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On consistency and ranking of alternatives in uncertain AHP*

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ABSTRACT

This paper introduces uncertainty theory to deal with non-deterministic factors in ranking alternatives. The uncertain variable method (UVM) and the definition of consistency for uncertainty comparison matrices are proposed. A simple yet pragmatic approach for testing whether or not an uncertainty comparison matrix is consistent is put forward. In cases where an uncertainty comparison matrix is inconsistent, an algorithm is used to generate consistent matrix. And then the consistent uncertainty comparison matrix can derive the uncertainty weights. The final ranking is given by uncertainty weighs if they are acceptable; otherwise we rely on the ranks of expected values of uncertainty weights instead. Three numerical examples including a hierarchical (AHP) decision problem are examined to illustrate the validity and practicality of the proposed methods.

Keywords: Uncertainty Theory; Uncertain Variable Method; Analytic Hierarchy Process; Consistency Test; Bounds Modification

1. INTRODUCTION

The Analysis Hierarchy Process (AHP) is a multicriteria decision making (MCDM) technique introduced by Saaty [1], that generates the relative weights of criteria. The advantages of this MCDM tool are its conceptual simplicity and its capability of handling subjective criteria and inconsistencies in the decision making process. As such the estimation of the relative weights of attributes plays a critical role in MCDM. Saaty [1] proposed to use single points as the elements of the comparison matrices. Each element reflects the degree of preference of one attributes over another and is taken from the $\{1/9,\cdots,1,\cdots,9\}$ ratio scale. However, due to the complexity and uncertainty involved in real-world decision problems and inherent subjective nature of human preference judgments, it is always unrealistic and infeasible to obtain exact judgments. Then the methods using fuzzy or interval judgments for parts or all of the judgments in a pairwise comparison matrix are proposed naturally. A number of techniques have been developed to use such a fuzzy or interval comparison matrix to derive weights. In the following section, we summarize the previous works and propose our new approach of uncertain variable method.

Van Laarhoven and Pedryce [2] considered treating entries in a comparison matrix as fuzzy numbers having triangular membership functions and proposed the logarithmic least squares method to generate fuzzy weights. Buckley [3] extended the method to trapezoidal membership functions. Modifying Van Laarhoven and Pedrycz's method (1983), Boender *et al*. [4] pointed out a fallacy in the normalization procedure for generating fuzzy weights. Kwiesielewicz [5] found the extension of Saaty's priority theory. Leung and Cao [6] proposed a fuzzy consistency definition with consideration of a tolerance deviation and determined fuzzy local and global weights using the extension principle. Xu and Da [7] introduced a least deviation method to obtain a priority vector of a fuzzy preference relation. However, most of the above fuzzy priority techniques take little account of inconsistent judgments.

Saaty and Vargas [8] employed interval judgments for the AHP method as a way to model subjective uncertainty and applied a Monte Carlo simulation approach to find out weight intervals from matrices of interval judgments. At the same time, they also pointed out difficulties in using this method. Arbel [9] proposed linear programming (LP) model as the prioritization process. Kress [10] pointed out that Arbel's method is inefficient for inconsistent interval comparison matrices because no feasible region exists in such situations. Islam [11] used lexicographic goal programming (LGP) to find out weights from pairwise inconsistent interval judgment matrices and an algorithm for identification and modification of inconsistent bounds was also provided. Linda M. Haines

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[12] employed a statistical approach to the AHP with interval judgments and constructed the distributions on feasible regions. Wang, Yang and Xu [13] proposed a two-stage logarithmic goal programming method (TLGP) for generating weights from interval comparison matrices and it was also applied to fuzzy comparison matrices when they were transformed into interval comparison matrices using α-level sets and the extension principle. Uncertainty in the preference judgments gave rise to uncertainty in the ranking of alternatives as well as difficulty in determining consistency of preferences.

As is well known, we must test the consistency of comparison matrices in the process of generation weights. Because of the complexity and uncertainty of real-world decision analysis problems and the subjectivity of human judgments, it is inevitable to generate inconsistent comparisons. Unreliable weights and ranking orders for alternatives may be caused by high inconsistency. Then it is necessary to check satisfactory consistency in order to ensure the rationality of decisions. Only comparison matrices passing the test of satisfactory consistency can be used to derive reliable weights. In this case, Saaty proposed the use of Consistency Ratio (CR). This consistency measure is obtained by taking the ratio between $\lambda_{\max}-n$ to its expected value over a large number of positive reciprocal matrices of order n, whose elements are randomly chosen in the $\{1/9,\cdots,1,\cdots,9\}$. Juan Aguarón and José María Moreno-Jiménez [14] employed Geometric Consistency Index (GCI) which can be seen as an average of the squared difference between the log of the errors and the log of unity. Leung and Cao [6] introduced the definition of fuzzy consistency and Wang *et al*. [15] proposed the consistency of interval comparison matrix both using the concept of feasible region S. Although we can test the consistency of matrices, how to modify the inconsistent matrices in order to derive the weights of attributes is still not solved. An algorithm provided by Islam *et al*. [11] for identification and modification of inconsistent bounds is used in our paper. With this algorithm the inconsistent matrices can be transformed into consistent ones without any more information. Due to the drawbacks of weights generation approaches mentioned above here we propose a new approach called uncertain variable method to derive weights from both consistent and inconsistent uncertainty comparison matrices.

