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Generalized Discrete Entropic Uncertainty Relations on Linear Canonical Transform

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ABSTRACT

Uncertainty principle plays an important role in physics, mathematics, signal processing and *et al*. In this paper, based on the definition and properties of discrete linear canonical transform (DLCT), we introduced the discrete Hausdorff-Young inequality. Furthermore, the generalized discrete Shannon entropic uncertainty relation and discrete Rényi entropic uncertainty relation were explored. In addition, the condition of equality via Lagrange optimization was developed, which shows that if the two conjugate variables have constant amplitudes that are the inverse of the square root of numbers of non-zero elements, then the uncertainty relations touch their lowest bounds. On one hand, these new uncertainty relations enrich the ensemble of uncertainty principles, and on the other hand, these derived bounds yield new understanding of discrete signals in new transform domain.

Keywords: Discrete Linear Canonical Transform (DLCT); Uncertainty Principle; Rényi Entropy; Shannon Entropy

1. Introduction

Uncertainty principle [1-20] plays an important role in physics, mathematics, signal processing and *et al*. Uncertainty principle not only holds in continuous signals, but also in discrete signals [1,2]. Recently, with the development of fractional Fourier transform (FRFT), continuous generalized uncertainty relations associated with FRFT have been carefully explored in some papers such as [3,4,16], which effectively enrich the ensemble of FRFT. However, up till now there has been no reported article covering the discrete generalized uncertainty relations associated with discrete linear canonical transform (DLCT) that is the generalization of FRFT. From the viewpoint of engineering application, discrete data are widely used. Hence, there is great need to explore discrete generalized uncertainty relations. DLCT is the discrete version of LCT [5,6], which is applied in practical engineering fields. In this article we will discuss the entropic uncertainty relations [7,8] on LCT.

In this paper, we made some contributions such as follows. The first contribution is that we extend the traditional Hausdorff-Young inequality to the DLCT domain with finite supports. It is shown that these bounds are connected with lengths of the supports and LCT parameters. The second contribution is that we derived the Shannon entropic uncertainty principle in LCT domain for discrete data, based on which we also derived the conditions when these uncertainty relations have the equalities via Lagrange optimization. The third contribution is that we derived the Renyi entropic uncertainty principle in DLCT domain. As far as we know, there have been no reported papers covering these generalized discrete entropic uncertainty relations on LCT.

2. Preliminaries

2.1. LCT and DLCT

Before discussing the uncertainty principle, we introduce some relevant preliminaries. Here we first briefly review the definition of LCT. For given analog signal $f(t) \in L^1(R) \cap L^2(R)$ and $\left\|f(t)\right\|_2 = 1$ (where $\left\|f(t)\right\|_2$ denotes the l_2 norm of function $f(t)$), its LCT [5,6] is defined as

$$F_A(u) = F_A(f(t)) = \int_{-\infty}^{\infty} f(t) K_A(u,t)\,\mathrm{d}t$$

$$= \begin{cases} \sqrt{1/i2b\pi} \cdot \mathrm{e}^{\frac{idu^2}{2b}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-iut}{b}} \mathrm{e}^{\frac{iat^2}{2b}} f(t)\,\mathrm{d}t & b \neq 0, ad + bc = 1 \\ \sqrt{d}\,\mathrm{e}^{icdu^2/2} f(du) & b = 0 \end{cases} \quad (1)$$

where $K_A(u,t)=\sqrt{1/i2b\pi}\cdot e^{\frac{idu^2}{2b}}e^{\frac{-iut}{b}}e^{\frac{iat^2}{2b}}$, $n\in Z$ and i is the complex unit, $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is the transform parameter defined as that in [5,6]. In addition,

$$F_A F_B\left(f(t)\right)=f(t).$$

If $A=B^{-1}$, $f(t)=\int_{-\infty}^{\infty}F_A(u)K_{A^{-1}}(u,t)\mathrm{d}u$, *i.e.*, the inverse LCT reads: $f(t)=\int_{-\infty}^{\infty}F_A(u)K_{A^{-1}}(u,t)\mathrm{d}u$.

