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The Integral Equation, Corresponding to the Ordinary Differential Equation

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Abstract

The differential operator of the ordinary differential equation (ODE) is represented as the sum of two operators: basic and supplementing operators. The order of the higher derivatives of a basic operator and ODE operator should coincide. If the basic operator has explicit system of fundamental solutions it is possible to make integral equation Volterra of II kind. For linear equations the approximate solutions of the integral equation are system of the approximate fundamental solutions of the initial ODE.

Keywords

Fundamental Solutions

Subject Areas: **Numerical Mathematics, Ordinary Differential Equation**

1. Introduction

Connection between the ordinary differential equations (ODE) and integral equations was considered for a long time, and the basic outcomes were included into textbooks and handbooks (for example [1]-[6]). For arbitrary the ODE there is a method allowing to construct integral equation Volterra of II kind. Such integral equation provides a solution of the Cauchy problem with arbitrary initial conditions. Mathematical procedures are familiar and it relieves of necessity to do extensive references to the literature.

The offered method differs from known. The operator of linear ODE is represented as the sum of two operators, one of which has explicit system of fundamental solutions. The integral equation is constructed by these solutions and solutions of the integral equation are the system (usually approximate) fundamental solutions of ODE. Such method can be considered also as a variant of a perturbation method.

2. Construction of the Integral Equation

Let's consider linear differential operator L with the maximum derivative *of* n-th order which forms the homo-

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geneous ODE

$$L\left(u\left(x\right)\right)=0. \tag{2.1}$$

We shall present L as a difference of operators

$$L=l_B-l_S. \tag{2.2}$$

We shall name: l_B is the basic operator and l_S is the supplementing operator. The basic operator should contain the highest derivative of operator L, the supplementing operator has no such restriction.

Let's consider the inhomogeneous equation with the basic operator

$$l_B\left(u\right)=F\left(x\right). \tag{2.3}$$

If the homogeneous equation, corresponding (2.3), has system of fundamental solutions $\varphi_i(x)$ and the inhomogeneous equation has private solution $W(F)$, where W is integral operator, the general solution of the Equation (2.3) with arbitrary constants C_i notes in the form

$$u=\sum_{i=1}^{n}C_i\varphi_i+W\left(F\left(x\right)\right). \tag{2.4}$$

We shall write the Equation (2.1) by means of (2.2) in the form

$$l_B\left(u\right)=l_S\left(u\right). \tag{2.5}$$

Comparing (2.3) and (2.5) and using (2.4), we receive the integro-differential equation for $u(x)$

$$u\left(x\right)=\sum_{i=1}^{n}C_i\varphi_i+W\left(l_S\left(u\right)\right). \tag{2.6}$$

The Equation (2.6) is equivalent to the Equation (2.1) on construction. Backwards, if to apply the operator l_B to the Equation (2.6) we shall receive the Equation (2.1). Thus, the Equations (2.1) and (2.6) are equivalent.

Integration by parts the integro-differential operator $W\left(l_S\left(u\right)\right)$ is led to an integral operator. It is convenient for numerical solutions.

The described algorithm is not absolutely new. There is the familiar example. The differential equation $y'-f\left(x,y\left(x\right)\right)=0$ is equivalent to an integral equation

$$y\left(x\right)=C+\int_a^x f\left(s,y\left(s\right)\right)\mathrm{d}s. \tag{2.7}$$

Hence, the described algorithm can be applied to the nonlinear equations.

The offered method enables constructions of system of fundamental solutions. If in the Equation (2.6) to put $C_k = 1$ and remaining $C_i = 0$ we receive the equation for definition of k-th fundamental solution of the equation (2.1)

$$u_k=\varphi_k+W\left(l_S\left(u_k\right)\right) \tag{2.8}$$

In an integral equation requirements to smoothness of functions under an integral are weakened and its solutions can be considered as the generalized solutions of the initial differential equation.

The supplementing operator it is possible to name perturbation operator in relation to a basic operator. Then the algorithm is admissible to consider as a variant of a method of perturbations.

Sometimes separation on operators under the formula (2.2) is defined either the mathematical form of the equation or physical sense of the problem.

