}{19595520}\left(672t^6+42t^5-120t^4+40t^3-48t^2+t\right)$$

$$\beta'_{\frac{1}{6}}(t)=\frac{1}{19595520}\left(14448t^6+20832t^5-9450t^4+19960t^3+11340t^2-1140t\right)$$

$$\beta'_{\frac{1}{3}}(t)=\frac{1}{19595520}\left(48720t^6+387912t^5-234360t^4+158480t^3+95256t^2+42942t\right)$$

$$\beta'_{\frac{1}{2}}(t)=\frac{1}{19595520}\left(104160t^6+77532t^5+432300t^4-209000t^3+170688t^2-42612t\right)$$

$$\beta'_{\frac{2}{3}}(t)=\frac{1}{19595520}\left(41776t^6+162792t^5+85800t^4-53120t^3+55488t^2-18612t\right)$$

$$\beta'_{\frac{5}{6}}(t)=\frac{1}{19595520}\left(1232t^6-4032t^5+1200t^4-2720t^3-768t^2+444t\right)$$

$$\beta'_{1}(t)=\frac{1}{19595520}\left(280t^6+516t^5-300t^4+160t^3-72t^2+48t\right) \tag{12}$$

The third derivative of $\alpha(t)$ and $\beta(t)$ in (9) gives

$$\alpha'''_0(t)=\frac{1}{8}(12)\ \alpha'''_{\frac{1}{3}}(t)=-\frac{1}{4}(48)$$

$$\alpha'''_{\frac{1}{2}}(t)=\frac{1}{2}(60)$$

$$\alpha'''_{\frac{2}{3}}(t)=-\frac{1}{4}(192)$$

$$\alpha'''_{\frac{5}{6}}(t)=\frac{1}{2}(18)$$

$$\beta'''_{0}(t)=\frac{1}{19595520}\left(4032t^5+210t^4-480t^3+120t^2-48t+1\right)$$

$$\beta'''_{\frac{1}{6}}(t)=\frac{1}{19595520}\left(86688t^5+104160t^4-37800t^3+59880t^2+22680t-1140\right)$$

$$\beta'''_{\frac{1}{3}}(t)=\frac{1}{19595520}\left(292320t^5+1939560t^4-937440t^3+475440t^2+190512t+42942\right)$$

$$\beta'''_{\frac{1}{2}}(t)=\frac{1}{19595520}\left(624960t^5+387660t^4+1729200t^3-672000t^2+341376t-42612\right)$$

$$\beta'''_{\frac{2}{3}}(t)=\frac{1}{19595520}\left(250656t^5+813960t^4+343200t^3-159360t^2+110976t-18612\right)$$

$$\beta'''_{\frac{5}{6}}(t)=\frac{1}{19595520}\left(7392t^5-20160t^4+4800t^3-8160t^2-1536t+444\right)$$

$$\beta'''_{1}(t)=\frac{1}{19595520}\left(1400t^5+2580t^4-1200t^3+480t^2-144t+48\right) \tag{13}$$

It is noted that the general fourth order odes involve the first, second and third derivatives. The derivatives can be obtained by imposing that:

$$y'(x)=\frac{1}{h}\sum_{j=0}^{k-1}\alpha_j(t)\,y_{n+j}+\frac{1}{h}\sum_{v_i}\alpha_{v_i}(t)\,y_{n+v_i}+h^3\left(\sum_{j=0}^{k}\beta_j(t)\,f'_{n+j}+\sum_{v_i}\beta_{v_i}(t)\,f'_{n+v_i}\right) \tag{14}$$

By using (14) and evaluating (11), (12) and (13) at $x=x_{n+j},\ j=0\left(\frac{1}{6}\right)1,$ we obtain the first, second and the third derivative scheme as follows:

$$hy'_n+47y_{n+\frac{1}{3}}-114y_{n+\frac{1}{2}}+93y_{n+\frac{2}{3}}-26y_{n+\frac{5}{6}}$$
$$=\frac{-h^4}{195955200}\left[7658f_n+435930f_{n+\frac{1}{6}}+1848915f_{n+\frac{1}{3}}+2845040f_{n+\frac{1}{2}}+696540f_{n+\frac{2}{3}}-14418f_{n+\frac{5}{6}}+1535f_{n+1}\right]$$

$$hy'_{n+\frac{1}{6}}+26y_{n+\frac{1}{3}}-57y_{n+\frac{1}{2}}+42y_{n+\frac{2}{3}}-11y_{n+\frac{5}{6}}$$
$$=\frac{-h^4}{195955200}\left[235f_n+4713f_{n+\frac{1}{6}}+454125f_{n+\frac{1}{3}}+1144150f_{n+\frac{1}{2}}+291225f_{n+\frac{2}{3}}-4935f_{n+\frac{5}{6}}+487f_{n+1}\right]$$