Uncertainty theory, proposed by Liu [16] in 2007 and refined by Liu [17] in 2010, provides a new approach to deal with non-deterministic factors. Firstly, uncertain variable (Liu [16]) instead of precise ratio is used to represent human judgments. Then uncertain measure (Liu [16]) is used to indicate the belief degree of an uncertain event. Subsequently uncertainty distribution (Liu [16]) is used to describe uncertain variables in an incomplete but easy-to-use way. Based on uncertainty comparison matrix, priorities are derived by using inverse uncertainty distribution (Liu [18]) without any optimization model and ranking with a certain confidence level is obtained. Finally, consistency of matrices is tested and the inconsistent matrices are modified. At the end, the reliable weights can be obtained. The decision problem model is called uncertain AHP [19] when uncertain variable is brought in. This paper is focused only on the generating uncertainty weights, ranks of alternatives and modification of inconsistent matrices.

This paper is organized as follows. Section 2 proposes the uncertain variable method (UVM) to generate uncertainty weights from both consistent and inconsistent uncertainty comparison matrices. In Section 3, the definition of consistency for uncertainty comparison and the theorem for checking consistency are expressed, and then we develop a simple yet pragmatic approach that can be used to test whether an uncertainty comparison matrix is consistent or not without solving any mathematical program. Moreover an algorithm for identification and modification of inconsistent bounds is discussed. Section 4 provides three numerical examples including a hierarchical (AHP) decision problem to show the simplicity and practicality of the proposed methods. The paper is concluded in Section 5.

2. AN UNCERTAIN VARIABLE METHOD (UVM) FOR GENERATING UNCERTAINTY WEIGHTS

As we all know, fuzzy ratios and interval ratios were used to describe uncertainty of human judgments in previous works. Here in our paper, we introduce uncertain variable (Liu [16]) with uncertainty distribution to reflect the uncertain nature of expert judgments.

Let ξ_{ij} be an uncertain variable represents the ratio criterion i over criterion j. We assume that its uncertainty distribution (Liu [16]) is linear uncertainty distribution and can be obtained by Delphi method. The consultation process is as follows:

Q1: What do you think is the minimum ratio criterion i over criterion j?

A1: a. (an expert's experimental data $(a,0)$ is acquired)

Q2: What do you think is the maximum ratio criterion i over criterion j?

A2: b. (an expert's experimental data $(b,1)$ is acquired)

where $(a,0)$, $(b,1)$ represents $\Phi(a)=\mathcal{M}\{\xi_{ij}\le a\}=0$, $\Phi(b)=\mathcal{M}\{\xi_{ij}\le b\}=1$, respectively. So the uncertainty ratio ξ_{ij} has an uncertainty distribution $\mathcal{L}_{ij}(a,b)$. Repeating the steps, we can get a matrix given by

$$\mathcal{A} = \left(\xi_{ij}\right)_{n\times n} = \begin{pmatrix} 1 & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ & 1 & \cdots & \mathcal{L}_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \quad (1)$$

Since uncertainty distributions $\mathcal{L}_{ji}(a,b)$ $(i=1,2,\cdots,n-1; j=i+1,\cdots,n)$ of inverse uncertainty ratios $\xi_{ji}(i=1,2,\cdots,n-1; j=i+1,\cdots,n)$ cannot be easily determined in the sense of definition of uncertain measure (Liu [16]). So, in this paper, we just need to obtain the uncertainty distributions of upper triangular uncertainty ratios, the lower triangular uncertainty ratios can be obtained by inverse uncertainty distribution (Liu [16]) below

$$\mathcal{L}_{ji}^{-1}(\alpha) = \frac{1}{\mathcal{L}_{ij}^{-1}(\alpha)} \quad (2)$$
$$(i=1,2,\cdots,n-1; j=i+1,\cdots,n)$$

where α is the uncertain measure with $\alpha\in[0,1]$. $\mathcal{A}$ is called uncertainty comparison matrix.

Crawford and Williams [20] suggested for the Row Geometric Mean Method (RGMM), where the priorities (without the normalization factor) are given by

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n} \quad i=1,2,\cdots,n \quad (3)$$

where w_i is the weight of the ith attribute. Liu [16] proposed the concept of strictly increasing function of uncertain variables. A real-valued function $f(x_1,x_2,\cdots,x_n)$ is said to be strictly increasing if

$$f(x_1,x_2,\cdots,x_n) \le f(y_1,y_2,\cdots,y_n)$$

whenever $x_i \le y_i$ for $i=1,2,\cdots,n$, and

$$f(x_1,x_2,\cdots,x_n) < f(y_1,y_2,\cdots,y_n)$$

whenever $x_i < y_i$ for $i=1,2,\cdots,n.$

Eq.2 can be expressed as

$$f(x_1,x_2,\cdots,x_n) = \left(\prod_{j=1}^{n} x_j\right)^{1/n}, \ x_j > 0$$

then f is a strictly increasing function.

