



Stability and Regularization Method for Inverse Initial Value Problem of Biparabolic Equation

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Abstract

We consider an inverse initial value problem of the biparabolic equation; this problem is ill-posed and the regularization methods are needed to stabilize the numerical computations. This paper firstly establishes a conditional stability of Holder type, then uses a modified regularization method to overcome its ill-posedness and gives the convergence estimate under an *a-priori* assumption for the exact solution. Finally, a numerical example is presented to show that this method works well.

Keywords

Inverse Initial Value Problem, Biparabolic Equation, Conditional Stability, Regularization Method, Convergence Estimate

Subject Areas: Numerical Mathematics, Partial Differential Equation

1. Introduction

Let H be a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, and $L(H)$ be the Banach algebra of bounded linear operators on H . Denote $A: D(A) \subset H \rightarrow H$ as a positive and self-adjoint operator with compact resolvent; $\lambda_n (n \geq 1)$ is the real eigenvalues of A ; $X_n \in H$ is the corresponding orthonormal basis of eigenvectors, and λ_n satisfies

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n = \infty. \quad (1)$$

This paper considers the inverse initial value problem for the biparabolic equation

$$\begin{cases} \left(\frac{d}{dt} + A\right)^2 u(t) = u''(t) + 2Au'(t) + A^2u(t) = 0, & t \in (0, T), \\ u(T) = g, u_t(0) = 0, \end{cases} \tag{2}$$

our purpose is to reconstruct the initial value $f := u(0)$ from the final measured data g^δ ; here δ denotes the noisy level.

In past years, many authors have considered the inverse initial value problem of classical parabolic equation $u_t = a\Delta u$ with $a > 0$ (see [1]-[4], etc.). However, it is well-known that the classical parabolic equation can not accurately describe the procedure of heat conduction [5] [6], so many models have been proposed to describe this procedure; among them the biparabolic model proposed in [7] can give a more adequate mathematical description for the process of heat conduction than the classical case. Meanwhile we note that, for the biparabolic model, up to now the literatures devoted to it are relatively scarce, except for [7]-[9]. On other models, we can see [10]-[13], etc.

Problem (2) is ill-posed and the regularization techniques are required to stabilize numerical computations [14] [15]. In 2015, [9] considered this problem and proved a condition stability result of Hölder type, and then applied the Kozlov-Maz'ya iteration method to deal with it; the corresponding convergence results have been given, but unfortunately the condition stability result in [9] is not useful for the case of $t = 0$. In this paper, we firstly establish a conditional stability of Hölder type, which is valid at the point $t = 0$, then use a modified regularization method to overcome its ill-posedness and give the convergence estimate under an *a-priori* assumption for the exact solution. On the similar references for this regularization method, we can refer to [16]-[18], etc.

This paper is constructed as follows. In Section 2, we establish the conditional stability of Hölder type for this problem, then use a modified regularization method to deal with it and derive the convergence estimate under an *a-priori* assumption for the exact solution in Section 3. Numerical results are given in Section 4. Some conclusions are made in Section 5.

2. The Ill-Posedness and Conditional Stability Estimate

From [9], we know that the unique formal solution of problem (2) can be expressed as

$$u(t) = \sum_{n=1}^{\infty} \left(\frac{1+t\lambda_n}{1+T\lambda_n} \right) e^{(T-t)\lambda_n} g_n X_n, \quad g_n = \langle g, X_n \rangle. \tag{3}$$

It can be noticed that, for $t \in [0, T)$, $\left(\frac{1+t\lambda_n}{1+T\lambda_n} \right) e^{(T-t)\lambda_n}$ tends to infinity as $n \rightarrow \infty$, so in order to recovery the stability of solution $u(t)$ given by (3), the coefficient g_n must decay rapidly. However, such a decay usually cannot occur for the measured data g^δ , thus we have to use a regularization technique to restore numerical stability.