3. The Example. Mathieu's Equation.

Let's write down the equation of Mathieu in a kind

$$y''+a=q\cos 2x\,y\left(x\right). \tag{3.1}$$

For it at $a=+b^2,-b^2,0$ are received three integral equations

$$
\begin{aligned}
& y(x) = C_1 \sin bx + C_2 \cos bx + \int_0^x q \sin(x-z)\cos 2z\, y(z)\,\mathrm{d}z \\
& y(x) = C_1 \operatorname{sh} bx + C_2 \operatorname{ch} bx + \int_0^x q \operatorname{sh}(x-z)\cos 2z\, y(z)\,\mathrm{d}z \quad . \\
& y(x) = C_1 + C_2 x + \int_0^x q(x-z)\cos 2z\, y(z)\,\mathrm{d}z
\end{aligned}
\tag{3.2}
$$

Let's consider a solution of the concrete equation

$$y'' + 1 = 2\cos 2x\, y(x). \tag{3.3}$$

Let's search for the first fundamental solution of the Equation (3.3) in the field of $x \geq 0$ for which according to (3.2) we have an integral equation

$$y(x) = \sin x + \int_0^x 2\sin(x-z)\cos 2z\, y(z)\,\mathrm{d}z\,. \tag{3.4}$$

The solution of the Equation (3.4) can be found the method of resolvent. Let's remind its essence. Integral equation

$$y(x) = f(x) + \int_a^x K(x,s)\, y(s)\,\mathrm{d}s \tag{3.5}$$

has the approximate solution

$$y_m(x) = f(x) + \int_a^x R_m(x,s)\, f(s)\,\mathrm{d}s \tag{3.6}$$

where $R_m(x,s) = \sum_{n=1}^{m} K_n(x,s)$ is the approximate resolvent.

K_n are iterated kernels defined by formulas

$$K_1(x,s) = K(x,s), \cdots, K_n(x,s) = \int_s^x K(x,t)\, K_{n-1}(t,s)\,\mathrm{d}t\,. \tag{3.7}$$

For the Equation (3.4) approximate resolvent kernel and solutions are expressed in elementary functions. The interval on which the approximate solution practically coincides with an exact solution, increases with magnification of the order of a resolvent kernel. We will cite the solution y_3.

$$
\begin{aligned}
y_3(x) = & -\frac{\sin 7x}{9216} + \frac{7\sin 5x}{4608} - \frac{5\sin 3x}{1536} + \frac{2033\sin x}{3072} - \frac{x^2 \sin 3x}{64} + \frac{7x^2 \sin x}{64} \\
& + \frac{169x\cos x}{384} + \frac{x^3\cos x}{48} + \frac{x\cos 5x}{384} - \frac{13x\cos 3x}{128}
\end{aligned}
\;. \tag{3.8}
$$

In **Figure 1** an exact solution (Mathieu's function) and the approximate solution y_3 are shown at $x \geq 3$. At $x < 3$ these solutions coincide. We will receive the same result, if we will compare y_3 and y_4.

It is simple to receive the approximate solutions of higher order. Even on a personal computer for small time (about 5 minutes) it manages to receive y_{15}, which practically coincides with the exact solution up to $x = 30$. (The author used the program of series *Maple*.) The program produces also an exact solution of the Equation (3.3) by the Mathieu's function. For this purpose the Cauchy problem with initial conditions $y(0) = 0,\ y'(0) = 1$ is solved.