Theorem 2.1 (Liu [17]) Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If f is a strictly increasing function, then

$$\xi = f(\xi_1,\xi_2,\cdots,\xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha)\right).$$

By the Theorem 2.1, we know that $w_i = f(\xi_{i1},\xi_{i2},\cdots,\xi_{in})$ is an uncertain variable with inverse uncertainty distribution

$$\Phi_i^{-1}(\alpha) = \left(\prod_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)\right)^{1/n} = \left(\prod_{j=1}^{n}\xi_{ij}\right)^{1/n} = w_i \quad (4)$$

where $\alpha\in[0,1]$ is the uncertain measure. w_i is defined as uncertainty weight of the i th attribute. We consider the weights acceptable when the ranking order does not alter if $\alpha \ge 0.5$. Otherwise experts will be asked to provide new judgments. Without any modification or new information, we rely on the ranks of expected values of uncertainty weights instead.

Note that the weight w_i is an uncertain variable and can not be used to test consistency of matrices. So which group of weights can be used to check the consistency will be asked. Liu and Ha [21] give the expected value $E[\xi]$ of monotone function of uncertain variables

$$E[\xi] = \int_0^1 \left(\prod_{j=1}^{n}\Phi_{ij}^{-1}(\alpha)\right)^{1/n} \mathrm{d}\alpha \quad (5)$$

which is a precise number representing expected value of uncertain variable w_i (priority of ith attribute) in **Eq.4**. However the integral can not be found, only the numerical solution of the expected value could be calculated by computer.

3. IDENTIFICATION AND MODIFICATION OF INCONSISTENT BOUNDS

Suppose decision maker (DM) provides uncertainty judgments instead of precise judgments for a pairwise comparison. It can be denoted by **Eq.1**. About the above uncertainty comparison matrix, we give the following definition and theorem:

Definition 3.1 Let $\mathcal{A}$ is an uncertainty comparison matrix defined by (1) with $\mathcal{L}_{ij}^{-1}(0) \le \xi_{ij} \le \mathcal{L}_{ij}^{-1}(1)$ and $\xi_{ii} = \mathcal{L}_{ii}^{-1}(0) = \mathcal{L}_{ii}^{-1}(1)$ for $i,j=1,2,\cdots,n$. If the convex feasible region

$$S_w = \Big\{ w=(w_1,w_2,\cdots,w_n) \Big| \mathcal{L}_{ij}^{-1}(0) \le w_i/w_j \le \mathcal{L}_{ij}^{-1}(1)$$
$$\sum_{i=1}^{n} w_i = 1, w_j > 0, j=1,2,\cdots,n \Big\}$$

is nonempty, then $\mathcal{A}$ is considered to be a consistent uncertainty comparison matrix.

Theorem 3.1 $\mathcal{A}$ is a consistent uncertainty comparison matrix if and only if it satisfies the following inequality constraints:

$$\max\left(\mathcal{L}_{ik}^{-1}(0)\cdot\mathcal{L}_{kj}^{-1}(0)\right) \le \min\left(\mathcal{L}_{ik}^{-1}(1)\cdot\mathcal{L}_{kj}^{-1}(1)\right) \quad (6)$$
$$\text{for all } k=1,2,\cdots,n$$

Proof. If $\mathcal{A}$ is a consistent uncertainty comparison matrix, it means the convex feasible region S_w is nonempty and there is no contradiction among the following inequality constraints:

$$\mathcal{L}_{ik}^{-1}(0) \le w_i/w_k \le \mathcal{L}_{ik}^{-1}(1) \quad i,k = 1,2,\cdots,n \tag{7}$$

$$\mathcal{L}_{kj}^{-1}(0) \le w_k/w_j \le \mathcal{L}_{kj}^{-1}(1) \quad k,j = 1,2,\cdots,n \tag{8}$$

Multiplying (7) by (8) leads to the following inequalities

$$\begin{gathered}\mathcal{L}_{ik}^{-1}(0)\cdot\mathcal{L}_{kj}^{-1}(0) \le \mathcal{L}_{ik}^{-1}(1)\cdot\mathcal{L}_{kj}^{-1}(1)\\ i,j,k = 1,2,\cdots,n\end{gathered} \tag{9}$$

Since (9) holds for any $k = 1,2,\cdots,n$, it follows that $\max\left(\mathcal{L}_{ik}^{-1}(0)\cdot\mathcal{L}_{kj}^{-1}(0)\right) \le \min\left(\mathcal{L}_{ik}^{-1}(1)\cdot\mathcal{L}_{kj}^{-1}(1)\right)$ holds for all $i,j,k = 1,2,\cdots,n$.

Inversely, if (6) holds for all i,j,k, then $\mathcal{L}_{ij}^{-1}(0) \le w_i/w_j \le \mathcal{L}_{ij}^{-1}(1)$ holds for any $i,j = 1,2,\cdots,n$. So, S_w is nonempty and $\mathcal{A}$ is a consistent uncertainty comparison matrix in the sense of Definition 3.1.

