Transient Solution of M/M/2/N System Subjected to Catastrophe cum Restoration

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Abstract

In this paper, we study the distribution of the number of times that a finite capacity with equal servers Markovian queuing model catastrophic-cum-restorations reaches its capacity in time *t*. The occurrence of a catastrophe makes the system empty instantly but the system takes its own time to be ready to accept new customers. This time is referred to as "restoration time". The aforesaid distribution is obtained as a marginal distribution of the joint distribution of the number of customers in the system at time *t* and the number of times system reaches its capacity in time *t* under the conditions of catastrophes and restorations.

Keywords

1. Introduction

Catastrophe modeling and analysis had been playing a vital role in various areas of science and technology. Chao [1] has modeled computer networks with a virus by queuing networks with catastrophes. Kumar *et al*. [2] and [3] studied the transient behaviour of the M/M/2 queue with catastrophes. Di Crescenzo *et al*. [4] made a continuous approximation of M/M/1 queue with catastrophe. Jain and Kumar [5] and [6] obtained the transient solution of a catastrophic-cum-restorative queueing problem with correlated arrivals and variable service capacity and special case of [7]-[10]. In all the above mentioned studies, the researchers have obtained the state probabilities in one way or the other and have computed various measures of performance. In this paper, the occurrence of a catastrophe makes the system empty instantly whenever the system is not empty but the system takes its own time to ready to accept new customers. This time is referred to as "the restoration time". The system subjected to catastrophes must take some time for its restoration after the occurrence of a catastrophe. We have obtained explicitly the distribution of the number of times the system reaches its capacity in time *t* under the effects of catastrophe

and restoration.

2. The Queue Model

We consider an M/M/2/N queueing system having two homogenous servers with FCFS discipline subject to catastrophes and restorations. The customers arrive at a counter in accordance with a Poisson process with mean arrival rate $\lambda > 0$. Each server serves one customer at a time if available. The service time distribution of a customer is negative exponential with mean rate $\mu > 0$. The queuing process starts at time zero with zero state of the system. Catastrophes occur according to Poisson process with mean rate ξ only when the system is not empty. The occurrence of a catastrophe destroys all the customers in the instants and affects the system as well. The system will require some sort of time to restarts in a normal way, which is taken as restoration time. The restoration times are independently, identically exponentially distributed with parameter $\beta > 0$. The customers arrive in the system during the restoration time as usual.

We define joint probability distribution

$$P_{m,n}(t) = \text{Prob.}\big[X(t) = m, Y(t) = n\big], \quad 0 \leq n \leq N \tag{1}$$

where $X(t) =$ the number of times the system reaches its capacity in time t;

$Y(t) =$ the number of customers in the system at time t;

$P_{m,00}(t) =$ the prob. that there are zero customers in the system at time t without the occurrence of catastrophe;

$Q_{m,00}(t) =$ the prob. that there are zero customers in the system at time t with the occurrence of catastrophe destroying all the customers.

3. Time Dependent Probabilities

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{m,00}(t) = -\lambda P_{m,00}(t) + \mu P_{m,1}(t) + \beta Q_{m,00}(t), \quad n = 0 \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{m,00}(t) = -(\lambda + \beta) Q_{m,00}(t) + \xi \sum_{n=1}^{N} P_{m,n}(t), \quad n = 0 \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{m,1}(t) = -(\lambda + \mu + \xi) P_{m,1}(t) + \lambda P_{m,0}(t) + 2\mu P_{m,2}(t), \quad n = 1, \, m \geq 0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{m,n}(t) = -(\lambda + 2\mu + \xi) P_{m,n}(t) + \lambda P_{m,n-1}(t) + 2\mu P_{m,n+1}(t), \quad 1 < n < N, \, m \geq 0 \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{m,N}(t) = -(2\mu + \xi) P_{m,N}(t) + \lambda P_{m-1,N-1}(t), \quad n = N, \, m \geq 1 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{0,N}(t) = 0 \tag{7}$$