Let

$$X=\{x_1,x_2,x_3,\cdots,x_N\}=\{x(1),x(2),x(3),\cdots,x(N)\}\in C^N$$

be a discrete time series with length N and $\|X\|_2=1$. Assume its DLCT (discrete FLCT) $\hat{X}_\alpha=\{\hat{x}_1,\hat{x}_2,\hat{x}_3,\cdots,\hat{x}_N\}\in C^N$ under the transform parameter α.

Then the DLCT [5] can be written as

$$\begin{aligned}\hat{x}(k)&=\sum_{l=1}^{N}\sqrt{1/ibN}\cdot e^{\frac{idk^2}{2b}}e^{\frac{-ikn}{Nb}}e^{\frac{ian^2}{2bN^2}}x(n)\\&=\sum_{l=1}^{N}u_A(k,n)\cdot x(n),\quad 1\le n,k\le N.\end{aligned}\tag{2}$$

Also, we can rewrite the definition (2) as

$$\hat{X}_A=U_A X,$$

where $U_A=\left[u_A(k,n)\right]_{N\times N}$.

Clearly, for DLCT we have the following property [5]: $\|\hat{X}_A\|_2=\|U_A X\|_2=1$.

In the following, we will assume that the transform parameter $b\ne 0$. Note the main difference between the discrete and analog definitions is the length: one is finite and discrete and the other one is infinite and continuous.

2.2. Shannon Entropy and Rényi Entropy

For any discrete random variable $x_n\ (n=1,\cdots,N)$ and its probability density function $p(x_n)$, the Shannon entropy [9] and the Rényi Entropy [10] are defined as, respectively

$$H(x_n)=\sum_{n=1}^{N}\left|p(x_n)\right|\ln\left|p(x_n)\right|,$$

$$H_\vartheta(x_n)=\frac{1}{1-\vartheta}\cdot\ln\left(\sum_{n=1}^{N}\left|p(x_n)\right|^{\vartheta}\right).$$

Hence, in this paper, we know that for any DLCT $\hat{X}_A=\{\hat{x}_1,\hat{x}_2,\hat{x}_3,\cdots,\hat{x}_N\}\in C^N$ (with $\|X\|_2=1$ and $\|\hat{X}_A\|_2=\|U_A X\|_2=1$), the Shannon entropy and the Rényi Entropy [13] associated with DLCT are defined as, respectively

$$H(\hat{x}_A)=\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^2\ln\left|\hat{x}_A(n)\right|^2,$$

$$H_\vartheta(\hat{x}_A)=\frac{1}{1-\vartheta}\cdot\ln\left(\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^{2\vartheta}\right).$$

Clearly, if $\vartheta\to 1$ as shown in [13],

$$H_\vartheta(\hat{x}_A)\to H(\hat{x}_A).$$

2.3. Discrete Hausdorff-Young Inequality on DLCT

Lemma 1: For any given discrete time series

$$X=\{x_1,x_2,x_3,\cdots,x_N\}=\{x(1),x(2),x(3),\cdots,x(N)\}\in C^N$$

with length N and $\|X\|_2=1$, $U_A(U_B)$ is the DLCT transform matrix associated with the transform parameter $A=\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ ($B=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$, respectively), then we can obtain the generalized discrete Hausdorff-Young inequality

$$\|U_A X\|_q\le\left(N\cdot\left|a_1b_2-a_2b_1\right|\right)^{\frac{p-2}{2p}}\|U_B X\|_p$$

with $1<p\le 2$ and

$$\frac{1}{p}+\frac{1}{q}=1.$$

Proof: Let

$$X=\{x_1,x_2,x_3,\cdots,x_N\}=\{x(1),x(2),x(3),\cdots,x(N)\}\in C^N$$

be a discrete time series with length N and its DLCT $\hat{X}_C=\{\hat{x}_1,\hat{x}_2,\hat{x}_3,\cdots,\hat{x}_N\}\in C^N$ with $\hat{X}_C=U_C X$ and the transform parameter $C=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

Since $\|X\|_2=1$, $\|\hat{X}_B\|_2=\|U_C X\|_2=1$ from Parseval's theorem. Here $\|X\|_2=\left[\sum_{n=1}^{N}|x_n|^2\right]^{\frac{1}{2}}$. Clearly, we can obtain the inequality [13]:

$$\|U_C X\|_\infty\le M_C\|X\|_1$$

with $M=\|U_C\|_\infty$.