Note that,

$$f := u(0) = \sum_{n=1}^{\infty} \frac{e^{T\lambda_n}}{1+T\lambda_n} g_n X_n, \quad g_n = \langle g, X_n \rangle. \tag{4}$$

In general, under an additional *a-priori* bound assumption, a stability of the solution on the data can be obtained, this is called the conditional stability. Let $p > 0$, $E > 0$, here we assume the exact solution $f := u(0)$ satisfy the *a-priori* condition

$$\sum_{n=1}^{\infty} \lambda_n^2 e^{pT\lambda_n} |f_n|^2 \leq E^2, \quad f_n = \langle f, X_n \rangle, \tag{5}$$

now we give a conditional stability estimate of Hölder type for $f := u(0)$.

Theorem 2.1. *Let f given by (4) is the exact solution of problem (2) with the exact data g , assume the *a priori* bound (5) is satisfied, then we have the following stability result*

$$\|f\| \leq C(\lambda_1, p, T) E^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}, \tag{6}$$

where, $C(\lambda_1, p, T) = \left(\frac{1}{\lambda_1^2 (1+T\lambda_1)^p}\right)^{\frac{1}{p+2}}$.

Proof. Using (4), Hölder inequality, (5), we have

$$\begin{aligned} \|f\|^2 &= \left\| \sum_{n=1}^{\infty} \left(\frac{e^{T\lambda_n}}{1+T\lambda_n} \right) g_n X_n \right\|^2 = \sum_{n=1}^{\infty} \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^2 |g_n|^2 = \sum_{n=1}^{\infty} \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^2 |g_n|^{\frac{4}{p+2}} |g_n|^{\frac{2p}{p+2}} \\ &\leq \left[\sum_{n=1}^{\infty} \left(\left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^2 |g_n|^{\frac{4}{p+2}} \right)^{\frac{p+2}{2}} \right]^{\frac{2}{p+2}} \left[\sum_{n=1}^{\infty} \left(|g_n|^{\frac{2p}{p+2}} \right)^p \right]^{\frac{p}{p+2}} \\ &= \left[\sum_{n=1}^{\infty} \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^{p+2} |g_n|^2 \right]^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p} = \left[\sum_{n=1}^{\infty} \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^p \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^2 |g_n|^2 \right]^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p} \\ &= \left[\sum_{n=1}^{\infty} \left| \frac{e^{T\lambda_n}}{1+T\lambda_n} \right|^p |f_n|^2 \right]^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p} = \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2 (1+T\lambda_n)^p} \lambda_n^2 e^{pT\lambda_n} |f_n|^2 \right]^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p} \\ &\leq \left(\frac{1}{\lambda_1^2 (1+T\lambda_1)^p} \right)^{\frac{2}{p+2}} \left[\sum_{n=1}^{\infty} \lambda_n^2 e^{pT\lambda_n} |f_n|^2 \right]^{\frac{2}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p} \leq \left(\frac{1}{\lambda_1^2 (1+T\lambda_1)^p} \right)^{\frac{2}{p+2}} E^{\frac{4}{p+2}} \|g\|_{\frac{2p}{p+2}}^{2p}, \end{aligned}$$

from the above estimate, the conditional stability result (6) can be established.

3. Regularization Method and Convergence Estimate

Let the exact and noisy data $g, g^\delta \in H$ and satisfy

$$\|g^\delta - g\| \leq \delta, \tag{7}$$

where $\|\cdot\|$ denotes the H -norm. Based on the ill-posedness analysis in Section 2, we define the following modified regularization solution

$$f_\alpha^\delta := u_\alpha^\delta(0) = \sum_{n=1}^{\infty} \frac{1}{1+T\lambda_n} \frac{e^{T\lambda_n}}{1+\alpha\lambda_n e^{T\lambda_n}} g_n^\delta X_n, \quad g_n = \langle g^\delta, X_n \rangle. \tag{8}$$

here, $\alpha > 0$ plays a role of the regularization parameter. In the following, we give the convergence estimate under an *a-priori* assumption for the exact solution f .