Taking Laplace transform of the Equations (2)-(6) w.r.t. t we have

$$s\bar{P}_{0,00}(s) = -\lambda \bar{P}_{0,00}(s) + \mu \bar{P}_{0,1}(s) + \beta \bar{Q}_{0,00}(s) + 1, \quad m = 0 \tag{8}$$

$$s\bar{Q}_{0,00}(s) = -(\lambda + \beta) \bar{Q}_{0,00}(s) + \xi \big[\bar{P}_{0,.}(s) - \bar{P}_{0,00}(s)\big], \quad m = 0 \tag{9}$$

$$s\bar{P}_{m,00}(s) = -\lambda \bar{P}_{m,00}(s) + \mu \bar{P}_{m,1}(s) + \beta \bar{Q}_{m,00}(s), \quad m \geq 1 \tag{10}$$

$$s\bar{Q}_{m,00}(s) = -(\lambda + \beta) \bar{Q}_{m,00}(s) + \xi \big[\bar{P}_{m,.}(s) - \bar{P}_{m,00}(s)\big], \quad m \geq 1 \tag{11}$$

$$s\bar{P}_{m,1}(s) = -(\lambda + \mu + \xi) \bar{P}_{m,1}(s) + \lambda \bar{P}_{m,0}(s) + 2\mu \bar{P}_{m,2}(s), \quad n = 1 \tag{12}$$

$$s\bar{P}_{m,n}(s) = -(\lambda + 2\mu + \xi) \bar{P}_{m,n}(s) + \lambda \bar{P}_{m,n-1}(s) + 2\mu \bar{P}_{m,n+1}(s), \quad 2 \leq n < N \tag{13}$$

$$s\bar{P}_{m,N}(s) = -(2\mu + \xi) \bar{P}_{m,N}(s) + \lambda \bar{P}_{m-1,N-1}(s), \quad n = N \tag{14}$$

$$s\overline{P}_{0,N}(s) = 0.$$

Since $P_{0,0}(0) = 1$
where,

$$\overline{P}_{m,n}(s) = \int_0^\infty e^{-st} P_{m,n}(t)\,dt.$$

Define the probability generating functions by

$$\overline{P}_n(x,s) = \sum_{m=0}^\infty \overline{P}_{m,n}(s) x^m \tag{15}$$

$$\overline{H}(x,y;s) = \sum_{n=0}^N \overline{P}_n(x,s) y^n \tag{16}$$

$$\overline{P}(x,s) = \sum_{m=0}^\infty \overline{P}_{m,.}(s) x^m. \tag{17}$$

Multiplying Equation (8) to (14) by x^m, summing over the ranges of m and using (15), we have

$$(s+\lambda+\xi)\overline{P}_0(x,s) = 1 + \mu\overline{P}_1(x,s) + \xi\overline{P}_.(x,s), \quad n = 0 \tag{18}$$

$$(s+\lambda+\mu+\xi)\overline{P}_1(x,s) = \lambda\overline{P}_0(x,s) + 2\mu\overline{P}_2(x,s), \quad n = 1 \tag{19}$$

$$(s+\lambda+2\mu+\xi)\overline{P}_n(x,s) = \lambda\overline{P}_{n-1}(x,s) + 2\mu\overline{P}_{n+1}(x,s), \quad n = 2,3,\cdots,N-1 \tag{20}$$

$$(s+2\mu+\xi)\overline{P}_N(x,s) = \lambda x\overline{P}_{N-1}(x,s), \quad n = N. \tag{21}$$