Here $\|U_C\|_\infty=\sup_l\left|u_C(l)\right|$ with

$$U_C=\{u_C(l)\},l=1,\cdots,N.$$

Hence, we have $\|U_C X\|_\infty\le M_C\|X\|_1$ with $M_C=\|U_C\|_\infty$.

Then from Riesz's theorem [11,12], we can obtain the discrete Hausdorff-Young inequality [11,12]

$$\|U_C X\|_q\le\left(M_C\right)^{\frac{2-p}{p}}\|X\|_p$$

with $1<p\le 2$ and

$$\frac{1}{p}+\frac{1}{q}=1.$$

Set $U_C = U_{AB^{-1}}$, then $U_C = U_{AB^{-1}} = U_A U_{B^{-1}}$ [5], we obtain

$$\left\|U_A U_{B^{-1}} X\right\|_q \le \left(M_C\right)^{\frac{2-p}{p}} \left\|X\right\|_p$$

with $M_C = \left\|U_{AB^{-1}}\right\|_\infty = \left\|U_A U_{B^{-1}}\right\|_\infty$.

Let $Y = U_{B^{-1}} X$, then $X = U_B Y$. In addition, from the property of DLCT [5] we can have

$$M_{AB^{-1}} = \left\|U_A U_{B^{-1}}\right\|_\infty = \sqrt{\frac{1}{N \cdot \left|a_1 b_2 - a_2 b_1\right|}}.$$

Hence we can obtain from the above equations

$$\left\|U_A Y\right\|_q \le \left(M_{AB^{-1}}\right)^{\frac{2-p}{p}} \left\|U_B Y\right\|_p$$

with

$$M_{AB^{-1}} = \sqrt{\frac{1}{N \cdot \left|a_1 b_2 - a_2 b_1\right|}}.$$

Since the value of X can be taken arbitrarily in C^N, Y can also be taken arbitrarily in C^N. Therefore, we can obtain the lemma.

Clearly, this lemma is the discrete version of Hausdorff-Young inequality. In the next sections, we will use this lemma to prove the new uncertainty relations.

3. The Uncertainty Relations

3.1. Shannon Entropic Principle

Theorem 1: For any given discrete time series

$$X = \left\{x_1, x_2, x_3, \cdots, x_N\right\} = \left\{x(1), x(2), x(3), \cdots, x(N)\right\} \in C^N$$

with length N and $\|X\|_2 = 1$, $\hat{x}_A\left(\hat{x}_B\right)$ is the DLCT series associated with the transform parameter $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ ($B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, respectively), $N_A\left(N_B\right)$ counts the non-zero elements of $\hat{x}_A$ ($\hat{x}_B$, respectively), then we can obtain the generalized discrete Shannon entropic uncertainty relation

$$H\left(\hat{x}_A(n)\right) + H\left(\hat{x}_B(m)\right) \ge \ln\left(\left|N \cdot \left(a_1 b_2 - a_2 b_1\right)\right|\right), \quad (n, m = 1, \cdots, N) \tag{3}$$

where

$$H\left(\hat{x}_A\right) = -\sum_{n=1}^{N}\left[\left(\ln\left|\hat{x}_A(n)\right|^2\right) \cdot \left|\hat{x}_A(n)\right|^2\right]$$

and

$$H\left(\hat{x}_B\right) = -\sum_{m=1}^{N}\left[\left(\ln\left|\hat{x}_B(m)\right|^2\right) \cdot \left|\hat{x}_B(m)\right|^2\right],$$

which are Shannon entropies. The equality in (3) holds iff $\left|\hat{x}_A\right| \equiv \frac{1}{\sqrt{N_A}}$ and $\left|\hat{x}_B\right| \equiv \frac{1}{\sqrt{N_B}}$.