Theorem 3.1. *Suppose that f given by (4) is the exact solution of problem (2) with the exact data g at $t = 0$, f_α^δ is the regularization solution defined by (8) with the measured data g^δ . Let the measured data g^δ satisfy (7), and the a priori bound (5) is satisfied. If the regularization parameter is chosen as*

$$\alpha = \left(\frac{\delta}{E} \right)^{\frac{2}{p+2}}, \tag{9}$$

then we have the following convergence estimate

$$\|f_\alpha^\delta - f\| \leq TE^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}} + TE \left(\ln \left(T / \left(\delta/E \right)^{\frac{2}{p+2}} \right) \right)^{-1}. \tag{10}$$

Proof. For $x > 0$, we define the function

$$h(x) = \frac{1}{\alpha x + e^{-xT}}, \tag{11}$$

it is easy to verify that $h(x)$ has a unique maximizer x_0 as $\alpha < T$ such that

$$h(x) \leq h(x_0) = h\left(\frac{\ln(T/\alpha)}{T}\right) = \frac{T}{\alpha(1 + \ln(T/\alpha))}. \tag{12}$$

Note that

$$\|f_\alpha^\delta - f\| \leq \|f_\alpha^\delta - f_\alpha\| + \|f_\alpha - f\| := I_1 + I_2. \tag{13}$$

We firstly give a estimate for I_1 . By (4), (5), (7), (8), using (12) and the fact $\frac{T}{\alpha(1 + \ln(T/\alpha))} \leq \frac{T}{\alpha}$, we get

$$\begin{aligned} I_1 &= \|f_\alpha^\delta - f_\alpha\| = \left\| \sum_{n=1}^{\infty} \left(\frac{1}{1 + T\lambda_n} \right) \frac{1}{e^{-T\lambda_n} + \alpha\lambda_n} (g_n^\delta - g_n) X_n \right\| \\ &\leq \left(\frac{1}{1 + T\lambda_1} \right) \left\| \sum_{n=1}^{\infty} \frac{1}{e^{-T\lambda_n} + \alpha\lambda_n} (g_n^\delta - g_n) X_n \right\| \\ &\leq \frac{T}{\alpha(1 + \ln(T/\alpha))} \left\| \sum_{n=1}^{\infty} (g_n^\delta - g_n) X_n \right\| \leq \frac{T}{\alpha} \left\| \sum_{n=1}^{\infty} (g_n^\delta - g_n) X_n \right\| \leq T \frac{\delta}{\alpha}. \end{aligned}$$

Below, we estimate I_2 . Using (4), (5), (8) with the exact data g , (12) and the inequality

$$\frac{T}{\alpha(1 + \ln(T/\alpha))} \leq \frac{T}{\alpha \ln(T/\alpha)}, \text{ one can obtain that}$$

$$\begin{aligned} I_2^2 &= \|f_\alpha - f\|^2 = \left\| \sum_{n=1}^{\infty} \left(\frac{1}{1 + T\lambda_n} \right) \left(\frac{1}{e^{-T\lambda_n} + \alpha\lambda_n} - e^{T\lambda_n} \right) g_n X_n \right\|^2 \\ &\leq \left\| \sum_{n=1}^{\infty} \left(\frac{1}{1 + T\lambda_n} \right) \frac{-\alpha\lambda_n e^{T\lambda_n}}{e^{-T\lambda_n} + \alpha\lambda_n} g_n X_n \right\|^2 \\ &\leq \sum_{n=1}^{\infty} \alpha^2 \left(\frac{1}{e^{-T\lambda_n} + \alpha\lambda_n} \right)^2 \lambda_n^2 e^{pT\lambda_n} \left(\frac{e^{T\lambda_n}}{1 + T\lambda_n} \right)^2 g_n^2 \\ &\leq \alpha^2 \left(\frac{T}{\alpha(1 + \ln(T/\alpha))} \right)^2 \sum_{n=1}^{\infty} \lambda_n^2 e^{pT\lambda_n} |f_n|^2 \leq \left(\frac{TE}{\ln(T/\alpha)} \right)^2. \end{aligned}$$

From the above estimates of I_1 , I_2 , and combining with (9), the triangle inequality (13), we can obtain the convergence result (10).

