Elementary Uncertain Renewal Reward Theorem and Its Strict Proof

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Abstract

Uncertain renewal reward process, of which the interarrival times and rewards (or costs) are regarded as uncertain variables, is an important branch of Liu's uncertainty theory. At present, there is a lack of strict proof for the elementary theorem. Therefore, the paper gives its strict proof with two lemmas by some techniques.

Keywords

1. Introduction

In probability theory, renewal process and renewal reward process are two important uncertain processes in which interarrival times and rewards are regarded as random variables.

Note that probability theory is applicable only when the obtained probability is close enough to the real frequency. Otherwise, some counterintuitive results will happen [1]. But in real life, we are often lack of observed data or historical data to estimate the probability distributions of interarrival times and reward, so we have to invite some domain experts to evaluate their belief degree of the interarrival times and reward. Since human tends to overweight unlikely events (Kahneman and Tversky [2]), the belief degree may have a much larger than the real frequency. Thus probability theory fails to model the renewal process and renewal reward process in this situation. In order to resolve these problems, an uncertainty theory is founded by Liu [3] and refined by Liu [4] based on normality, duality, subadditivity and product axioms. Nowadays, uncertainty theory has been applied to uncertain programming [5] [6], uncertain process [7]-[10] etc. [11] [12], uncertainty theory. In the framework of uncertainty theory, Liu [13] first assumed the interarrival times and reward of an renewal process as uncertain variables, and proposed an uncertain renewal process. Then Liu [4] also proposed an

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uncertain renewal reward process which interarrival times and rewards were both regarded as uncertain variables and gave the an elementary renewal reward theorem. At present, there is a lack of strict proof for the elementary theorem. Therefore, the paper will give its strict proof with two lemmas by some techniques.

2. Preliminary

Definition 1. (Liu [3]) *Let* $\mathcal{L}$ *be a* σ *-algebra on nonempty set* Γ. *A set function* $\mathcal{M}$ *is called an uncertain measure if it satisfies the following axioms*:
Axiom 1. (Normality) $M\{\Gamma\} = 1$; for the universal set Γ;
Axiom 2. (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\{\Lambda\}, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.
In [14], Liu further presented the following axiom:
Axiom 4. (Product Axiom) Let $(\{\Lambda\}_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{n}\Lambda_k\right\} = \min_{1\le k\le n}\mathcal{M}_k\{\Lambda_k\}.$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.
Definition 2. (Liu [3]) *An uncertain variable is a measurable function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any Borel set B of real numbers, the set* $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ *is an event*.
Definition 3. (Liu [3]) *The uncertainty distribution* Φ *of an uncertain variable* ξ *is defined by* $\Phi(x) = \mathcal{M}\{\xi \le x\}$ *for any real number x.*
Definition 4. (Liu [4]) *An uncertainty distribution* Φ *is said to be regular if its inverse function* $\Phi^{-1}(\alpha)$ *exists and is unique for each* $\alpha \in (0,1)$.
Definition 5. (Liu [14]) *The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_m$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\le i\le m}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.
Definition 6. (Liu [3] (2007)) *The expected value of uncertain variable* ξ *is defined by*

$$E[\xi] = \int_0^{\infty}\mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0\mathcal{M}\{\xi \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.
Theorem 1. (Liu [4]) *Let* ξ *be an uncertain variable with uncertainty distribution* Φ. *If the expected value exists, then* $E[\xi] = \int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha.$

Theorem 2. (Liu [14]) *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$, *respectively. If* $f(x_1, x_2, \cdots, x_n)$ *is strictly increasing with respect to* $x_1, x_2, \cdots, x_m$ *and strictly decreasing with respect to* $x_{m+1}, x_2, \cdots, x_n$, *then* $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ *is an uncertain variable with uncertainty distribution*

$$\Psi(x) = \sup_{f(x_1, x_2, \cdots, x_n)=x}\left(\min_{1\le i\le m}\Phi_i(x_i) \wedge \min_{m+1\le i\le n}(1-\Phi_i(x_i))\right)$$

and inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)\right).$$

In particular, if $\xi_1, \xi_2, \cdots, \xi_n$ have a common uncertainty distribution Φ, then $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ have a uncertainty distribution $\Psi(x) = \Phi\left(\dfrac{x}{n}\right)$.

Definition 7. (Liu [3]) *Let* $\xi, \xi_1, \xi_2, \cdots$ *be a sequence of uncertain variables with uncertainty distributions* $\Phi, \Phi_1, \Phi_2, \cdots$, *respectively, then* ξ_i *is said to converge in distribution to* ξ *if* $\lim\limits_{i \to \infty} \Phi_i(x) = \Phi(x)$ *at every continuous point x of* Φ.