Proof: From lemma 1, we have

$$\frac{\left(N \cdot \left|a_1 b_2 - a_2 b_1\right|\right)^{\frac{p-2}{2p}} \left(\sum_{m=1}^{N}\left|\hat{x}_B(m)\right|^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}} \ge 1.$$

Take natural logarithm in both sides in above inequality, we can obtain

$$T(p) \ge 0,$$

where

$$T(p) = \frac{p-2}{2p}\ln\left(N \cdot \left|a_1 b_2 - a_2 b_1\right|\right) + \frac{1}{p}\ln\left(\sum_{m=1}^{N}\left|\hat{x}_B(m)\right|^p\right) - \frac{p-1}{p}\ln\left(\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^{\frac{p}{p-1}}\right)$$

Since $1 < p \le 2$ and $\|X\|_2 = 1$ and Parseval equality, we know $T(2) = 0$. Note $T(p) \ge 0$ if $1 < p \le 2$. Hence, $T'(p) \le 0$ if $p = 2$. Since

$$T'(p) = \frac{1}{p^2}\ln\left(N \cdot \left|a_1 b_2 - a_2 b_1\right|\right) - \frac{1}{p^2}\ln\left(\sum_{m=1}^{N}\left|\hat{x}_B(m)\right|^p\right) + \frac{1}{p}\frac{\sum_{m=1}^{N}\left[\left(\ln\left|\hat{x}_B(m)\right|\right) \cdot \left|\hat{x}_B(m)\right|^p\right]}{\sum_{m=1}^{N}\left|\hat{x}_B(m)\right|^p} - \frac{1}{p^2}\ln\left(\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^{\frac{p}{p-1}}\right) + \frac{1}{p(p-1)}\frac{\sum_{n=1}^{N}\left[\left|\hat{x}_A(n)\right|^{\frac{p}{p-1}} \cdot \ln\left(\left|\hat{x}_A(n)\right|\right)\right]}{\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^{\frac{p}{p-1}}},$$

we can obtain the final result in theorem 1 by setting p = 2.

Now consider when the equality holds. From theorem 1, that the equality holds in (3) implies that $H\left(\hat{x}_A\right) + H\left(\hat{x}_B\right)$ reaches its minimum bound, which means that Minimize $H\left(\hat{x}_A\right) + H\left(\hat{x}_B\right)$ subject to $\left\|\hat{x}_A\right\|_2 = \left\|\hat{x}_B\right\|_2 = 1$, *i.e.*

Minimize

$$-\sum_{n=1}^{N}\left[\left(\ln\left|\hat{x}_A(n)\right|^2\right) \cdot \left|\hat{x}_A(n)\right|^2\right] - \sum_{m=1}^{N}\left[\left(\ln\left|\hat{x}_B(m)\right|^2\right) \cdot \left|\hat{x}_B(m)\right|^2\right],$$

subject to $\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^2 = \sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^2 = 1$.

To solve this problem let us consider the following Lagrangian

$$
\begin{aligned}
L = & -\sum_{n=1}^{N}\left[\left(\ln\left|\hat{x}_A(n)\right|^2\right)\cdot\left|\hat{x}_A(n)\right|^2\right] \\
& -\sum_{m=1}^{N}\left[\left(\ln\left|\hat{x}_B(m)\right|^2\right)\cdot\left|\hat{x}_B(m)\right|^2\right] \\
& +\lambda_1\left(\sum_{n=1}^{N}\left|\hat{x}_A(n)\right|^2-1\right)+\lambda_2\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^2-1\right)
\end{aligned}.
$$