$$hy'_{n+\frac{1}{3}}+11y_{n+\frac{1}{3}}-18y_{n+\frac{1}{2}}+9y_{n+\frac{2}{3}}-2y_{n+\frac{5}{6}}$$
$$=\frac{h^4}{195955200}\left[7f_n-14f_{n+\frac{1}{6}}-1000f_{n+\frac{1}{3}}-17960f_{n+\frac{1}{2}}-6445f_{n+\frac{2}{3}}+238f_{n+\frac{5}{6}}-26f_{n+1}\right]$$

$$hy'_{n+\frac{1}{2}}+2y_{n+\frac{1}{3}}+3y_{n+\frac{1}{2}}-6y_{n+\frac{2}{3}}+y_{n+\frac{5}{6}}$$
$$=\frac{-h^4}{195955200}\left[134f_n-1101f_{n+\frac{1}{6}}+3960f_{n+\frac{1}{3}}-49270f_{n+\frac{1}{2}}-30750f_{n+\frac{2}{3}}+1611f_{n+\frac{5}{6}}-184f_{n+1}\right]$$

$$hy'_{n+\frac{2}{3}} - y_{n+\frac{1}{3}} + 6y_{n+\frac{1}{2}} - 3y_{n+\frac{2}{3}} - 2y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{195955200}\left[134 f_n - 1122 f_{n+\frac{1}{6}} + 4425 f_{n+\frac{1}{3}} - 35440 f_{n+\frac{1}{2}} - 44580 f_{n+\frac{2}{3}} + 1146 f_{n+\frac{5}{6}} - 163 f_{n+1}\right]$$

$$hy'_{n+\frac{5}{6}} + 2y_{n+\frac{1}{3}} - 9y_{n+\frac{1}{2}} + 18y_{n+\frac{2}{3}} - 11y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{195955200}\left[7 f_n - 75 f_{n+\frac{1}{6}} + 385 f_{n+\frac{1}{3}} - 6690 f_{n+\frac{1}{2}} - 17715 f_{n+\frac{2}{3}} - 1147 f_{n+\frac{5}{6}} + 35 f_{n+1}\right]$$

$$hy'_{n+1} + 11y_{n+\frac{1}{3}} - 42y_{n+\frac{1}{2}} + 57y_{n+\frac{2}{3}} - 26y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{195955200}\left[235 f_n - 1158 f_{n+\frac{1}{6}} + 283000 f_{n+\frac{1}{2}} 1152375 f_{n+\frac{2}{3}} + 449190 f_{n+\frac{5}{6}} + 6358 f_{n+1}\right]$$

$$h^2 y''_n - 144y_{n+\frac{1}{3}} + 396y_{n+\frac{1}{2}} - 360y_{n+\frac{2}{3}} + 108y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{10886400}\left[19823 f_n + 322282 f_{n+\frac{1}{6}} + 575180 f_{n+\frac{1}{3}} + 728600 f_{n+\frac{1}{2}} + 140915 f_{n+\frac{2}{3}} + 2902 f_{n+\frac{5}{6}} - 502 f_{n+1}\right]$$

$$h^2 y''_{n+\frac{1}{6}} - 108y_{n+\frac{1}{3}} + 288y_{n+\frac{1}{2}} - 252y_{n+\frac{2}{3}} + 72y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{10886400}\left[802 f_n - 25137 f_{n+\frac{1}{6}} - 308500 f_{n+\frac{1}{3}} - 442510 f_{n+\frac{1}{2}} - 109290 f_{n+\frac{2}{3}} + 2983 f_{n+\frac{5}{6}} - 348 f_{n+1}\right]$$

$$h^2 y''_{n+\frac{1}{3}} - 72y_{n+\frac{1}{3}} + 180y_{n+\frac{1}{2}} - 144y_{n+\frac{2}{3}} + 36y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{10886400}\left[148 f_n - 2038 f_{n+\frac{1}{6}} + 30285 f_{n+\frac{1}{3}} + 196160 f_{n+\frac{1}{2}} + 53530 f_{n+\frac{2}{3}} - 978 f_{n+\frac{5}{6}} + 93 f_{n+1}\right]$$