We can check the matrices whether they are consistent or not by the Definition 3.1, however it computes inefficiently when the order of matrix is large. So an algorithm proposed by Islam *et al*. [11],

$$a = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(p_{ij} + p'_{ij}\right) \tag{10}$$

subject to

$$\begin{gathered}-w_i + \mathcal{L}_{ij}^{-1}(0)w_j + n_{ij} - p_{ij} = 0,\\ w_i - \mathcal{L}_{ij}^{-1}(1)w_j + n'_{ij} - p'_{ij} = 0,\\ i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n,\end{gathered}$$

where $n_{ij}p_{ij} = 0, n'_{ij}p'_{ij} = 0, p_{ij}p'_{ij} = 0$.

After calculating the weights w_i by UVM from uncertainty comparison matrix, the ratios w_i/w_j for all i and j may or may not belong to the interval $\left[\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1)\right]$. The p_{ij} or p'_{ij} reflects the deviation w_i/w_j exceeding the set $\left[\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1)\right]$. It is clear that the matrix is consistent if and only if $a = 0$. With the bounds of the set being modified, the value of a can approximate 0. The algorithm proposed by Islam *et al*. [11] for modification of inconsistent bounds can find out the most inconsistent bound. Then modify the lower bound or upper bound or both lower and upper bounds and calculate the value of *a*, if $a > 0$, we repeat the steps below, otherwise we stop. Finally the consistent judgments are constructed. The steps are as follows:

Step 1. Calculate the expected weights of alternatives by (5). If $a > 0$, then go to Step 2, otherwise stop.

Step 2. Find out the matrices $L = \left(\mathcal{L}_{ij}^{-1}(0)\right)_{n\times n}$ and $U = \left(\mathcal{L}_{ij}^{-1}(1)\right)_{n\times n}$ taking the lower and upper bounds from the $\mathcal{A}$, respectively.

Step 3. Bring the matrices $C = \left(c_{ij}\right)_{n\times n}$ and $D = \left(d_{ij}\right)_{n\times n}$ where

$$\begin{gathered}C = \left(\prod_{k=1}^{n}\mathcal{L}_{ik}^{-1}(0)\cdot\mathcal{L}_{kj}^{-1}(0)\right)^{1/n}\\ i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n\\ c_{ji} = 1/c_{ij}\end{gathered}$$

and

$$\begin{gathered}D = \left(\prod_{k=1}^{n}\mathcal{L}_{ik}^{-1}(1)\cdot\mathcal{L}_{kj}^{-1}(1)\right)^{1/n}\\ i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n\\ d_{ji} = 1/d_{ij}\end{gathered}$$

Four cases may appear,

(i) $\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1) \ge 1 \quad c_{ij},d_{ij} \ge 1$

(ii) $\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1) \ge 1 \quad c_{ij},d_{ij} < 1$

(iii) $\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1) < 1 \quad c_{ij},d_{ij} \ge 1$

(iv) $\mathcal{L}_{ij}^{-1}(0),\mathcal{L}_{ij}^{-1}(1) < 1 \quad c_{ij},d_{ij} < 1$

for $i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$.

Step 4. Construct the matrices

$$\mathcal{L}_{ij}^{-1}(0)' = \begin{cases}\mathcal{L}_{ij}^{-1}(0) & \text{if } \mathcal{L}_{ij}^{-1}(0) \ge 1\\ 1/\mathcal{L}_{ij}^{-1}(0) & \text{if } \mathcal{L}_{ij}^{-1}(0) < 1\end{cases}$$

$$c'_{ij} = \begin{cases}c_{ij} & \text{if } c_{ij} \ge 1\\ 1/c_{ij} & \text{if } c_{ij} < 1\end{cases}$$

Same notations are used for the upper bound case.

Step 5. Calculate the absolute deviations $\rho_{ij} = \left|\mathcal{L}_{ij}^{-1}(0)' * c'_{ij} - \gamma_{ij}\right|$ and $\eta_{ij} = \left|\mathcal{L}_{ij}^{-1}(1)' * d'_{ij} - \delta_{ij}\right|$, for all $i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$, where "*" denotes the "–" in cases (i) and (iv) and the "+" in cases (ii) and (iii) in Step 3. γ_{ij} and δ_{ij} are 0 in cases (i) and (iv) and 2 in cases (ii) and (iii).

Step 6. Find out the maximum of all the deviations ρ_{ij} and η_{ij}, $i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$. If ρ_{pq} is maximum, then modify $\mathcal{L}_{pq}^{-1}(0)$, otherwise if η_{pq} gives the maximum value, vary $\mathcal{L}_{pq}^{-1}(1)$. If $\mathcal{L}_{pq}^{-1}(0) - c_{pq} < 0$, then $\mathcal{L}_{pq}^{-1}(0)$ should be increased, otherwise the modified value should be decreased. Similar treatment follows for the matrix of upper bounds.

Go to Step 1.