In order to simplify the computation, we set $\left|\hat{x}_A(n)\right|^2 = p_n^A$ and $\left|\hat{x}_B(n)\right|^2 = p_n^B$. Hence we have

$$\frac{\partial L}{\partial p_n^A} = -\ln p_n^A - 1 + \lambda_1 = 0,$$

$$\frac{\partial L}{\partial p_n^B} = -\ln p_n^B - 1 + \lambda_2 = 0,$$

$$\sum_{n=1}^{N} p_n^A = 1,$$

$$\sum_{n=1}^{N} p_n^B = 1.$$

Solving the above equations, we finally obtain $\left|\hat{x}_A(n)\right| = \frac{1}{\sqrt{N_A}}$, $\left|\hat{x}_B(n)\right| = \frac{1}{\sqrt{N_B}}$. From the definition of Shannon entropy, we know that if $H(\hat{x}_A) = \ln N_A$ and $H(\hat{x}_B) = \ln N_B$, then

$$H(\hat{x}_A) + H(\hat{x}_B) = \ln\left(\left|N\cdot(a_1b_2 - a_2b_1)\right|\right).$$

In addition, we also can obtain $N_A N_B = \left|N\cdot(a_1b_2 - a_2b_1)\right|$. From the above proof, we know that $\left|\hat{x}_A(n)\right| = \frac{1}{\sqrt{N_A}}$ and $\left|\hat{x}_B(n)\right| = \frac{1}{\sqrt{N_B}}$ imply that $\hat{x}_A(m)$ and $\hat{x}_B(n)$ can be complex values, and only if their amplitudes are constants, the equality will hold. Now we can obtain the following corollary out of above analysis.

Corollary 1: For any given discrete time series

$$
\begin{aligned}
\tilde{X} &= \{x_1, x_2, x_3, \cdots, x_N\} \\
&= \{x(1), x(2), x(3), \cdots, x(N)\} \in C^N
\end{aligned}
$$

with length N and $\|\tilde{X}\|_2 = 1$, $\hat{x}_A(\hat{x}_B)$ is the DLCT series associated with the transform parameter $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ ($B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, respectively), $N_A(N_B)$ counts the non-zero elements of $\hat{x}_A$ ($\hat{x}_B$, respectively), if

$$\left|\hat{x}_A(n)\right| = \frac{1}{\sqrt{N_A}}$$

and

$$\left|\hat{x}_B(n)\right| = \frac{1}{\sqrt{N_B}},$$

then we have $N_A \cdot N_B = N\cdot\left|a_1b_2 - a_2b_1\right|$ and

$$H(\hat{x}_A) + H(\hat{x}_B) = \ln\left(\left|N\cdot(a_1b_2 - a_2b_1)\right|\right).$$

3.2. Rényi Entropic Principle

Theorem 2: For any given discrete time series

$$
\begin{aligned}
X &= \{x_1, x_2, x_3, \cdots, x_N\} \\
&= \{x(1), x(2), x(3), \cdots, x(N)\} \in C^N
\end{aligned}
$$

with length N and $\|X\|_2 = 1$, $\hat{x}_A(\hat{x}_B)$ is the DLCT series associated with the transform parameter $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ ($B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, respectively), $N_A(N_B)$ counts the non-zero elements of $\hat{x}_A$ ($\hat{x}_B$, respectively), then we can obtain the generalized discrete Renyi entropic uncertainty relation

$$
\begin{aligned}
& H_\vartheta(\hat{x}_A) + H_\zeta(\hat{x}_B) \geq \ln\left(\left|N\cdot(a_1b_2 - a_2b_1)\right|\right) \\
& \text{with } \frac{1}{2} < \zeta \leq 1 \text{ and } \frac{1}{\zeta} + \frac{1}{\vartheta} = 2
\end{aligned} \tag{4}
$$

where

$$H_\vartheta(\hat{x}_A) = \frac{1}{1-\vartheta}\cdot\ln\left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right),$$

$$H_\zeta(\hat{x}_B) = \frac{1}{1-\zeta}\cdot\ln\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right),$$

which are Rényi entropies.