$$h^2 y''_{n+\frac{1}{2}} - 36y_{n+\frac{1}{3}} + 72y_{n+\frac{1}{2}} - 36y_{n+\frac{2}{3}}$$
$$= \frac{-h^4}{10886400}\left[7 f_n - 97 f_{n+\frac{1}{6}} + 1165 f_{n+\frac{1}{3}} + 23050 f_{n+\frac{1}{2}} + 1165 f_{n+\frac{2}{3}} - 97 f_{n+\frac{5}{6}} + 7 f_{n+1}\right]$$

$$h^2 y''_{n+\frac{2}{3}} - 36y_{n+\frac{1}{2}} + 72y_{n+\frac{2}{3}} - 36y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{10886400}\left[7 f_n - 42 f_{n+\frac{1}{6}} + 50 f_{n+\frac{1}{3}} + 920 f_{n+\frac{1}{2}} + 23295 f_{n+\frac{2}{3}} + 1018 f_{n+\frac{5}{6}} - 48 f_{n+1}\right]$$

$$h^2 y''_{n+\frac{5}{6}} + 36y_{n+\frac{1}{3}} - 144y_{n+\frac{1}{2}} + 180y_{n+\frac{2}{3}} - 72y_{n+5/6}$$
$$= \frac{h^4}{10886400}\left[148 f_n - 943 f_{n+\frac{1}{6}} + 2130 f_{n+\frac{1}{3}} + 48350 f_{n+\frac{1}{2}} + 20134 f_{n+\frac{2}{3}} + 27177 f_{n+\frac{5}{6}} - 1002 f_{n+1}\right]$$

$$h^2 y''_{n+1} + 72y_{n+\frac{1}{3}} - 252y_{n+\frac{1}{2}} + 288y_{n+\frac{2}{3}} - 108y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{10886400}\left[802 f_n - 5962 f_{n+\frac{1}{6}} + 19825 f_{n+\frac{1}{3}} - 137360 f_{n+\frac{1}{2}} - 414440 f_{n+\frac{2}{3}} - 325342 f_{n+\frac{5}{6}} - 19523 f_{n+1}\right]$$

$$h^3y'''_n + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{725760}\left[36799f_n + 176608f_{n+\frac{1}{6}} + 55219f_{n+\frac{1}{3}} + 156920f_{n+\frac{1}{2}} - 10459f_{n+\frac{2}{3}} + 9608f_{n+\frac{5}{6}} - 1335f_{n+1}\right]$$

$$h^3y'''_{n+\frac{1}{6}} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{725760}\left[1375f_n - 46384f_{n+\frac{1}{6}} - 148141f_{n+\frac{1}{3}} - 81912f_{n+\frac{1}{2}} - 29963f_{n+\frac{2}{3}} + 3016f_{n+\frac{5}{6}} - 391f_{n+1}\right]$$

$$h^3y'''_{n+\frac{1}{3}} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+2/3} - 216y_{n+5/6}$$
$$= \frac{-h^4}{725760}\left[351f_n - 3872f_{n+\frac{1}{6}} + 54163f_{n+\frac{1}{3}} + 114424f_{n+\frac{1}{2}} + 15365f_{n+\frac{2}{3}} + 1160f_{n+\frac{5}{6}} - 151f_{n+1}\right]$$

$$h^3y'''_{n+\frac{1}{2}} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{725760}\left[191f_n - 1648f_{n+\frac{1}{6}} + 7475f_{n+\frac{1}{3}} + 39416f_{n+\frac{1}{2}} - 28907f_{n+\frac{2}{3}} + 2056f_{n+\frac{5}{6}} - 231f_{n+1}\right]$$

$$h^3y'''_{n+\frac{2}{3}} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{725760}\left[191f_n - 1568f_{n+\frac{1}{6}} + 6067f_{n+\frac{1}{3}} - 35592f_{n+\frac{1}{2}} - 32731f_{n+\frac{2}{3}} + 3464f_{n+\frac{5}{6}} - 311f_{n+1}\right]$$

$$h^3y'''_{n+\frac{5}{6}} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{h^4}{725760}\left[351f_n - 2608f_{n+\frac{1}{6}} + 8531f_{n+\frac{1}{3}} - 3080f_{n+\frac{1}{2}} + 126709f_{n+\frac{2}{3}} + 46792f_{n+\frac{5}{6}} - 1415f_{n+1}\right]$$

$$h^3y'''_{n+1} + 216y_{n+\frac{1}{3}} - 648y_{n+\frac{1}{2}} + 648y_{n+\frac{2}{3}} - 216y_{n+\frac{5}{6}}$$
$$= \frac{-h^4}{725760}\left[1375f_n - 10016f_{n+\frac{1}{6}} + 31891f_{n+\frac{1}{3}} - 78088f_{n+\frac{1}{2}} - 33787f_{n+\frac{2}{3}} - 177016f_{n+\frac{5}{6}} - 36759f_{n+1}\right] \quad (15)$$