4. The Example. The Equivalent Equations

The equation of undamped oscillations of a simple pendulum has form

$$y'' + \sin y(x) = 0\,. \tag{4.1}$$

This equation brings to integral equation

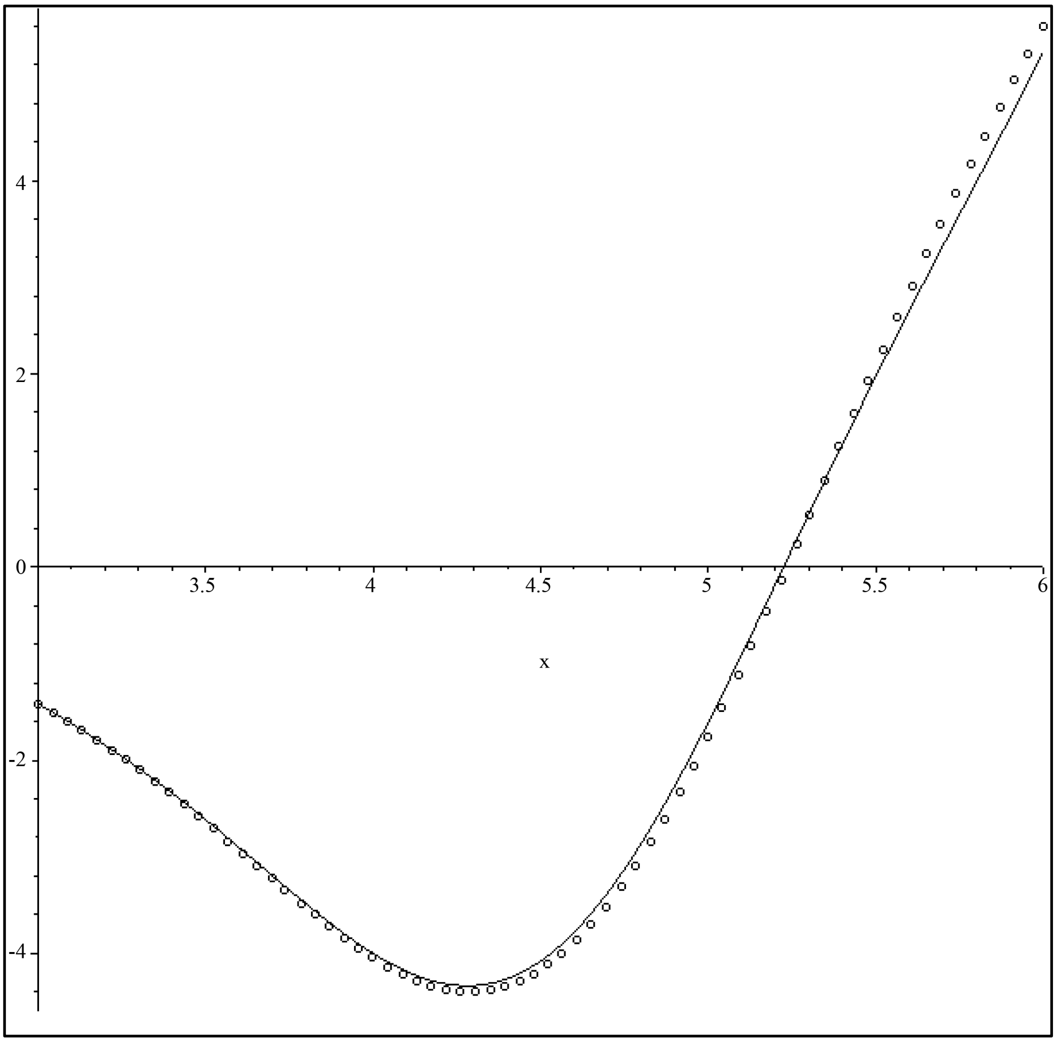

Figure 1. Comparison of the exact and approximate solutions of the Equation (3.3). Line is the exact solution, circles are approximate solution.

$$y(x) = y(0) + y'(0)x - \int_0^x (x-s)\sin(y(s))\mathrm{d}s . \tag{4.2}$$

It is possible to write the Equation (4.1) in the equivalent form

$$y'' + y(x) = y(x) - \sin y(x) . \tag{4.3}$$

The Equation (4.3) has physical sense: the left part is operator of small oscillations, right part is deviation from small oscillations. (4.3) brings to integral equation

$$y(x) = y(0)\cos x + y'(0)\sin x + \int_0^x \sin(x-z)(y(z) - \sin y(z))\mathrm{d}z . \tag{4.4}$$

In spite of the fact that the Equations (4.2) and (4.4) are various, their solutions under identical initial conditions coincide. The reader can establish it if he will solve these equations by means of the quadrature trapezoid rule. We see that variants are possible at decomposition of operator L by the formula (2.2).

5. Conclusions

The approximate methods cannot replace the general analysis of the equation. However, when existence of a solution of a problem is already proved, the approximate method can appear useful even at presence exact, but a complicated solution.

The offered algorithm is simple and effective and formally it can be applied to any linear ODE. However, if factors of ODE have singularities, it demands special research.

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