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A New Algorithm for the Determinant and the Inverse of Banded Matrices

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Abstract

In the current article, the authors present a new recurrence formula for the determinant of a banded matrix. An algorithm for inverting general banded matrices is derived.

Keywords

Banded Matrices, Determinants

Subject Areas: **Mathematical Analysis, Numerical Mathematics**

1. Introduction

The aim of this work is to extend the algorithm presented in [1] to obtain the inverse of the banded matrix

$$\boldsymbol{B} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,p+1} & 0 & \cdots & 0 \\ \vdots & \ddots & & a_{2,p+2} & & \vdots \\ \vdots & & \ddots & & & \\ a_{q+1,1} & \ddots & & & \ddots & \\ 0 & a_{q+2,2} & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & & & \cdots & \cdots \end{pmatrix},$$

Such matrices arise frequently in many problems such as computing the condition number or the discretization of partial differential equation in $2D$ or $3D$ by finite difference [2]. In practice, the bandwidth $p+q$

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is much less than n.

Standard method to compute the inverse of the matrix $\boldsymbol{B}$ is to use the LU based methods [3] and there are special algorithms taking into account the special form of the matrix [4]. Other methods were proposed more recently [5].

For the $n \times n$ banded matrix $\boldsymbol{B}$, where we assume that $a_{i,p+i} \neq 0$ for $i = 1, \cdots, n-p$, we associate the sequence $\left(A_{i,k}\right)_{1 \le i \le n+p, 1 \le k \le p}$ defined by the following relations for $k = 1, \cdots, p$:

$$A_{i,k} = \delta_{i,k} \text{ for } i = 1, \cdots, p, \tag{1}$$

and for $i = 1, \cdots, n-p$:

$$-a_{i,p+i} A_{p+i,k} = \sum_{s=1}^{p+i-1} a_{i,s} A_{s,k}, \tag{2}$$

and for $i = n-p+1, \cdots, n$:

$$A_{p+i,k} = \sum_{s=1}^{n} a_{i,s} A_{s,k}. \tag{3}$$

Here $\delta_{i,k}$ is the *Kronecker symbol* and we put $a_{i,j} = 0$ if $i - j > q$.

The relations above can be written in the matrix form

$$\boldsymbol{B} A_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{n+1,k} \\ A_{n+2,k} \\ \vdots \\ A_{n+p,k} \end{pmatrix},$$

where

$$A_k = \left[A_{1,k}, A_{2,k}, \cdots, A_{n,k}\right]^{\mathrm{T}}.$$

We shall note $\boldsymbol{Q}$ the $p \times p$ matrix $\left(A_{n+i,j}\right)_{1 \le i,j \le p}$.

The determinant of the matrix $\boldsymbol{Q}$ is related to the determinant of the matrix $\boldsymbol{B}$. This is the purpose of the next section.

2. Remark

Suppose that we have $a_{i,p+i} = 0$ for some $i = 1, \cdots, n-p,$. Let $\boldsymbol{B}(\varepsilon)$ be the matrix obtained from $\boldsymbol{B}$ by replacing $a_{i,p+i}$ such that $a_{i,p+i} = \varepsilon$ with $\varepsilon \succ 0$. We have $\lim_{\varepsilon \to 0} \boldsymbol{B}(\varepsilon) = \boldsymbol{B}$.

Then, we can compute $\boldsymbol{B}^{-1}$ tending ε e to 0 in $\boldsymbol{B}(\varepsilon)$.

3. Determinant of a Banded Matrix

Theorem. *Let the banded matrix* $\boldsymbol{B}$ *and the associated sequence* $\left(A_{i,k}\right)_{1 \le i \le n+p, 1 \le k \le p}$. *Then*

$$\det(\boldsymbol{B}) = (-1)^{p(n-p)} \left(\prod_{i=1}^{n-p} a_{i,i+p}\right) \det(\boldsymbol{Q}).$$

Proof. Let $\boldsymbol{T}$ the $n \times n$ triangular matrix

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{I}_p & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{I}_{n-p} \end{pmatrix}$$

where $\boldsymbol{I}_r$ denotes the $r \times r$ identity matrix and $\boldsymbol{D}$ is the $(n-p) \times p$ matrix

$$\begin{pmatrix} A_{p+1,1} & A_{p+1,2} & \cdots & A_{p+1,p} \\ A_{p+2,1} & A_{p+2,2} & \cdots & A_{p+2,p} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,p} \end{pmatrix}.$$

Report that

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,p} \end{pmatrix} = \boldsymbol{I}_p,$$

and thus using the relation Equation 1 we get

$$\boldsymbol{BT} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{C} \\ \boldsymbol{Q} & \boldsymbol{R} \end{pmatrix},$$

where $\boldsymbol{C}$ is the $(n-p) \times (n-p)$ triangular submatrix

$$\begin{pmatrix} a_{1,p+1} & 0 & \cdots & 0 \\ a_{2,p+1} & a_{2,p+2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n-p,p+1} & a_{n-p,p+2} & \cdots & a_{n-p,n} \end{pmatrix}$$

and $\boldsymbol{R}$ is a $p \times (n-p)$ matrix.