4. Numerical Implementations

In this section, we use a numerical example to verify how this method works for the reconstruction of initial data f . Consider the following forward problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right)^2 u(x, t) = 0, & x \in (0, \pi), t \in (0, T), \\ u_t(x, 0) = 0, & x \in [0, \pi], \\ u(x, 0) = \sin(x) + \sin(2x), & x \in [0, \pi], \\ u(0, t) = u(\pi, t) = 0, & t \in [0, T], \end{cases} \tag{14}$$

where $H = L^2(0, \pi)$, $A = -\frac{\partial^2}{\partial x^2}$ with the domain $D(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \subset H$, its eigenvalue and the eigenfunction are $\lambda_n = n^2$, $X_n = \sqrt{\frac{2}{\pi}} \sin(nx)$, respectively.

By the method of separation of variables, it is easy to obtain that the solution of problem (14) can be expressed as

$$u(x, t) = \sum_{n=1}^{\infty} c_n (1 + tn^2) e^{-tn^2} \sin(nx), \tag{15}$$

where, $c_n = \frac{2}{\pi} \int_0^{\pi} (\sin(x) + \sin(2x)) \sin(nx) dx$. We take the exact data as

$$g(x) = u(x, T) = \sum_{n=1}^m c_n (1 + Tn^2) e^{-Tn^2} \sin(nx), \tag{16}$$

the measured data is chosen as $g^\delta(x) = g(x) + \varepsilon \text{rand}(\text{size}(g))$, where ε is the error level.

In the computational procedure, the exact and regularization solutions are computed by (4), (8), respectively. The regularization parameter α is chosen by (9) with $p = 1, E = 1$. For $\varepsilon = 0.001$, the numerical results for $f(x) = u(x, 0)$, $f_\alpha^\delta(x) = u_\alpha^\delta(x, 0)$ constructed from $g^\delta = u^\delta(x, T)$ with $T = 0.5, 1, 2, 2.5$ are shown in **Figure 1**. For $\varepsilon = 0.0001, 0.001, 0.005, 0.01$, the numerical results for $f(x)$, $f_\alpha^\delta(x)$ constructed from $T = 1$ are shown in **Figure 2**.

From **Figure 1** and **Figure 2**, we can see that this method is effective and feasible. **Figure 1** indicates that, with the increase of T , the construction effects become worse, this is because the information of final value data will become less when T becomes big. **Figure 2** shows that the smaller ε is, the better the computed efficiency is, this is a normal phenomena in the inverse initial value problem of parabolic equation.

5. Conclusion

An inverse initial value problem of the bipolarabolic equation is investigated. We firstly establish a conditional stability of Hölder type for this problem, then use a modified regularization method to regularize it and derive