By combining the schemes (10), the first, second, third derivatives schemes (15) together and write them in block form, using the definition of implicit block method in [9] to obtain the block formula describe as follows:

$$h^p\sum_{j=0}^{q} a_{i,j}y^{\lambda}_{n+j} = h^{\lambda}\sum_{j=0}^{q} e_{i,j}y^{\lambda}_n + h^{p-\lambda}\left(\sum_{j+1}^{q} d_{i,j}f_n + \sum_{j=1}^{q} b_{i,j}f_{n+j}\right), \quad i = 0,1,\cdots,q \quad (16)$$

λ is the power of the derivative of the continuous method and p is the order of the problem to solved: $q = r + s$.

This equation is solved and we obtained values for $y_{n+v_i}, y_{n+1}, y'_{n+v_i}, y'_{n+1}, y''_{n+v_i}, y''_{n+1}, y'''_{n+v_i}$ and y'''_{n+1} as follows:

$$y_{n+\frac{1}{6}} = y_n + \frac{1}{6}hy'_n + \frac{1}{72}h^2y''_n + \frac{1}{1296}h^3y'''_n + \frac{h^4}{4702924800}\left[95929f_n + 112028f_{n+\frac{1}{6}}\right.$$
$$\left. - 115165f_{n+\frac{1}{3}} + 97320f_{n+\frac{1}{2}} - 53465f_{n+\frac{2}{3}} + 16876f_{n+\frac{5}{6}} - 2323f_{n+1}\right]$$

$$y_{n+\frac{1}{3}} = y_n + \frac{1}{3}hy'_n + \frac{1}{18}h^2y''_n + \frac{1}{162}h^3y'''_n + \frac{h^4}{18370800}\Big[4127f_n + 8782f_{n+\frac{1}{6}} - 6965f_{n+\frac{1}{3}} + 5820f_{n+\frac{1}{2}} - 3175f_{n+\frac{2}{3}} + 998f_{n+\frac{5}{6}} - 137f_{n+1}\Big]$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + \frac{h^4}{6451200}\Big[5471f_n + 15228f_{n+\frac{1}{6}} - 8775f_{n+\frac{1}{3}} + 8120f_{n+\frac{1}{2}} - 4455f_{n+\frac{2}{3}} + 14040f_{n+\frac{5}{6}} - 193f_{n+1}\Big]$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}hy'_n + \frac{2}{9}h^2y''_n + \frac{4}{81}h^3y'''_n + \frac{h^4}{1148175}\Big[1220f_n + 3904f_{n+\frac{1}{6}} - 1580f_{n+\frac{1}{3}} + 1920f_{n+\frac{1}{2}} - 1015f_{n+\frac{2}{3}} + 320f_{n+\frac{5}{6}} - 44f_{n+1}\Big]$$

$$y_{n+\frac{5}{6}} = y_n + \frac{5}{6}hy'_n + \frac{25}{72}h^2y''_n + \frac{125}{1296}h^3y'''_n + \frac{h^4}{188116992}\Big[807125f_n + 2807500f_{n+\frac{1}{6}} - 790625f_{n+\frac{1}{3}} + 1425000f_{n+\frac{1}{2}} - 653125f_{n+\frac{2}{3}} + 214256f_{n+\frac{5}{6}} - 29375f_{n+1}\Big]$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{h^4}{25200}\Big[191f_n + 702f_{n+\frac{1}{6}} - 135f_{n+\frac{1}{3}} + 380f_{n+\frac{1}{2}} - 135f_{n+\frac{2}{3}} + 54f_{n+\frac{5}{6}} - 7f_{n+1}\Big] \tag{17}$$

$$y'_{n+\frac{1}{6}} = hy'_n + \frac{1}{6}h^2y''_n + \frac{1}{72}h^3y'''_n + \frac{h^3}{26123963443200}\Big[11458543529f_n + 1688294016f_{n+\frac{1}{6}} - 16488356235f_{n+\frac{1}{3}} + 13670222640f_{n+\frac{1}{2}} + 7552518045f_{n+\frac{2}{3}} + 2378490756f_{n+\frac{5}{6}} - 326924161f_{n+1}\Big]$$

$$y'_{n+\frac{1}{3}} = hy'_n + \frac{1}{3}h^2y''_n + \frac{1}{18}h^3y'''_n + \frac{h^3}{6123600}\Big[13774f_n + 35976f_{n+\frac{1}{6}} - 24465f_{n+\frac{1}{3}} + 20800f_{n+\frac{1}{2}} - 11370f_{n+\frac{2}{3}} + 3576f_{n+\frac{5}{6}} - 491f_{n+1}\Big]$$