After introducing the UVM and algorithm for modifycation of inconsistent bounds, we show the whole process below. The whole process introduced for generating uncertainty weights and expected weights from uncertainty comparison matrices is summarized in **Figure 1**.

4. NUMERICAL EXAMPLES

In this section, we offer three numerical examples that

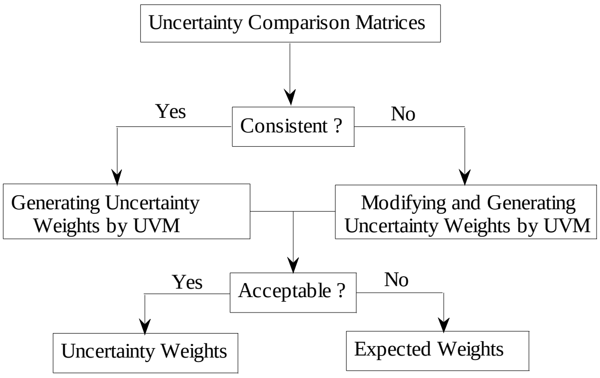

Figure 1. Process for generating priorities from uncertainty comparison matrices.

are examined using the proposed UVM and modification algorithm and show their applications. The uncertainty distributions are obtained by Delphi method. Some comparisons with other existing methods will be made whenever necessary.

Example 1 Consider the following interval comparison matrix that was examined by Arbel and Vargas [22] and Wang *et al*. [13].

$$A=\begin{pmatrix} 1 & [2,5] & [2,4] & [1,3] \\ [1/5,1/2] & 1 & [1,3] & [1,2] \\ [1/4,1/2] & [1/3,1] & 1 & [1/2,1] \\ [1/3,1] & [1/2,1] & [1,2] & 1 \end{pmatrix}$$

It has been known that A is a consistent interval judgments matrix, which was confirmed using the goal programming model given by Wang *et al*. [13]. Subsequently Wang *et al*. derived the interval weights and ranking by two-stage logarithmic goal programming method (TLGP), $w_1 \overset{100\%}{\succ} w_2 \overset{63.62\%}{\succ} w_4 \overset{76.42\%}{\succ} w_3$. The percentage is called degree of preference which represents criterion i is preferred over criterion j. We can find out that the interval comparison matrices can be transformed easily into uncertainty comparison matrices as given in **Eq.1**, so the corresponding uncertainty comparison matrix of A is

$$\mathcal{A}=\begin{pmatrix} 1 & \mathcal{L}_{12}(2,5) & \mathcal{L}_{13}(2,4) & \mathcal{L}_{14}(1,3) \\ & 1 & \mathcal{L}_{23}(1,3) & \mathcal{L}_{24}(1,2) \\ & & 1 & \mathcal{L}_{34}(1/2,1) \\ & & & 1 \end{pmatrix}$$

Before testing the consistency of $\mathcal{A}$, we should derive the expected values of uncertainty weights using **Eq.5**. So the normalized expected values of uncertainty weights are $0.4514, 0.2134, 0.1388, 0.1963$. Using the formula **Eq.10**, $a=0$ is obtained and $\mathcal{A}$ is consistent uncertainty comparison matrix. So we can directly use the UVM. By the **Eqs.2** and **4**, **Table 1** shows the

Table 1. Uncertainty weights in Example 1.

Criterion weights	α						
	0	0.1	0.2	…	0.8	0.9	1
Criterion 1	0.3047	0.3738	0.4032	…	0.5274	0.5422	0.5560
Criterion 2	0.2026	0.2073	0.2103	…	0.2114	0.2103	0.2091
Criterion 3	0.1703	0.1609	0.1525	…	0.1159	0.1115	0.1073
Criterion 4	0.2865	0.2580	0.2341	…	0.1453	0.1360	0.1276

weights with different uncertain measures, from which it is clear that criterion 1 is the most important because its weight is greater than weights of all the other criteria with $\alpha\in[0,1]$. To provide a complete ranking order for the four uncertainty weights, a directed diagram is depicted in **Figure 2**, from which it is quite clear that the ranking order is generated to be $w_1>w_2>w_4>w_3$ $(if\ \ \alpha>0.4)$, which is same as rankings given by Wang *et al*. [13] using TLGP, but our ranking order provides the information about uncertain measure of preference, which reflects uncertain nature of the ranking. We can say that the ranking is credible.

Example 2 Consider the following decision analysis given by interval comparison matrix, which was investigated by Kress [10], Islam *et al*. [11] and Wang *et al*. [9]

$$A=\begin{pmatrix} 1 & [1,2] & [1,2] & [2,3] \\ [1/2,1] & 1 & [3,5] & [4,5] \\ [1/2,1] & [1/5,1/3] & 1 & [6,8] \\ [1/3,1/2] & [1/5,1/4] & [1/8,1/6] & 1 \end{pmatrix}$$

Naturally, the uncertainty comparison matrix can be expressed as

$$\mathcal{A}=\begin{pmatrix} 1 & \mathcal{L}_{12}(1,2) & \mathcal{L}_{13}(1,2) & \mathcal{L}_{14}(2,3) \\ & 1 & \mathcal{L}_{23}(3,5) & \mathcal{L}_{24}(4,5) \\ & & 1 & \mathcal{L}_{34}(6,8) \\ & & & 1 \end{pmatrix}$$

Kress [10] and and Islam *et al*. [11] showed that this interval comparison matrix A is inconsistent, which can be further confirmed using the algorithm proposed above. The problem of weight determination from $\mathcal{A}$ is calculated expected weights are (0.3150, 0.3922, 0.2226, 0.0701), respectively, with $a=0.6973$. Using the UVM proposed in Section 2, the preference relations without modification can be obtained and plotted in **Figure 3**.