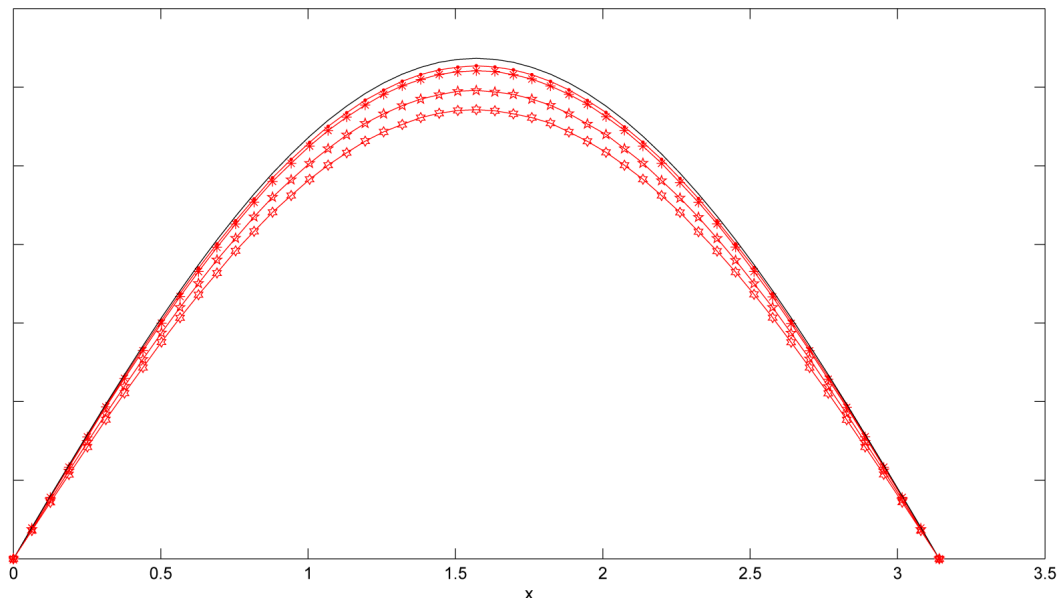


Figure 1. $\varepsilon = 0.001$, $f(x), f_\alpha^\delta(x)$ from different T (dot: $T = 0.5$, star: $T = 1$, pentagram: $T = 2$, hexagram: $T = 2.5$).

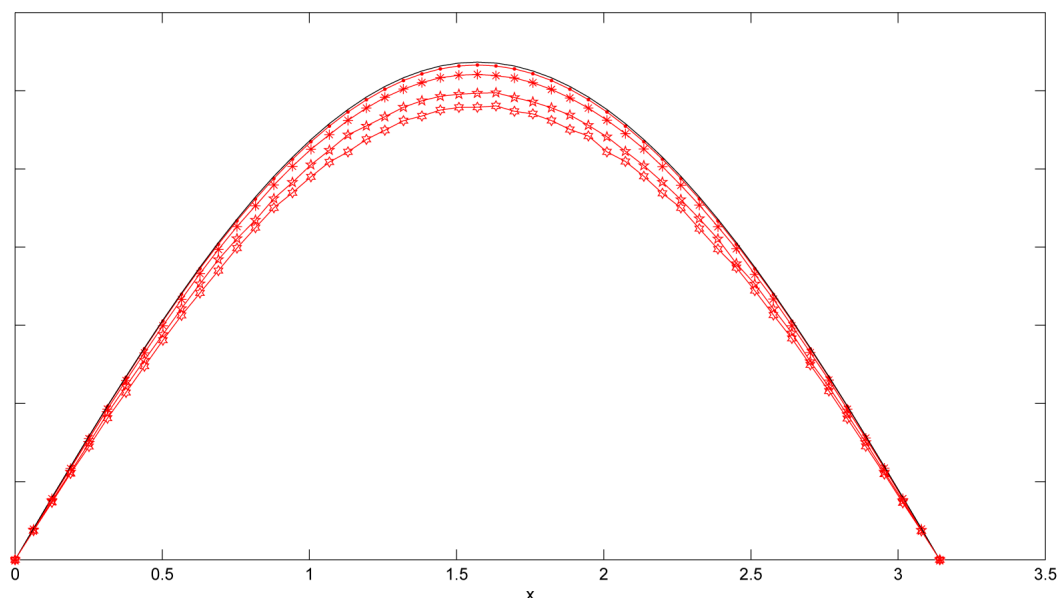


Figure 2. $f(x), f_{\alpha}^{\delta}(x)$ from $T=1$ under different ε (dot: $\varepsilon=0.0001$, star: $\varepsilon=0.001$ pentagram: $\varepsilon=0.005$, hexagram: $\varepsilon=0.01$).

the convergence estimate under an *a-priori* assumption for the exact solution. Numerical results show that this method is stable and feasible.