3. Uncertain Renewal Reward Process

Definition 8. (Liu [13]) *Let* T *be a index set and let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space. An uncertain process is a measurable function from* $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers, i.e., for any* $t \in T$ *and any Borel set B of real numbers, the set* $\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\}$ *is an event.*

Definition 9. (Liu [13]) *Let* $\xi_1, \xi_2, \cdots$ *be independent and identical distribution(iid) positive uncertain variables. Define* $S_0 = 0$ *and* $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ *for* $n \geq 1$. *Then the uncertain process* $N_t = \max\limits_{n \geq 0}\{n \mid S_n \leq t\}$ *is called a renewal process.*

Note that event $\{N_t \leq k\}$ is same with event $\{S_{K+1} \leq t\}$.

For an uncertain renewal process, Liu [4] proved that $\dfrac{N_t}{t}$ converges in mean to $\dfrac{1}{\xi_1}$, *i.e.*,

$$\lim_{t \to \infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right].$$

Definition 10. (Liu [4]) *Let* $\xi_1, \xi_2, \cdots$ *be iid uncertain interarrival times, and let* $\eta_1, \eta_2, \cdots$ *be uncertain rewards. It is also assume that* $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ *are independent. Then*

$$R_t = \sum_{i=1}^{N_t} \eta_i$$

is called a renewal reward process, where N_t *is the renewal process.*

Theorem 3. (Liu [4]) *Let* R_t *be a renewal reward process with uncertain interarrival times* $\xi_1, \xi_2, \cdots$ *and uncertain rewards* $\eta_1, \eta_2, \cdots$. *If those interarrival times* $\xi_1, \xi_2, \xi_3 \cdots$ *and rewards* $\eta_1, \eta_2, \eta_3, \cdots$ *have uncertainty distributions* Φ *and* Ψ, *then* R_t *has an uncertainty distribution*

$$\Upsilon_t(x) = \max_{k \geq 0}\left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).$$

Here we set $x/k = \infty$ and $\Phi(x/k) = 1$ when $k = 0$.

Liu gave an elementary uncertain renewal reward theorem in the book [4] (see latter Theorem 4). But, it is not strict to proof of the theorem. Therefore, in the following we strict prove it by two lemmas.

Lemma 1. If $\Phi(x)$ and $\Psi(x)$ are nonnegative continuous strict increasing functions on $[0, +\infty)$, and $\Phi(0) = \Psi(0) = 0$, $\lim\limits_{x \to +\infty} \Phi(x) = \lim\limits_{x \to +\infty} \Psi(x) = 1$, then

(i) for given $x \in [0, +\infty)$, there exists $s_x \in (0, +\infty)$ such that

$$\sup_{s \geq 0}\left((1 - \Phi(s)) \wedge \Psi(sx)\right) = \max_{s \geq 0}\left((1 - \Phi(s)) \wedge \Psi(sx)\right)$$

$$= (1 - \Phi(s_x)) \wedge \Psi(s_x x) = 1 - \Phi(s_x) = \Psi(s_x x)$$

(ii)

$$\lim_{t \to +\infty} \max_{k \geq 1} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right) = \sup_{s \geq 0} \left(1 - \Phi(s)\right) \wedge \Psi(sx)$$

Proof. Proof of (i) is easy. In following we prove (ii). Note that we have the following facts:

$$\frac{t}{k+1} \leq \frac{t}{k} \leq s_x, \quad \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{t}{k}x\right) = \Psi\left(\frac{t}{k}x\right) \leq \Psi(s_x x),$$

$$s_x \leq \frac{t}{k+1} \leq \frac{t}{k}, \quad \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{t}{k}x\right) = \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \leq \Psi(s_x x),$$

$$\frac{t}{k+1} \leq s_x \leq \frac{t}{k}, \quad \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{t}{k}x\right) \geq \Psi(s_x x).$$

For given $t \in [s_x + 1, +\infty)$, there exists $k_0 = \left[\frac{t}{s_x}\right]$ such that

$$\frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right] + 1} s_x \leq s_x \leq \frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right]} s_x$$

and for any integer $k \neq_0 k$ such that

$$\frac{t}{k+1} < \frac{t}{\left[\frac{t}{s_x}\right] + 1} < s_x < \frac{t}{\left[\frac{t}{s_x}\right]} < \frac{t}{k}$$

Thus, when $t > s_x + 1$,

$$\max_{k \geq 1} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right) = \left(1 - \Phi\left(\frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right] + 1} s_x\right)\right) \wedge \Psi\left(\frac{\frac{t}{s_x}x}{\left[\frac{t}{s_x}\right]}\right).$$