Since the value of a is positive, we apply the algorithm for identification of the inconsistent bounds. The maximum values among all ρ_{ij} and η_{ij} are $\rho_{34}=3.6834$ and $\eta_{34}=3.7705$, respectively. Perhaps the experts gave the judgment $A(3,4)$ carelessly. Changing this judgment $\mathcal{L}_{34}(6,8)$ to $\mathcal{L}_{34}(2,3)$ the modified set of weights as (0.3246, 0.4042, 0.1773, 0.0939) with $a=0.2608$ can be got. Since a is still positive, we compute the

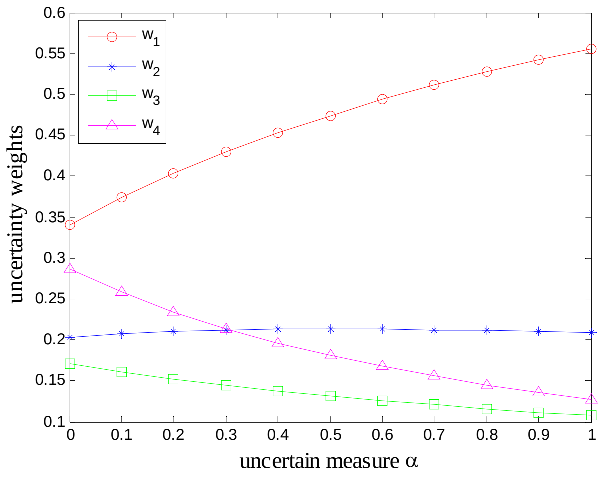

Figure 2. Preference relations in Example 1.

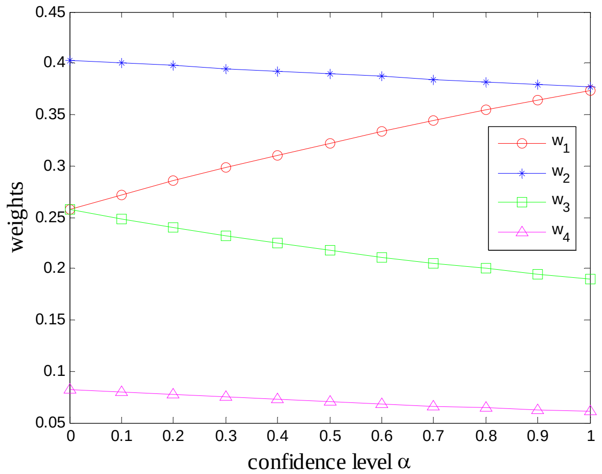

Figure 3. Preference relations without modification in Example 2.

deviationsρ_{ij} and η_{ij} and it is found that $\rho_{23} = 1.4349$ and $\eta_{14} = 1.8206$ are the maximum among the deviations. We take the new bounds $\mathcal{L}_{23}^{-1}(0) = 2$ and $\mathcal{L}_{14}^{-1}(1) = 4$. Repeating the steps above, the final modified set of expected weights are obtained as (0.3469, 0.3452, 0.2243, 0.0836) with $a = 0$. The modified uncertainty comparison is

$$\mathcal{A}' = \begin{pmatrix} 1 & \mathcal{L}_{12}(1,2) & \mathcal{L}_{13}(1,2) & \mathcal{L}_{14}(2,7) \\ & 1 & \mathcal{L}_{23}(1,5) & \mathcal{L}_{24}(2,7) \\ & & 1 & \mathcal{L}_{34}(2,6) \\ & & & 1 \end{pmatrix}$$

Until now we can apply the UVM to calculate the uncertainty weights with different uncertain measures and the results are recorded in **Table 2**. A directed diagram is depicted in **Figure 4**, from which it is quite clear that the ranking order is given by $w_1 > w_2 > w_3 > w_4$, $(if\ \alpha \geq 0.4)$ which is different from the preference relations without

Table 2. Modified uncertainty weights in Example 2.