Multiplying Equation (18) to (21) by y^n, summing over the ranges of n and using Equations (16) (17), we have on simplification:

$$\overline{H}(x,y;s)$$
$$= \frac{y[s+\xi(2-y)] - s(1-y)[2\mu + y(s+\lambda+\xi)]\overline{P}_0(x,s) + \lambda\left[sy(x-1)(2-y) + sy^{N+1}\left\{(x-1) + \frac{\lambda x(1-y)}{s+2\mu+\xi}\right\}\right]\overline{P}_{N-1}(x,s)}{s[(s+\xi)y + (1-y)(\lambda y - 2\mu)]}. \tag{22}$$

The zeros of the denominator in (22) are given by

$$\alpha_i(s) = \frac{(s+\lambda+2\mu+\xi) \pm \sqrt{(s+\lambda+2\mu+\xi)^2 - 4\times 2\lambda\mu}}{2\lambda}, \quad i = 1,2. \tag{23}$$

The existence of $\overline{H}(x,y;s)$ is only possible if numerator vanishes for α_1 and α_2 the two zeros of the denominator. This will give rise two equations, solving them we have:

$$\overline{P}_0(x,s) = \frac{\overline{A}_1(x,s)}{\overline{A}(x,s)} \tag{24}$$

$$\overline{P}_{N-1}(x,s) = \frac{\overline{A}_2(.,s)}{\overline{A}(x,s)} \tag{25}$$

where

$$\overline{A}_1(x,s) = 2\mu\lambda\left[\frac{\lambda x}{s+2\mu+\xi}\left[(s+2\xi)\{V(N+1)-V(N)\} - \xi\gamma^{-2}\{V(N)-V(N-1)\}\right]\right.$$

$$\left. + (1-x)\left[s\{V(1)-V(N)\} - \xi\{V(N)-\gamma^{-2}V(N-1)\}\right]\right],$$

$$\overline{A}_2(.,s) = 2\mu\left[(s+\xi)(s+2\lambda) + 2\mu\xi\right]V(1)$$

$$\bar{A}(x,s) = 2\mu\lambda^2 s\left[(1-x)\left[2V(1)+\{V(N+1)-\gamma^{-2}V(N)\}+\frac{s+\lambda+\xi}{\lambda}\{V(N)-\gamma^{-2}V(N-1)\}\right]\right.$$

$$+\frac{x}{s+2\mu+\xi}\left[\lambda\{V(N+2)-\gamma^{-2}V(N+1)\}+(s+\xi)\{V(N+1)-\gamma^{-2}V(N)\}\right.$$

$$\left.\left.-(s+\lambda+\xi)\{V(N)-\gamma^{-2}V(N-1)\}\right]\right],$$

$$V(r)=\alpha_1^r(s)-\alpha_2^r(s),\quad r=0,1,2,\cdots.$$

Now $\bar{P}_n(x,s)$ is the coefficient of y^n in (16). Comparing the coefficients of y^n on both sides, we have:

$$\bar{P}_n(x,s)=\frac{\gamma^{2n}}{s\lambda V(1)}\left[\{2\mu s\gamma^2 T(n)+s(s+\lambda+\xi)T(n-1)\}\bar{P}_0(x,s)-\xi T(n-1)-(s+\xi)V(n)\right],0\le n\le N. \quad (26)$$

Applying the Leibniz differentiation theorem to (24), setting $x=0$ and dividing both sides by $m!$. On simplification, we have:

$$\bar{P}_{m,00}(s)=\frac{(s+\lambda+\beta)\bar{P}_{m,0}(s)}{(s+\lambda+\beta-\xi)}$$

$$-\frac{\xi\left[\sum_{n=0}^{N}\gamma^{2n}\left[s\lambda V(1)\right]^{-1}\left[\{2\mu s\gamma^2 T(n)+s(s+\lambda+\xi)T(n-1)\}\bar{P}_{m,0}(s)-\xi T(n-1)-(s+\xi)V(n)\right]\right]}{(s+\lambda+\beta-\xi)}, \quad (27)$$