Also,

$$\lim_{t \to +\infty} \frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right] + 1} = \lim_{t \to +\infty} \frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right]} = 1,$$

and function Ψ at s_x is continuous, thus

$$\lim_{t \to +\infty} \max_{k \geq 1} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right) = \lim_{t \to +\infty} \left(1 - \Phi\left(\frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right] + 1} s_x\right)\right) \wedge \Psi\left(\frac{\frac{t}{s_x}}{\left[\frac{t}{s_x}\right]} s_x x\right) = \Psi(s_x x).$$

i.e.,

$$\lim_{t \to +\infty} \max_{k \geq 1} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right) = \Psi(s_x x) = \sup_{s \geq 0} \left(1 - \Phi(s)\right) \wedge \Psi(sx).$$

Lemma 2. If conditions of Lemma 1 are satisfied, and

$$\int_0^{+\infty}\left(1-\sup_{s\geq0}\left(1-\Phi(s)\right)\wedge\Psi(sx)\right)dx$$

converge, then

$$\int_0^{+\infty}\left(1-\max_{k\geq1}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\left(\frac{tx}{k}\right)\right)dx$$

consistent convergent on $[0,+\infty)$ about t.

Proof. It follows from process of proof of Lemma 1 that, for any $t\in[0,+\infty)$,

$$\sup_{s\geq0}\left(1-\Phi(s)\right)\wedge(sx)\leq\max_{k\geq1}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{tx}{k}\right).$$

Therefore, for any $t\in[0,+\infty)$,

$$1-\max_{k\geq1}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{tx}{k}\right)\leq1-\sup_{s\geq0}\left(1-\Phi(s)\right)\wedge\Psi(sx).$$

also,

$$\int_0^{+\infty}\left(1-\sup_{s\geq0}\left(1-\Phi(s)\right)\wedge\Psi(sx)\right)dx,$$

is convergent, then

$$\int_0^{+\infty}\left(1-\max_{k\geq1}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{tx}{k}\right)\right)dx$$

is consistent convergent on $[0,+\infty)$ about t.

Theorem 4. (Elementary uncertain renewal reward theorem, Liu [4]) *Let R_t be a renewal reward process with uncertain interarrival times $\xi_1,\xi_2,\cdots$ and uncertain rewards $\eta_1,\eta_2,\cdots$ If $E[\eta_1/\xi_1]$ exists, then $\lim_{t\to\infty}\frac{E[R_t]}{t}=E\left[\frac{\eta_1}{\xi_1}\right]$. If those interarrival times $\xi_1,\xi_2,\cdots$ and rewards $\eta_1,\eta_2,\cdots$ have regular uncertainty distribution Φ and Ψ satisfy the following conditions $\Phi(0)=\Psi(0)=0$, and $\lim_{x\to+\infty}\Phi(x)=\lim_{x\to+\infty}\Psi(x)=1$, then*

$$\lim_{t\to\infty}\frac{E[R_t]}{t}=\int_0^1\frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)}d\alpha.$$

Proof. Firstly, note that uncertainty distribution of η_1/ξ_1 is

$$\gamma_{\eta_1/\xi_1}(x)=\sup_{s\geq0}\left(1-\Phi(s)\right)\wedge\Psi(sx),$$

and

$$E[\eta_1/\xi_1]=\int_0^{+\infty}\left(1-\sup_{s\geq0}\left((1-\Phi(s))\wedge\Psi(sx)\right)\right)dx=\int_0^1\frac{\Psi^{-1}(\alpha)}{1-\Phi^{-1}(\alpha)}d\alpha.$$

Since the uncertainty distribution of R_t is

$$G_t(x)=\max_{k\geq0}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{x}{k}\right),$$

and the uncertainty distribution of R_t/t is

$$G_t(x)=\max_{k\geq0}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{tx}{k}\right),$$

using **Lemma 2** we have

$$\lim_{t \to +\infty} \frac{E[R_t]}{t} = \lim_{t \to +\infty} \int_0^{+\infty} \left(1 - \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{t}{k}x\right)\right) \mathrm{d}x$$

$$= \int_0^{+\infty} \left(1 - \sup_{s \geq 0}(1 - \Phi(s)) \wedge \Psi(sx)\right) \mathrm{d}x = \int_0^1 \frac{\Psi^{-1}(\alpha)}{1 - \Phi^{-1}(\alpha)} \mathrm{d}\alpha = E\left[\frac{\eta_1}{\xi_1}\right].$$

2. Conclusion

This paper provides a strict proof of elementary uncertain renewal reward theorem by some technics.

Acknowledgements

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