Criterion weights	α					
	0	0.1	0.2	…	0.8	0.9	1
Criterion 1	0.2857	0.3081	0.2857	…	0.4026	0.4120	0.4207
Criterion 2	0.2857	0.3121	0.2857	…	0.3709	0.3728	0.3741
Criterion 3	0.2857	0.2611	0.2857	…	0.1746	0.1675	0.1610
Criterion 4	0.1429	0.1187	0.1429	…	0.0518	0.0477	0.0442

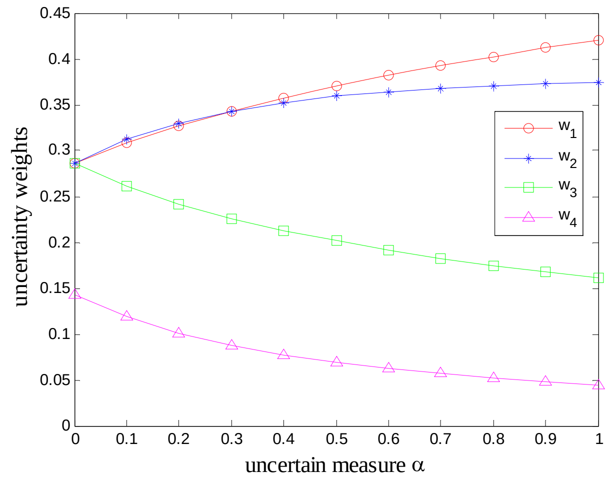

Figure 4. Modified preference relations in Example 2.

modification. Islam *et al*. [11] gave the ranking $w_1 = w_2 > w_3 > w_4$. In Islam's lexicographic goal programming (LGP) method, the upper triangular judgments of A are used for priorities. However the lower triangular judgments of A are applied, which provide completely the same information, then a different point estimate would be obtained. It is likely to be flawed. Wang *et al*. [13] showed the final ranking $w_2 \overset{74.75\%}{\succ} w_1 \overset{76.07\%}{\succ} w_3 \overset{100\%}{\succ} w_4$. In that TLGP method, the ranks of alternatives is based on inconsistent interval comparison matrices and minimizing the deviation variables but no optimal minimum is given. We fail to rank the two alternatives when the degrees of preference are equal to 50%. The results given by TLGP are doubtful.

Example 3 The problem is about a government agency's goal (G) to rank chemicals A_1, A_2, A_3 in terms of their level of harm to the environment. The goal is affected by three criteria with criterion C_1: Air, C_2: Water, and C_3: Soil, see **Figure 5**.

The uncertainty distribution matrices for the three criteria as well and for the three alternatives are summarized in **Table 3**. Before checking the consistency of the matrices, we can not derive the priorities using any methods. However the expected weights can be obtained by **Eq.5** and shown in **Table 4**, from which four uncertainty comparison matrices all turn out to be inconsistent.

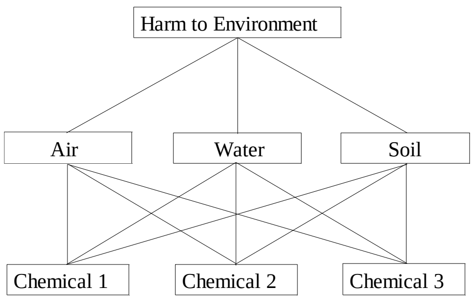

Figure 5. Hierarchy structure in Example 3.

Table 3. Uncertainty comparison matrices in Example 3.

$$G:\begin{pmatrix} 1 & \mathcal{L}_{12}(1,2) & \mathcal{L}_{13}(2,3) \\ & 1 & \mathcal{L}_{23}(1/2,2) \\ & & 1 \end{pmatrix}$$

$$C_1:\begin{pmatrix} 1 & \mathcal{L}_{12}(1/4,1/2) & \mathcal{L}_{13}(1/5,1/3) \\ & 1 & \mathcal{L}_{23}(1/3,1/2) \\ & & 1 \end{pmatrix}$$

$$C_2:\begin{pmatrix} 1 & \mathcal{L}_{12}(2,4) & \mathcal{L}_{13}(3,5) \\ & 1 & \mathcal{L}_{23}(1/2,2) \\ & & 1 \end{pmatrix} \quad C_3:\begin{pmatrix} 1 & \mathcal{L}_{12}(1,2) & \mathcal{L}_{13}(3,5) \\ & 1 & \mathcal{L}_{23}(2,4) \\ & & 1 \end{pmatrix}$$

Table 4. Expected weights and deviation α in Example 3.

Chemicals	C1	C2	C3	a
	0.4334	0.3040	0.2626	0.0917
A_1	0.1653	0.5416	0.4671	
A_2	0.3110	0.2389	0.3651	
A_3	0.5237	0.2195	0.1678	
a	0.0590	0.1169	0.0364	

The proposed algorithm is used to modify the bounds of matrices and new uncertainty comparison matrices are recorded in **Table 5**. **Table 6** gives both local and global expected weights. It is can be seen that the expected weights indicate the ranking $A_1 \succ A_3 \succ A_2$. Based on the consistent matrices, the UVM are used to generate the local and global priorities. In this paper, $\Phi_{ij}^{-1}(\alpha)$ can be treated as precise number with $\alpha = \alpha_0$. So the syn- thesis of weights is similar with that in conven- tional- AHP. Comparisons with original matrices without modi- fication are depicted in **Figures 6** and **7**.