$$\bar{Q}_{m,00}(s)=\frac{\xi\left[\sum_{n=0}^{N}\gamma^{2n}\left[s\lambda V(1)\right]^{-1}\left[\{2\mu s\gamma^2 T(n)+s(s+\lambda+\xi)T(n-1)\}\bar{P}_{m,0}(s)-\xi T(n-1)-(s+\xi)V(n)\right]\right]}{(s+\lambda+\beta-\xi)}$$

$$-\frac{\xi\bar{P}_{m,0}(s)}{(s+\lambda+\beta-\xi)}. \quad (28)$$

From (27), we have

$$\bar{P}_{m,n}(s)=\frac{\gamma^{2n}}{s\lambda V(1)}\left[\{2\mu s\gamma^2 T(n)+s(s+\lambda+\xi)T(n-1)\}\bar{P}_{m,0}(s)-(s+\xi)V(n)-\xi T(n-1)\right],\quad m\ge 1 \quad (29)$$

where

$$\bar{P}_{m,0}(s)$$

$$=\left[\frac{\begin{array}{l}\{sV(1)-(s+\xi)V(N)+\xi\gamma^{-2}V(N-1)\}\{2\lambda\gamma^2(s+2\mu+\xi)V(1)-\gamma^2\lambda^2 T(N+1)\\ -(s-\lambda+2\mu+\xi)[(s+\xi)\gamma^2 T(N-1)-\lambda V(N-1)]\}\\ \times\{2\lambda\gamma^2[(s+2\mu+\xi)V(1)+\mu T(N)]+\gamma^2 T(N-1)[(s+\lambda+\xi)^2+2\mu(s+\lambda+\xi)]-\gamma^2\lambda^2 V(N+2)\}^{m-1}\end{array}}{s\gamma^{2m}(s+2\mu+\xi)^m\{2\lambda V(1)+\lambda T(N)+(s+\lambda+\xi)T(N-1)\}^{m+1}}\right].$$

We know that,

$$(1-a/b)^{-(j+1)}=\sum_{i=0}^{\infty}\left(C_i^{j+i}\right)\frac{a^i}{b^i},\quad (a+b)^{j+1}=a(a+b)^j+b(a+b)^j\quad\text{and}\quad (a-b)^j=\sum_{i=0}^{j}(-1)^i\left(C_i^j\right)a^{j-i}b^i.$$

Using these identities in above equations, We are now in a position to complete the solution for the joint distribution of $X(t)$ and $Y(t)$. Taking the Inverse Laplace transform, using the tables [11]-[13], we have:

$$P_{0,n}(t) = \sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{l+h}\binom{j+1}{l}\binom{l}{h}\binom{j+l+r}{r}2^{-(j+1)}\lambda^{-(j-l+2)}\gamma^{n-2(j+k-l)}$$

$$\times\Bigg[\Big\{2\mu\gamma^{-1}\big[I_{(g+j+h-l-n-1)}-I_{(g+j+h-l+n+1)}\big]-\gamma^{-1}\big[I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\big]\Big\}e^{-(\lambda+2\mu+\xi)t}$$

$$+\int_0^t\frac{(t-w)^{(k-l-1)}}{(k-l-1)!}\Big[\big\{\big[I_{(g+j+h-l-n)}-I_{(g+j+h-l+n)}\big]-\gamma^{-1}\big[I_{(g+j+h-l+n+1)}-I_{(g+j+h-l+n-1)}\big]\big\}\big(2\sqrt{2\lambda\mu}w\big)e^{-(\lambda+2\mu+\xi)w}\Big]$$

$$\times e^{-(\lambda+\xi)(t-w)}dw\Bigg]-\sum_{i=0}^{N-1}\gamma^{-(N-1)}(2i-N+1)I_{(2i-N+1)}t^{-1}e^{-(\lambda+2\mu+\xi)t}$$