Proof: In lemma 1, set $q = 2\vartheta$ and $p = 2\zeta$, we have $\frac{1}{2} < \zeta \leq 1$ and $\frac{1}{\zeta} + \frac{1}{\vartheta} = 2$. Then from lemma 1, we obtain

$$
\begin{aligned}
& \left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right)^{\frac{1}{2\vartheta}} \\
& \leq \left(N\cdot\left|a_1b_2 - a_2b_1\right|\right)^{\frac{\zeta-1}{2\zeta}}\cdot\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right)^{\frac{1}{2\zeta}}
\end{aligned}.
$$

Take the square of the above inequality, we have

$$\left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right)^{\frac{1}{\vartheta}} \leq \left(N\cdot\left|a_1b_2 - a_2b_1\right|\right)^{\frac{\zeta-1}{\zeta}}\cdot\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right)^{\frac{1}{\zeta}}.$$

Take the power $\frac{\zeta}{1-\zeta}$ of both sides in above inequallity, we obtain

$$\left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right)^{\frac{1}{\vartheta-1}} \le \left(N\cdot\left|a_1b_2-a_2b_1\right|\right)^{-1}\cdot\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right)^{\frac{1}{1-\zeta}},$$

i.e.,

$$\frac{\left(N\cdot\left|a_1b_2-a_2b_1\right|\right)^{-1}\cdot\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right)^{\frac{1}{1-\zeta}}}{\left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right)^{\frac{1}{\vartheta-1}}} \ge 1. \quad (5)$$

Take the natural logarithm on both sides of (5), we can obtain

$$\frac{1}{1-\zeta}\cdot\ln\left(\sum_{n=1}^{N}\left|\hat{x}_B(n)\right|^{2\zeta}\right)+\frac{1}{1-\vartheta}\cdot\ln\left(\sum_{m=1}^{N}\left|\hat{x}_A(m)\right|^{2\vartheta}\right) \ge \ln\left(\left|N\cdot\left(a_1b_2-a_2b_1\right)\right|\right).$$

Clearly, as $\zeta\to1$ and $\vartheta\to1$, the Renyi entropy reduces to Shannon entropy, thus the Renyi entropic uncertainty relation in (4) reduces to the Shannon entropic uncertainty relation (3). Hence the proof of equality in theorem 2 is trivial according to the proof of theorem 1.

Note that although Shannon entropic uncertainty relation can be obtained by Rényi entropic uncertainty relation, we still discuss them separately in the sake of integrality.

3.3. Another Shannon Entropic Principle via Sampling

The discrete Shannon entropy can be defined as

$$E\left(\rho(s)\right)=-\sum_{k=-\infty}^{\infty}\rho_k(s)\ln\rho_k(s) \quad (6)$$

where $\rho_k(x)$ is the density function of variable s.

Discrete Rényi entropy can be defined as follows:

$$H_\mu\left(\rho(x)\right)=\frac{1}{1-\mu}\ln\left(\sum_{k=-\infty}^{\infty}\left[\rho_k(x)\right]^\mu\right) \quad (7)$$

when $\mu\to1$, discrete Rényi entropy tend to discrete Shannon entropy.

In order to obtain the discrete spectrum, the sampling must be done. For two continuous functions' DLCT $F_A(u)$ and $F_B(v)$ with the transform parameter $A=\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ ($B=\begin{bmatrix}a_2 & b_2\\ c_2 & d_2\end{bmatrix}$, respectively), we set the sampling periods T_1 and T_2 and assume that they satisfy the Shannon sampling theorem [16]. Set

$$\begin{aligned}\rho_k(u)&=\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left|F_A(u)\right|^2\mathrm{d}u\\ \rho_l(v)&=\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left|F_B(v)\right|^2\mathrm{d}v\end{aligned} \quad (8)$$

Therefore

$$\int_{-\infty}^{\infty}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u=\sum_{k=-\infty}^{\infty}\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u \quad (9)$$

$$\int_{-\infty}^{\infty}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v=\sum_{l=-\infty}^{\infty}\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v \quad (10)$$