$$y'_{n+\frac{1}{2}} = hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{8}h^3y'''_n + \frac{h^3}{1075200}\Big[5877f_n + 19188f_{n+\frac{1}{6}} - 8055f_{n+\frac{1}{3}} + 8960f_{n+\frac{1}{2}} - 4905f_{n+\frac{2}{3}} + 1548f_{n+\frac{5}{6}} - 213f_{n+1}\Big]$$

$$y'_{n+\frac{2}{3}} = hy'_n + \frac{2}{3}h^2y''_n + \frac{2}{9}h^3y'''_n + \frac{h^3}{382725}\Big[3863f_n + 13992f_{n+\frac{1}{6}} - 3390f_{n+\frac{1}{3}} + 6800f_{n+\frac{1}{2}} - 3255f_{n+\frac{2}{3}} + 1032f_{n+\frac{5}{6}} - 142f_{n+1}\Big]$$

$$y'_{n+5/6}=hy'_n+\frac{5}{6}h^2y''_n+\frac{25}{72}h^3y'''_n+\frac{h^3}{31352832}\Big[505625f_n+1945500f_{n+\frac{1}{6}}-256875f_{n+\frac{1}{3}}$$
$$+1070000f_{n+\frac{1}{2}}-358125f_{n+\frac{2}{3}}+136500f_{n+\frac{5}{6}}-18625f_{n+1}\Big]$$

$$y'_{n+1}=hy'_n+h^2y''_n+\frac{1}{2}h^3y'''_n+\frac{h^3}{8400}\Big[198f_n+792f_{n+\frac{1}{6}}-45f_{n+\frac{1}{3}}+480f_{n+\frac{1}{2}}-90f_{n+\frac{2}{3}}+72f_{n+\frac{5}{6}}-7f_{n+1}\Big] \tag{18}$$

$$y''_{n+\frac{1}{6}}=h^2y''_n+\frac{1}{6}h^3y'''_n$$
$$+\frac{h^2}{4354560}\Big[28549f_n+57750f_{n+\frac{1}{6}}-51453f_{n+\frac{1}{3}}+42484f_{n+\frac{1}{2}}-23109f_{n+\frac{2}{3}}+7254f_{n+\frac{5}{6}}-995f_{n+1}\Big]$$

$$y''_{n+\frac{1}{3}}=h^2y''_n+\frac{1}{3}h^3y'''_n+\frac{h^2}{68040}\Big[1027f_n+3492f_{n+\frac{1}{6}}-1680f_{n+\frac{1}{3}}+1576f_{n+\frac{1}{2}}-873f_{n+\frac{2}{3}}+276f_{n+\frac{5}{6}}-38f_{n+1}\Big]$$

$$y''_{n+\frac{1}{2}}=h^2y''_n+\frac{1}{2}h^3y'''_n+\frac{h^2}{53760}\Big[1265f_n+4950f_{n+\frac{1}{6}}-801f_{n+\frac{1}{3}}+2100f_{n+\frac{1}{2}}-1089f_{n+\frac{2}{3}}+342f_{n+\frac{5}{6}}-47f_{n+1}\Big]$$

$$y''_{n+\frac{2}{3}}=h^2y''_n+\frac{2}{3}h^3y'''_n+\frac{h^2}{8505}\Big[272f_n+1128f_{n+\frac{1}{6}}-18f_{n+\frac{1}{3}}+656f_{n+\frac{1}{2}}-210f_{n+\frac{2}{3}}+72f_{n+\frac{5}{6}}-10f_{n+1}\Big]$$

$$y''_{n+\frac{5}{6}}=h^2y''_n+\frac{5}{6}h^3y'''_n+\frac{h^2}{870912}\Big[1409f_n+6030f_{n+\frac{1}{6}}-375f_{n+\frac{1}{3}}+4100f_{n+\frac{1}{2}}-225f_{n+\frac{2}{3}}+462f_{n+\frac{5}{6}}-55f_{n+1}\Big]$$

$$y''_{n+1}=h^2y''_n+h^3y'''_n+\frac{h^2}{840}\Big[41f_n+180f_{n+\frac{1}{6}}+18f_{n+\frac{1}{3}}+136f_{n+\frac{1}{2}}+9f_{n+\frac{2}{3}}+36f_{n+\frac{5}{6}}\Big] \tag{19}$$