$$+\xi\Bigg[\int_0^t\Bigg[\Bigg[\Big[\sum_{i=0}^{N-2}\gamma^{-N}(2i-N+2)I_{(2i-N+2)}\Big]-\Big[\sum_{i=0}^{N-1}\gamma^{-(N-1)}(2i-N+1)I_{(2i-N+1)}\Big]$$

$$+\lambda^{-1}\Big[\Big[\sum_{i=0}^{n-2}\gamma^n(2i-n+2)I_{(2i-n+2)}\Big]-3\Big[\sum_{i=0}^{n-1}\gamma^{n+1}(2i-n+1)I_{(2i-n+1)}\Big]\Big]\Bigg]\big(2\sqrt{2\lambda\mu}w\big)w^{-1}e^{-(\lambda+2\mu+\xi)w}\Bigg]e^{-(\lambda+\xi)(t-w)}dw\Bigg],$$

$$(30)$$

$$P_{m,00}(t) = \sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{p=0}^{h}\sum_{q=0}^{j-l}\sum_{r=0}^{\infty}(-1)^{i+l+h+p+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+k}{k}\binom{j+i+1}{l}\binom{l}{h}\binom{h}{p}$$

$$\times\binom{j-l}{p}\binom{j+l+q+r}{r}\lambda^{m-i-l+p}2^{m-i-l+p-q}\mu^l\gamma^{-(2m+j-i-q)}\int_0^t\frac{(t-w)^{m-i-l+p-1}}{(m-i-l+p-1)!}\Bigg[\int_0^w\frac{(w-u)^{m-i-l+k-1}}{(m-i-l+k-1)!}$$

$$\times\Big[\big\{(\lambda+\beta)e^{-(\lambda+\beta-\xi)u}-(\lambda+\beta-\xi)e^{-(\lambda+\beta-\xi)u}\big\}\big[I_{(g+j+i+h+p+1)}-\gamma I_{(g+j+i+h+p+2)}\big]$$

$$\times\big(2\sqrt{2\lambda\mu u}\big)e^{-(\lambda+2\mu+\xi)u}\Big]e^{-(\lambda+\xi)u}du\Bigg]e^{-(2\mu+\xi)(t-w)}dw$$

$$-\xi\int_0^t\frac{(t-w)}{1!}\Bigg[\Big[\sum_{i=0}^{N-2}\gamma^{-(N-1)}(2i-N+3)I_{(2i-N+3)}-\lambda^{-1}\Big[\sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^n(2i-n+2)I_{(2i-n+2)}\Big]\Big]$$

$$\times\big(2\sqrt{2\lambda\mu}w\big)w^{-1}e^{-(\lambda+2\mu+\xi)w}\Bigg]e^{-(t-w)}dw$$

$$-\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{p=0}^{h}\sum_{q=0}^{j-l}\sum_{r=0}^{\infty}\sum_{n=0}^{N}(-1)^{i+l+h+p+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+k}{k}\binom{j+i+1}{l}\binom{l}{h}$$

$$\times\binom{h}{p}\binom{j-l}{p}\binom{j+l+q+r}{r}\lambda^{m-i-l+p-1}2^{m-i-l+p-q}\mu^l\gamma^{-(2m+j-i-q-n)}\int_0^t\frac{(t-w)^{m-i-l+p-1}}{(m-i-l+p-1)!}\Bigg[\Big[\int_0^w\frac{(w-u)^{m-i-l+k-1}}{(m-i-l+k-1)!}$$

$$\times\Big[\big\{2\mu\gamma^{-1}(B-n)I_{(B-n)}+2\mu(B+n+3)I_{(B+n+3)}+(B-n+3)I_{(B-n+3)}+\gamma^{-1}(B+n)I_{(B+n)}$$

$$+\gamma\big[1-2\mu\gamma^{-2}\big](B+n+2)I_{(B+n+2)}-\gamma(B-n+2)I_{(B-n+2)}+2\mu(B-n+1)I_{(B-n+1)}\big\}$$