Obviously, $\det(\boldsymbol{T}) = 1$ and $\det(\boldsymbol{BT}) = (-1)^{p(n-p)} \det(\boldsymbol{C}) \det(\boldsymbol{Q})$. The result follows.

Example. *For* $p = q = 2$, *formula of the determinant of the pentadiagonal matrix is presented in* [6].

4. Inverse of a Banded Matrix

Assume that the matrix $\boldsymbol{B}$ is invertible and let us denote by C_j the j-th column vector of the inverse matrix $\boldsymbol{B}^{-1}$.

From the relation $\boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}_n$, where $\boldsymbol{I}_n$ denotes the identity matrix of order n, we get the relations:

$$C_{n-j-p} = \frac{1}{a_{n-j-p,n-j}} \left(E_{n-j} - \sum_{i=n-j-p+1}^{n} a_{i,n-j} C_i \right),$$

for $j = 0, \cdots, n-p-1$, where $E_j = \left[\left(\delta_{i,j} \right)_{1 \le i \le n} \right]^{\mathrm{T}} \in \mathbb{C}^n$ is the vector of order j of the canonical basis of $\mathbb{C}^n$.

It follow from r1 that knowing the p last columns $C_n, C_{n-1}, \cdots, C_{n-p+1}$ determine recursively all other columns $C_{n-p}, C_{n-p-1}, \cdots, C_1$. We give a straightforward recurrence formulae for computing $C_n, C_{n-1}, \cdots, C_{n-p+1}$.

Since the matrix $\boldsymbol{B}$ is invertible, we obtain from the previous section that the matrix $\boldsymbol{Q}$ is invertible too. We shall denotes $X_j = \left[x_{1,j}, \cdots, x_{p,j} \right]^{\mathrm{T}}$ the j-th column vector of the matrix $\boldsymbol{Q}^{-1}$.

Theorem. *The* j-th *vector column of the inverse matrix* $\boldsymbol{B}^{-1}$, $1 \le j \le p$, *is given by*

$$C_{n-p+j} = \sum_{k=1}^{p} x_{k,j} \boldsymbol{A}_k$$

Proof. We get from the relation Equation (1):

$$\boldsymbol{B}\cdot\sum_{k=1}^{p}x_{k,j}\boldsymbol{A}_k=\begin{pmatrix}0\\ \vdots\\ 0\\ \sum_{k=1}^{p}x_{k,j}A_{n+1,k}\\ \sum_{k=1}^{p}x_{k,j}A_{n+2,k}\\ \vdots\\ \sum_{k=1}^{p}x_{k,j}A_{n+p,k}\end{pmatrix}.$$

The result follows from the fact that

$$\begin{pmatrix}\sum_{k=1}^{p}x_{k,j}A_{n+1,k}\\ \sum_{k=1}^{p}x_{k,j}A_{n+2,k}\\ \vdots\\ \sum_{k=1}^{p}x_{k,j}A_{n+p,k}\end{pmatrix}=\boldsymbol{Q}\begin{pmatrix}x_{1,j}\\ x_{2,j}\\ \vdots\\ x_{p,j}\end{pmatrix}=\left[\left(\delta_{i,j}\right)_{1\le i\le p}\right]^{\mathrm{T}}.$$

We will use the relations Equation (2) and r1 to explicit the coefficients $c_{s,j}$ for the inverse $\boldsymbol{B}^{-1}$. we have for $1\le j\le p$ and $1\le s\le n$:

$$c_{s,n-p+j}=\sum_{k=1}^{p}x_{k,j}A_{s,k} \tag{4}$$

and for $j=0,\cdots,n-p-1$:

$$c_{s,n-j-p}=\frac{1}{a_{n-j-p,n-j}}\left(\delta_{s,n-j}-\sum_{i=n-j-p+1}^{n}a_{i,n-j}c_{s,i}\right). \tag{5}$$

Here are the different steps of our algorithm. The implementation is left to the readers' choice.

- Compute $A_{i,k}$ by the relations (1), (2) and (3). If $\det\left(\left(A_{n+i,j}\right)_{1\le i,j\le p}\right)=0$ then $\boldsymbol{B}$ is non invertible.
- Compute the inverse $\left(x_{i,j}\right)_{1\le i,j\le p}$ of the matrix $\left(A_{n+i,j}\right)_{1\le i,j\le p}$.
- Compute $c_{s,n+j-p}$ for $1\le j\le p$ and $1\le s\le n$ by the relation (4).
- Compute $c_{s,n-j-p}$ for $j=0,\cdots,n-p-1$ and $1\le s\le n$ by the relation (5).

5. Conclusion

If we fix the bandwidth $p+q$, one can show easily that the complexity of the algorithm is $O\left(n^2\right)$. Of course, other algorithms of similar complexity exist (methods based on LU decomposition for example). However, the new method provides more benefits to others: First, the recursive formulas are simple and can be implemented effectively in a parallel machine to reduce the cost. Also, we can solve these relations for some matrices to get the explicit determinant of those matrices.

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