$$y'''_{n+\frac{1}{6}}=h^3y'''_n+\frac{h}{362880}\Big[19087f_n+65112f_{n+\frac{1}{6}}-46461f_{n+\frac{1}{3}}+37504f_{n+\frac{1}{2}}-20211f_{n+\frac{2}{3}}+6312f_{n+\frac{5}{6}}-863f_{n+1}\Big]$$

$$y'''_{n+\frac{1}{3}}=h^3y'''_n+\frac{h}{22680}\Big[1139f_n+5640f_{n+\frac{1}{6}}+33f_{n+\frac{1}{3}}+1328f_{n+\frac{1}{2}}-807f_{n+\frac{2}{3}}+264f_{n+\frac{5}{6}}-37f_{n+1}\Big]$$

$$y'''_{n+\frac{1}{2}}=h^3y'''_n+\frac{h}{13440}\Big[685f_n+3240f_{n+\frac{1}{6}}+561f_{n+\frac{1}{3}}+2176f_{n+\frac{1}{2}}-729f_{n+\frac{2}{3}}+216f_{n+\frac{5}{6}}-29f_{n+1}\Big]$$

$$y'''_{n+\frac{2}{3}}=h^3y'''_n+\frac{h}{45360}\Big[2288f_n+11136f_{n+\frac{1}{6}}+3072f_{n+\frac{1}{3}}+7583f_{n+\frac{1}{2}}+1392f_{n+\frac{2}{3}}+384f_{n+\frac{5}{6}}-64f_{n+1}\Big]$$

$$y'''_{n+\frac{5}{6}}=h^3y'''_n+\frac{h}{72576}\Big[3715f_n+17400f_{n+\frac{1}{6}}+6375f_{n+\frac{1}{3}}+15384f_{n+\frac{1}{2}}+11625f_{n+\frac{2}{3}}+5640f_{n+\frac{5}{6}}-275f_{n+1}\Big]$$

$$y'''_{n+1}=h^3y'''_n+\frac{h}{840}\Big[41f_n+216f_{n+\frac{1}{6}}+27f_{n+\frac{1}{3}}+272f_{n+\frac{1}{2}}+27f_{n+\frac{2}{3}}+216f_{n+\frac{5}{6}}+41f_{n+1}\Big] \tag{20}$$

3. Analysis of the Properties of the Block

In this section, we carry out the analysis of the basic properties of the new method.

3.1. Order of the Method

3.1.1. Order of the Block (17)

The linear operator of the block (17) is defined as:

$$L\{y(x):h\} = Y_m - ey_m + h^{\mu-\lambda} df(y_m) + h^{\mu-\lambda} bF(y_m) \tag{21}$$

By expanding $y(x_n + ih)$ and $f(x_n + jh)$ in Taylor series, (21) becomes:

$$L\{y(x):h\} = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) \tag{22}$$

The block (17) and associated linear operator are said to have order p if

$$C_0 = C_1 = \cdots = C_{p+3} = 0, C_{p+4} \neq 0.$$ See [10].

The term C_{p+4} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+4} h^{(p+4)} y^{(p+4)}(x_n) + 0h^{(p+5)} \tag{23}$$

Hence the block (17) has order 7 with error constant:

$$C_{p+4} = \begin{bmatrix} \frac{15739}{10861273143705600} & \frac{733}{33941478574080} & \frac{1}{11496038400} \\ \frac{37}{165729875850} & \frac{198125}{434450925748224} & \frac{1}{1231718400} \end{bmatrix}^{\mathrm{T}}$$

3.1.2. Order and Error Constant of the Main Method (10c)

By rewriting the main method (10c) in the form:

$$\begin{aligned} & y_{n+1} - 4y_{n+\frac{5}{6}} + 6y_{n+\frac{2}{3}} - 4y_{n+\frac{1}{2}} + y_{n+\frac{1}{3}} \\ & -\frac{h^4}{19595520}\left[5f_n - 30f_{n+\frac{1}{6}} + 54f_{n+\frac{1}{3}} + 2504f_{n+\frac{1}{2}} + 10029f_{n+\frac{2}{3}} + 2574f_{n+\frac{5}{6}} - 16f_{n+1}\right] = 0 \end{aligned} \tag{24}$$