$$\times\big(2\sqrt{2\lambda\mu u}\big)u^{-1}e^{-(\lambda+2\mu+\xi)u}\Big]e^{-(\lambda+\xi)(w-u)}du+\int_0^w\frac{(w-u)^{m-i-l+k-2}}{(m-i-l+k-2)!}\Big[(B-n+1)I_{(B-n+1)}-2(B+n+1)I_{(B+n+1)}$$

$$+(B-n+3)I_{(B-n+3)}+\gamma^{-1}(B+n)I_{(B+n)}+\gamma\big[(B+n+2)I_{(B+n+2)}-(1+\gamma^{-2})(B-n+2)I_{(B-n+2)}\big]$$

$$\times\big(2\sqrt{2\lambda\mu u}\big)u^{-1}e^{-(\lambda+2\mu+\xi)u}\Big]e^{-(\lambda+\xi)(w-u)}du\Big]e^{-(2\mu+\xi)w}\Bigg]e^{-(\lambda+\beta-\xi)(t-w)}dw$$

$$-\xi\int_0^t\frac{(t-w)}{1!}\Bigg[\Big[\sum_{i=0}^{N-2}\gamma^{-(N-1)}(2i-N+3)I_{(2i-N+3)}-\lambda^{-1}\Big[\sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)}-\sum_{i=0}^{n-2}\gamma^n(2i-n+2)I_{(2i-n+2)}\Big]\Big]$$

$$(31)$$

$$\times\big(2\sqrt{2\lambda\mu}w\big)w^{-1}e^{-(\lambda+2\mu+\xi)w}\Bigg]e^{-(t-w)}dw$$

$$-\xi\lambda^{-1}\int_0^t\Bigg[\Big[\sum_{i=0}^{N-1}\gamma^{-(N-2)}(2i-N+2)I_{(2i-N+2)}+\sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)}\Big]\big(2\sqrt{2\lambda\mu}w\big)w^{-1}e^{-(\lambda+2\mu+\xi)w}\Bigg]e^{-(t-w)}dw,$$

$$Q_{m,00}(t) = \left[\sum_{i=0}^{m-1} \sum_{j=0}^{\infty} \sum_{l=0}^{j+i+1} \sum_{h=0}^{l} \sum_{k=0}^{\infty} \sum_{p=0}^{h} \sum_{q=0}^{j-l} \sum_{r=0}^{\infty} \sum_{n=0}^{N} (-1)^{i+l+h+p+q} \binom{m-1}{i} \binom{m+j}{j} \binom{j+i+k}{k} \binom{j+i+1}{l} \binom{l}{h} \binom{h}{p} \binom{j-l}{p} \right.$$

$$\times \binom{j+l+q+r}{r} \lambda^{m-i-l+p-1} 2^{m-i-l+p-q} \mu^{l} \gamma^{-(2m+j-i-q-n)} \int_{0}^{t} \frac{(t-w)^{m-i-l+p-1}}{(m-i-l+p-1)!} \left[\left[\int_{0}^{w} \frac{(w-u)^{m-i-l+k-1}}{(m-i-l+k-1)!} \right. \right.$$

$$\times \left[\left\{ 2\mu\gamma^{-1}(B-n)I_{(B-n)} + 2\mu(B+n+3)I_{(B+n+3)} + (B-n+3)I_{(B-n+3)} + \gamma^{-1}(B+n)I_{(B+n)} \right. \right.$$

$$+ \gamma\left[1-2\mu\gamma^{-2}\right](B+n+2)I_{(B+n+2)} - \gamma(B-n+2)I_{(B-n+2)} + 2\mu(B-n+1)I_{(B-n+1)} \Big\}$$