Since when $\gamma>1$, $f(x)=x^\gamma$ is a convex function, and when $\theta<1$, $g(y)=y^\theta$ is a concave function, we have the following inequalities

$$\left(\frac{1}{T_1}\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left|F_A(u)\right|^2\mathrm{d}u\right)^\gamma \le \frac{1}{T_1}\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u \quad (11)$$

$$\frac{1}{T_2}\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v \le \left(\frac{1}{T_2}\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left|F_B(v)\right|^2\mathrm{d}v\right)^\theta \quad (12)$$

Therefore

$$\int_{-\infty}^{\infty}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u=\sum_{k=-\infty}^{\infty}\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u \ge T_1\sum_{k=-\infty}^{\infty}\left(\frac{1}{T_1}\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left|F_A(u)\right|^2\mathrm{d}u\right)^\gamma=T_1^{1-\gamma}\sum_{k=-\infty}^{\infty}\left(\rho_k(u)\right)^\gamma.$$

$$\int_{-\infty}^{\infty}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v=\sum_{l=-\infty}^{\infty}\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v \le T_2\sum_{l=-\infty}^{\infty}\left(\frac{1}{\delta v}\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left|F_B(v)\right|^2\mathrm{d}v\right)^\theta=T_2^{1-\theta}\sum_{l=-\infty}^{\infty}\left(\rho_l(v)\right)^\theta,$$

i.e.,

$$\int_{-\infty}^{\infty}\left(\left|F_A(u)\right|^2\right)^\gamma\mathrm{d}u \ge T_1^{1-\gamma}\sum_{k=-\infty}^{\infty}\left(\rho_k(u)\right)^\gamma \quad (13)$$

$$\int_{-\infty}^{\infty}\left(\left|F_B(v)\right|^2\right)^\theta\mathrm{d}v \le T_2^{1-\theta}\sum_{l=-\infty}^{\infty}\left(\rho_l(v)\right)^\theta. \quad (14)$$

Therefore

$$\left(\frac{1}{\left|a_1b_2-a_2b_1\right|}\right)^{\frac{1}{\gamma}}\left|a_1b_2-a_2b_1\right|\left(T_1^{1-\gamma}\sum_{k=-\infty}^{\infty}\left(\rho_k(u)\right)^\gamma\right)^{1/\gamma} \le \left(\left(\frac{\theta}{\pi}\right)^{1/\theta}\Big/\left(\frac{\gamma}{\pi}\right)^{1/\gamma}\right)^{1/2}\left(T_2^{1-\theta}\sum_{l=-\infty}^{\infty}\left(\rho_l(v)\right)^\theta\right)^{1/\theta} \quad (15)$$

Take the power of $\frac{\theta}{1-\theta}$ on the both sides of above equation and use the relation between θ and γ, we have

$$\frac{\left(T_1\cdot T_2\right)\left(\frac{\theta}{\pi}\right)^{\frac{1}{2(1-\theta)}}\left(\sum_{l=-\infty}^{\infty}\left(\rho_l\left(v\right)\right)^{\theta}\right)^{\frac{1}{1-\theta}}}{\left(\frac{1}{\left|a_1b_2-a_2b_1\right|}\right)^{\frac{1}{\gamma-1}}\left|a_1b_2-a_2b_1\right|^{\frac{\gamma}{\gamma-1}}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2(\gamma-1)}}\left(\sum_{k=-\infty}^{\infty}\left(\rho_k\left(u\right)\right)^{\gamma}\right)^{\frac{1}{\gamma-1}}}\geq 1 \tag{16}$$

Take logarithm on both sides of above equation

$$\frac{1}{1-\theta}\ln\left(\sum_{l=-\infty}^{\infty}\left(\rho_l\left(v\right)\right)^{\theta}\right)+\frac{1}{1-\gamma}\ln\left(\sum_{k=-\infty}^{\infty}\left(\rho_k\left(u\right)\right)^{\gamma}\right)\geq\frac{\ln\left(\theta/\pi\right)}{2\left(\theta-1\right)}+\frac{\ln\left(\gamma/\pi\right)}{2\left(\gamma-1\right)}+\ln\left(\frac{\left|a_1b_2-a_2b_1\right|}{T_1\cdot T_2}\right) \tag{17}$$