Note that, in **Figure 7**, the ranking alters with $\alpha > 0.7$, which is unacceptable. It is clear that A_1 is preferred over A_2 and A_3, but whether A_3 is preferred over A_2 is ambiguous. The reason may be that the consistent matrices include contradictory information. Kwiesielewicz and Van Uden [23] considered that the consistency test is performed to ensure that judgments are neither random nor illogical. They pointed out that even if a matrix will pass a consistency check successfully, it can be

Table 5. Modified uncertainty comparison matrices in Example 3.

$$G:\begin{pmatrix} 1 & \mathcal{L}_{12}(1,3) & \mathcal{L}_{13}(1,3) \\ & 1 & \mathcal{L}_{23}(1/2,2) \\ & & 1 \end{pmatrix}$$

$$C_1:\begin{pmatrix} 1 & \mathcal{L}_{12}(1/4,1/2) & \mathcal{L}_{13}(1/6,1/3) \\ & 1 & \mathcal{L}_{23}(1/3,1) \\ & & 1 \end{pmatrix}$$

$$C_2:\begin{pmatrix} 1 & \mathcal{L}_{12}(2,5) & \mathcal{L}_{13}(2,5) \\ & 1 & \mathcal{L}_{23}(1/2,2) \\ & & 1 \end{pmatrix} \quad C_3:\begin{pmatrix} 1 & \mathcal{L}_{12}(1,2) & \mathcal{L}_{13}(2,6) \\ & 1 & \mathcal{L}_{23}(2,4) \\ & & 1 \end{pmatrix}$$

Table 6. Local expected weights and global expected weights in Example 3.

Chemicals	C1	C2	C3	Global weights
	0.4260	0.2870	0.2870	
A_1	0.1597	0.5361	0.4619	0.3545
A_2	0.3421	0.2320	0.3627	0.3164
A_3	0.4982	0.2320	0.1725	0.3283

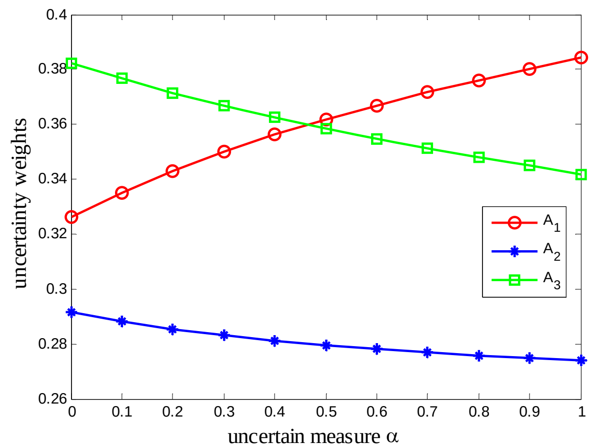

Figure 6. Preference relations without modification in Example 3.

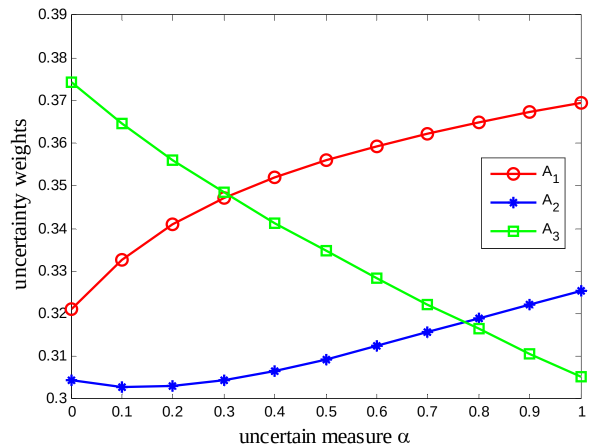

Figure 7. Modified preference relations in Example 3.

contradictory. Perhaps the judgments for A_3 have been given carelessly. Without any new information, we take the ranking $A_1 \succ A_3 \succ A_2$ given by expected weights as the final ranks of alternatives.

5. SUMMARY AND CONCLUDING REMARKS

In multiple criteria decision analysis problem, human judgments are required in order to generate relative weights of criteria. Due to complexity of real world decision problems and subjective nature of human judgments, uncertainty comparison matrices can provide a more realistic framework to explain such uncertainty. However how to check contradictions and avoid modified bounds exceeding the bounded $(1/9-9)$ ratio scale are still subject to further investigation.

In this paper, uncertain variable method (UVM), a new approach to consistency and inconsistency of uncertainty comparison matrices was proposed to generate uncertainty weights. The definition of consistency for uncertainty comparison matrices was provided and a simple yet pragmatic approach for testing whether or not an uncertainty comparison matrix is consistent was put forward without having to solve any optimization model. In case of consistent uncertainty comparison matrices, the UVM was recommended to generate consistent uncertainty weights; otherwise the uncertainty comparison matrices would be modified via the algorithm proposed and the uncertainty weights were derived from modified matrices. We consider the uncertainty weights as final ranking if they were acceptable. Otherwise the ranks of expected weights would be adopted. Three numerical examples illustrated the simplicity and wide applicability of the proposed methods. Since interval comparison matrices may be transformed into uncertainty comparison matrices, the proposed methods are applicable to interval comparison matrices. Therefore they can be widely used to deal with decision analysis problems.

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