$$\times \left(2\sqrt{2\lambda\mu u}\right)u^{-1}e^{-(\lambda+2\mu+\xi)u}\right]e^{-(\lambda+\xi)(w-u)}du + \int_{0}^{w} \frac{(w-u)^{m-i-l+k-2}}{(m-i-l+k-2)!}\left[(B-n+1)I_{(B-n+1)} - 2(B+n+1)I_{(B+n+1)}\right.$$

$$+ (B-n+3)I_{(B-n+3)} + \gamma^{-1}(B+n)I_{(B+n)} + \gamma\left[(B+n+2)I_{(B+n+2)} - (1+\gamma^{-2})(B-n+2)I_{(B-n+2)}\right]$$

$$\times \left(2\sqrt{2\lambda\mu u}\right)u^{-1}e^{-(\lambda+2\mu+\xi)u}\right]e^{-(\lambda+\xi)(w-u)}du\right]e^{-(2\mu+\xi)w}\right]e^{-(\lambda+\beta-\xi)(t-w)}dw$$

$$- \xi\int_{0}^{t} \frac{(t-w)}{1!}\left[\sum_{i=0}^{N-2}\gamma^{-(N-1)}(2i-N+3)I_{(2i-N+3)} - \lambda^{-1}\left[\sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)} - \sum_{i=0}^{n-2}\gamma^{n}(2i-n+2)I_{(2i-n+2)}\right]\right]$$

$$\times \left(2\sqrt{2\lambda\mu w}\right)w^{-1}e^{-(\lambda+2\mu+\xi)w}\right]e^{-(t-w)}dw - \xi\lambda^{-1}\int_{0}^{t}\left[\left[\sum_{i=0}^{N-1}\gamma^{-(N-2)}(2i-N+2)I_{(2i-N+2)} + \sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)}\right]\right.$$

$$\times \left(2\sqrt{2\lambda\mu w}\right)w^{-1}e^{-(\lambda+2\mu+\xi)w}\right]e^{-(t-w)}dw\right]$$

$$- \left[\sum_{i=0}^{m-1}\sum_{j=0}^{\infty}\sum_{l=0}^{j+i+1}\sum_{h=0}^{l}\sum_{k=0}^{\infty}\sum_{p=0}^{h}\sum_{q=0}^{j-l}\sum_{r=0}^{\infty}(-1)^{i+l+h+p+q}\binom{m-1}{i}\binom{m+j}{j}\binom{j+i+k}{k}\binom{j+i+1}{l}\binom{l}{h}\binom{h}{p}\binom{j-l}{p}\binom{j+l+q+r}{r}\right.$$

$$\times \lambda^{m-i-l+p} 2^{m-i-l+p-q} \mu^{l} \gamma^{-(2m+j-i-q)} \int_{0}^{t} \frac{(t-w)^{m-i-l+p-1}}{(m-i-l+p-1)!}\left[\int_{0}^{w}\frac{(w-u)^{m-i-l+k-1}}{(m-i-l+k-1)!}\left[\left[I_{(g+j+i+h+p+1)} - \gamma I_{(g+j+i+h+p+2)}\right]\right.\right.$$

$$\times \left(2\sqrt{2\lambda\mu u}\right)e^{-(\lambda+2\mu+\xi)u}\right]e^{-(\lambda+\xi)u}du\right]e^{-(2\mu+\xi)(t-w)}dw$$

$$- \xi\int_{0}^{t}\frac{(t-w)}{1!}\left[\sum_{i=0}^{N-2}\gamma^{-(N-1)}(2i-N+3)I_{(2i-N+3)} - \lambda^{-1}\left[\sum_{i=0}^{n-1}\gamma^{(n+1)}(2i-n+1)I_{(2i-n+1)} - \sum_{i=0}^{n-2}\gamma^{n}(2i-n+2)I_{(2i-n+2)}\right]\right]$$

$$\times \left(2\sqrt{2\lambda\mu w}\right)w^{-1}e^{-(\lambda+2\mu+\xi)w}\right]e^{-(\lambda+\beta-\xi)(t-w)}dw\right],$$