That is,

$$H_\theta^B+H_\gamma^A\geq\frac{\ln\left(\theta/\pi\right)}{2\left(\theta-1\right)}+\frac{\ln\left(\gamma/\pi\right)}{2\left(\gamma-1\right)}+\ln\left(\frac{\left|a_1b_2-a_2b_1\right|}{T_1\cdot T_2}\right) \tag{18}$$

If

$$\left(a_1,b_1,c_1,d_1\right)=\left(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha\right)$$

and

$$\left(a_2,b_2,c_2,d_2\right)=\left(\cos\beta,\sin\beta,-\sin\beta,\cos\beta\right),$$

then we have

$$H_\theta^\alpha+H_\gamma^\beta\geq\frac{\ln\left(\theta/\pi\right)}{2\left(\theta-1\right)}+\frac{\ln\left(\gamma/\pi\right)}{2\left(\gamma-1\right)}+\ln\left(\frac{\left|\cos\alpha\sin\beta-\cos\beta\sin\alpha\right|}{T_\alpha\cdot T_\beta}\right)^{1/2}\left(T_2^{1-\theta}\sum_{l=-\infty}^{\infty}\left(\rho_l\left(v\right)\right)^{\theta}\right)^{1/\theta}$$

when $\alpha=2n\pi+\pi/2\ \left(n\in\mathbf{Z}\right)$ and $\beta=2l\pi\ \left(l\in\mathbf{Z}\right)$, we have the traditional case

$$H_\theta+H_\gamma\geq\frac{\ln\left(\theta/\pi\right)}{2\left(\theta-1\right)}+\frac{\ln\left(\gamma/\pi\right)}{2\left(\gamma-1\right)}-\ln\left(T_1\cdot T_2\right). \tag{19}$$

Specially, when $\theta\to 1$, $\gamma\to 1$, have

$$E_{\left(a_1,b_1,c_1,d_1\right)}+E_{\left(a_2,b_2,c_2,d_2\right)}\geq\ln\left(2\pi\right)+1-\ln\left(\frac{2T_1\cdot T_2}{\left|a_1b_2-a_2b_1\right|}\right) \tag{20}$$

where,

$$E_A=-\sum_{k=-\infty}^{\infty}\left(\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left|F_A\left(u\right)\right|^2\mathrm{d}u\right)\cdot\ln\left(\int_{k\cdot T_1}^{(k+1)\cdot T_1}\left|F_A\left(u\right)\right|^2\mathrm{d}u\right),$$

$$E_B=-\sum_{l=-\infty}^{\infty}\left(\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left|F_B\left(v\right)\right|^2\mathrm{d}v\right)\cdot\ln\left(\int_{l\cdot T_2}^{(l+1)\cdot T_2}\left|F_B\left(v\right)\right|^2\mathrm{d}v\right).$$

4. Conclusion

In this article, we extended the entropic uncertainty relations in DLCT domains. We first introduced the generalized discrete Hausdorff-Young inequality. Based on this inequality, we derived the discrete Shannon entropic uncertainty relation and discrete Rényi entropic uncertainty relation. Interestingly, when the variable's amplitude is equal to the constant, *i.e.* the inverse of the square root of number of non-zero elements, the equality holds in the uncertainty relation. In addition, the product of the two numbers of non-zero elements is equal to $N\cdot\left|a_1b_2-a_2b_1\right|$, *i.e.*, $N_\alpha N_\beta=N\cdot\left|a_1b_2-a_2b_1\right|$. On one hand, these new uncertainty relations enrich the ensemble of uncertainty principles, and on the other hand, these derived bounds yield new understanding of discrete signals in new transform domain.

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