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References

- [1] Cheng, J. and Liu, J.J. (2008) A Quasi Tikhonov Regularization for a Two-Dimensional Backward Heat Problem by a Fundamental Solution. *Inverse Problems*, **24**, Article ID: 065012. <http://dx.doi.org/10.1088/0266-5611/24/6/065012>
- [2] Feng, X.L., Qian, Z. and Fu, C.L. (2008) Numerical Approximation of Solution of Nonhomogeneous Backward Heat Conduction Problem in Bounded Region. *Mathematics and Computers in Simulation*, **79**, 177-188. <http://dx.doi.org/10.1016/j.matcom.2007.11.005>
- [3] Liu, J.J. (2002) Numerical Solution of Forward and Backward Problem for 2-d Heat Conduction Equation. *Journal of Computational and Applied Mathematics*, **145**, 459-482. [http://dx.doi.org/10.1016/S0377-0427\(01\)00595-7](http://dx.doi.org/10.1016/S0377-0427(01)00595-7)
- [4] Qian, Z., Fu, C.L. and Shi, R. (2007) A Modified Method for a Backward Heat Conduction Problem. *Applied Mathematics and Computation*, **185**, 564-573. <http://dx.doi.org/10.1016/j.amc.2006.07.055>
- [5] Fichera, G. (1992) Is The Fourier Theory of Heat Propagation Paradoxical? *Rendiconti del Circolo Matematico di Palermo*, **41**, 5-28.
- [6] Joseph, L. and Preziosi, D.D. (1989) Heat Waves. *Reviews of Modern Physics*, **61**, 41. <http://dx.doi.org/10.1103/revmodphys.61.41>
- [7] Fushchich, V.L., Galitsyn, A.S. and Polubinskii, A.S. (1990) A New Mathematical Model of Heat Conduction Processes. *Ukrainian Mathematical Journal*, **42**, 210-216. <http://dx.doi.org/10.1007/BF01071016>
- [8] Atakhadzaev, M.A. and Egamberdiev, O.M. (1990) The Cauchy Problem for the Abstract Bicoloric Equation. *Sibirskii Matematicheskii Zhurnal*, **31**, 187-191.
- [9] Lakhdari, A. and Boussetila, N. (2015) An Iterative Regularization Method for an Abstract Ill-Posed Biparabolic Problem. *Boundary Value Problems*, **55**, 1-17. <http://dx.doi.org/10.1186/s13661-015-0318-4>

- [10] Ames, K.A. and Straughan, B. (1997) *Non-Standard and Improperly Posed Problems*. Academic Press, New York.
- [11] Carasso, A.S. (2010) Bochner Subordination, Logarithmic Diffusion Equations, and Blind Deconvolution of Hubble Space Telescope Imagery and Other Scientific Data. *SIAM Journal on Imaging Sciences*, **3**, 954-980. <http://dx.doi.org/10.1137/090780225>
- [12] Payne, L.E. (2006) On a Proposed Model for Heat Conduction. *IMA Journal of Applied Mathematics*, **71**, 590-599. <http://dx.doi.org/10.1093/imat/hxh112>
- [13] Wang, L., Zhou, X. and Wei, X. (2008) *Heat Conduction: Mathematical Models and Analytical Solutions*. Springer-Verlag, Berlin.
- [14] Engl, H.W., Hanke, M. and Neubauer, A. (1996) *Regularization of Inverse Problems*, Volume 375 of *Mathematics and Its Applications*. Kluwer Academic Publishers Group, Dordrecht.
- [15] Kirsch, A. (1996) *An Introduction to the Mathematical Theory of Inverse Problems*. Volume 120 of *Applied Mathematical Sciences*. Springer-Verlag, New York. <http://dx.doi.org/10.1007/978-1-4612-5338-9>
- [16] Ames, K.A., Clark, G.W., Epperson, J.F. and Oppenheimer, S.F. (1998) A Comparison of Regularizations for an Ill-Posed Problem. *Mathematics of Computation*, **67**, 1451-1472. <http://dx.doi.org/10.1090/S0025-5718-98-01014-X>
- [17] Clark, G.W. and Oppenheimer, S.F. (1994) Quasireversibility Methods for Non-Well-Posed Problems. *Electronic Journal of Differential Equations*, 1-9.
- [18] Denche, M. and Bessila, K. (2005) A Modified Quasi-Boundary Value Method for Ill-Posed Problems. *Journal of Mathematical Analysis and Applications*, **301**, 419-426. <http://dx.doi.org/10.1016/j.jmaa.2004.08.001>