$$(32)$$

$$P_{m,n}(t)$$

$$= \left[\sum_{i=0}^{m-1} \sum_{j=0}^{\infty} \sum_{l=0}^{j+i+1} \sum_{h=0}^{l} \sum_{k=0}^{\infty} \sum_{p=0}^{h} \sum_{q=0}^{j-l} \sum_{r=0}^{\infty} (-1)^{i+l+h+p+q} \binom{m-1}{i} \binom{m+j}{j} \binom{j+i+k}{k} \binom{j+i+1}{l} \binom{l}{h} \binom{h}{p} \binom{j-l}{p} \right.$$

$$\times \binom{j+l+q+r}{r} \lambda^{(m-i-l+p-1)} 2^{(m-i-l+p-q)} \mu^l \gamma^{-(2m+j-i-q-n)} \int_0^t \left[\int_0^w \frac{(w-u)^{m-i+k-l-1}}{(m-i+k-l-1)!} \{ 2\mu\gamma^{-1} (A-n) I_{(A-n)} \right.$$

$$+ 2\mu(A+n+3) I_{(A+n+3)} + (A-n+3) I_{(A-n+3)} + \gamma^{-1}(A+n) I_{(A+n)} + \gamma^{-1} \left[1 - 2\mu\gamma^{-2} \right] (A+n+2) I_{(A+n+2)}$$

$$- \gamma(A-n+2) I_{(A-n+2)} + 2\mu(A-n+1) I_{(A-n+1)} \} \left(2\sqrt{2\lambda\mu}u \right) u^{-1} e^{-(\lambda+2\mu+\xi)u} du$$

$$+ \int_0^w \frac{(w-u)^{m-i+k-l-2}}{(m-i+k-l-2)!} \left[(A-n+1) I_{(A-n+1)} - 2(A+n+1) I_{(A+n+1)} + (A-n+3) I_{(A-n+3)} + \gamma^{-1}(A+n) I_{(A+n)} \right.$$

$$+ \gamma \left[(A+n+2) I_{(A+n+2)} - \left(1 + \gamma^{-2} \right) (A-n+2) I_{(A-n+2)} \right] \left. \right] \left(2\sqrt{2\lambda\mu}u \right) u^{-1} e^{-(\lambda+2\mu+\xi)u} du \right] dw$$

$$- \xi \int_0^t (t-w) \left[\sum_{i=0}^{N-2} \gamma^{-(N-1)} (2i-N+3) I_{(2i-N+3)} - \lambda^{-1} \left[\sum_{i=0}^{n-1} \gamma^{n+1} (2i-n+1) I_{(2i-n+1)} - \sum_{i=0}^{n-2} \gamma^n (2i-n+2) I_{(2i-n+2)} \right] \right]$$

$$\times \left(2\sqrt{2\lambda\mu}w \right) w^{-1} e^{-(\lambda+2\mu+\xi)w} dw$$

$$- \xi\lambda^{-1} \int_0^t \left[\sum_{i=0}^{N-1} \gamma^{-(N-2)} (2i-N+2) I_{(2i-N+2)} + \sum_{i=0}^{n-1} \gamma^{n+1} (2i-n+1) I_{(2i-n+1)} \right] \left(2\sqrt{2\lambda\mu}w \right) w^{-1} e^{-(\lambda+2\mu+\xi)w} dw, \tag{33}$$

where,

$$A = \left[g+j+i+p \right], \quad I_v(\alpha t) \equiv I_v, \quad B = \left[j+i+2N(l+k)+p+(2N+3)h \right], \text{ and } \alpha = 2\sqrt{2\lambda\mu}.$$

4. Conclusion

This paper discussed the transient solution of M/M/2/N queuing system with catastrophic and restorations effects on the number of times system reached its capacity in time t. This model finds its application in computer-network communications, telecommunications etc.

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