Expanding (24) in Taylor series in the form:

$$\begin{aligned} & \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^{(j)} - 4\sum_{j=0}^{\infty}\left(\frac{5}{6}\right)^j \frac{h^j}{j!} y_n^{(j)} + 6\sum_{j=0}^{\infty}\left(\frac{2}{3}\right)^j \frac{h^j}{j!} y_n^{(j)} - 4\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j \frac{h^j}{j!} y_n^{(j)} + \sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^j \frac{h^j}{j!} y_n^{(j)} \\ & -\sum_{j=0}^{\infty} \frac{h^{(j+4)}}{j!} y_n^{(j+4)} \left[\frac{-30}{19595520}\left(\frac{1}{6}\right)^j + \frac{54}{19595520}\left(\frac{1}{3}\right)^j + \frac{2504}{19595520}\left(\frac{1}{2}\right)^j\right. \\ & \left. + \frac{10029}{19595520}\left(\frac{2}{3}\right)^j + \frac{2574}{19595520}\left(\frac{5}{6}\right)^j - \frac{16}{19595520}\right] - \frac{5}{19595520} h^4 y_n^{(4)} = 0 \end{aligned} \tag{25}$$

Since $C_0, \cdots, C_{10} = 0$ but $C_{11} \neq 0$ see [10]; then the main scheme is of order 7 and the error constant is:

$$C_{p+4} = \frac{1}{1097098297344}.$$

3.2. Zero Stability of the Block

The block (17) is said to be Zero stable if the roots $z_s = 1, 2, \cdots, N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$, satisfies $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover as

$$h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1),$$

where μ is the order of the differential equation, for the block (19), $r = 10, \mu = 4$

$$\rho(z) = \lambda^5(\lambda - 1) = 0 \Rightarrow \lambda = 0,0,0,0,0,1$$

Hence our method is Zero stable.

3.3. Consistency of the Main Method (10c)

From main method (10c), the first and second characteristics polynomials of the method are given by:

$$\rho(r) = r - 4r^{\frac{5}{6}} + 6r^{\frac{2}{3}} - 4r^{\frac{1}{2}} + r^{\frac{1}{3}}$$

and

$$\sigma(r) = \frac{5}{19595520} - \frac{30}{19595520}r^{\frac{1}{6}} + \frac{54}{19595520}r^{\frac{2}{3}} + \frac{2504}{19595520}r^{\frac{1}{2}} + \frac{10029}{19595520}r^{\frac{1}{3}} + \frac{2574}{19595520}r^{\frac{5}{6}} - \frac{16}{19595520}r$$

the method (10c) is consistent since it satisfies the following conditions:

1. The order of the method is $p = 7 \geq 1$ which is obvious.
2. For the method $\alpha_1 = 1$, $\alpha_{\frac{5}{6}} = -4$, $\alpha_{\frac{2}{3}} = 6$, $\alpha_{\frac{1}{2}} = -4$ and $\alpha_{\frac{1}{3}} = 1$, thus $\sum_j \alpha_j = 1 - 4 + 6 - 4 + 1 = 0,\ j = \frac{1}{3}\left(\frac{1}{6}\right)1.$
3. $\rho(r) = r - 4r^{\frac{5}{6}} + 6r^{\frac{2}{3}} - 4r^{\frac{1}{2}} + r^{\frac{1}{3}}$.
4. it follows from here that $\rho(1) = 0 = \rho'(1)$ showing that the condition (3) is satisfied as well.
5. Note that:

$$\rho^{(iv)}(r) = \frac{455}{324}r^{-\frac{13}{6}} - \frac{112}{27}r^{-\frac{10}{3}} + \frac{15}{4}r^{-\frac{7}{2}} - \frac{80}{81}r^{-\frac{11}{3}}$$

$$\Rightarrow \rho^{(iv)}(1) = 4!\sigma(1).$$

For the principal root $r = 1$: it is observed that the last condition above is satisfied, hence the main method is consistent.

3.4. Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be consistent and Zero stable. Thus since it has been successfully shown from the above condition, it could be seen that our method is convergent.

3.5. Region of Absolute Stability of the Method

We consider the stability polynomial written in general form:

$$\pi(r,\bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0$$

where $\bar{h} = h^2\lambda$ and $\lambda = \frac{df}{\delta y}$ is assumed constant. The stability polynomial of the main method (10c) becomes:

$$\left(r - 4r^{\frac{5}{6}} + 6r^{\frac{2}{3}} - 4r^{\frac{1}{2}} + r^{\frac{1}{3}}\right) - \bar{h}\left(\frac{5}{19595520}r^0 - \frac{30}{19595520}r^{\frac{1}{6}} + \frac{54}{19595520}r^{\frac{1}{3}} + \frac{2504}{19595520}r^{\frac{1}{2}} + \frac{10029}{19595520}r^{\frac{2}{3}} + \frac{2574}{19595520}r^{\frac{5}{6}} - \frac{16}{19595520}r\right) = 0 \tag{26}$$

Adopting the boundary locus method whose equation is given by:

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} \tag{27}$$

By inserting the values of $\rho(r)$ and $\sigma(r)$ into (27) and evaluate, we obtain the following results as displayed in the table below:

θ	0°	30°	60°	90°	120°	150°	180°
$\bar{h}(\theta)$	0	−0.075	−1.203	−6.088	−19.241	−46.976	−96.407

From here, it could be seen that the region of absolute stability of the method is given by $x(\theta) = (-96.407, 0)$ which satisfies the condition for A-stability, similarly the interval of periodicity lies in interval $x(\theta) = (-\infty, 0)$.

4. Numerical Experiments

To test the accuracy, workability and suitability of the method, I adopted our method to solving some initial value problems of fourth order ordinary differential equations.

Test Problem 1

I consider special fourth order problem:

$$y^{iv} = x$$
$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0, h = 0.1$$

Whose exact solution is:

$$y(x) = \frac{x^5}{120} + x$$

My method was used to solve the problem and result compared with [6]. The result is as shown in **Table 1**.

Test Problem 2

I consider a linear fourth order problem

$$y^{iv} + y'' = 0, 0 \le x \le \frac{\pi}{2}$$

$$y(0) = 0, y'(0) = \frac{1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y''' = \frac{1.2}{144 - 100\pi}, h = \frac{0.1}{32}$$

Whose exact solution is given by:

$$y(x) = \frac{1 - x - \cos x - 1.2 \sin x}{144 - 100\pi}$$

My method was used to solve the problem and result compared with [8]. The result is as shown in **Table 2**.

Numerical Results

I make use of the following Notations in the table of results:
XVAL: Value of the independent variable where numerical value is taken.
ERC: Exact result at XVAL.
NRC: Our Numerical result at XVAL.
ERR: Error of our result at XVAL.

5. Discussion of Results

In this paper, I propose an accurate five off-step points modified implicit block algorithm for the numerical solution of initial value problems of fourth order ordinary differential equations. For better performance of the method, step size is chosen within the stability interval.

Table 1. Showing results for problem 1.

XVAL	ERC	NRC	ERR P = 7 K = 1	ERR in [6] P = 4 K = 6
0.1	0.1000000833333340	0.10000008333349980	1.658E−13	7.000E−10
0.2	0.20000266666666690	0.20000266666998294	3.316E−12	8.999E−10
0.3	0.300020250000000004	0.30002025000718312	7.183E−12	2.999E−09
0.4	0.400008533333333333	0.40000853339982528	6.649E−11	5.100E−09
0.5	0.500260416666666665	0.50026041667657280	9.906E−11	7.799E−09
0.6	0.600648000000000007	0.60064800003216824	3.217E−11	1.180E−08
0.7	0.701400583333333344	0.70140058343576487	2.432E−10	1.240E−08
0.8	0.802730666666666670	0.80273066698686870	3.202E−10	1.410E−08
0.9	0.904920750000000005	0.90492075025408587	2.540E−10	1.880E−08
1.0	1.00833333333333300	1.00833333359573400	2.024E−10	2.600E−08

Table 2. Showing results for problem 2.

XVAL	ERC	NRC	ERR P = 7 K = 1	ERR in [8] P = 6 K = 4
0.103150	0.001300799589367158	0.001300799589367196	0.38142683E−18	0.49873299E−15
0.206250	0.002531773700195635	0.002531773700195672	0.37184370E−17	0.67654215E 15
0.306250	0.003652478978884993	0.003652478978887675	0.26822346E−16	0.31350790E−14
0.406250	0.004695953223180484	0.004695953223180513	0.29384802E−16	0.94360283E−13
0.506250	0.005657642360803446	0.005657642360803864	0.41813224E−15	0.22116856E−13
0.603125	0.006507754608034524	0.00650775460803811	0.38734880E−15	0.43379362E 13
0.703125	0.007298314767638522	0.007298314767638809	0.28714827E−15	0.77870869E−13
0.803125	0.007998520222728983	0.007998520222737657	0.86740034E−14	0.12863494E 12
0.903125	0.008607246703302495	0.008607246703309575	0.70802448E−14	0.19927115E−12
1.003125	0.009124283967030094	0.009124283967034006	0.35121472E−14	0.29323245E 12

The order of my method is of order 7 higher than that of [6] of order 4, which collaborates the principle, that the higher the order of a method is, the more accurate it is. The absolute errors in [6] are more than those of the new methods; this also means that the new methods are accurate than [6] which is of order 4 and implemented in block mode.

The results of my new method when also compared with the block method proposed by [8] showed that my method is more accurate.

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