

# A t-Norm Fuzzy Logic for Approximate Reasoning

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**How to cite this paper:** Tserkovny, A. (2017) A t-Norm Fuzzy Logic for Approximate Reasoning. *Journal of Software Engineering and Applications*, 10, 639-662. <https://doi.org/10.4236/jsea.2017.107035>

**Received:** March 6, 2017

**Accepted:** June 26, 2017

**Published:** June 29, 2017

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## Abstract

A t-norm fuzzy logic is presented, in which a triangular norm (t-norm) plays the role of a graduated conjunction operator. Based on this fuzzy logic we develop methods for fuzzy reasoning in which antecedents and consequents involve fuzzy conditional propositions of the form “*If x is A then y is B*”, with *A* and *B* being fuzzy concepts (fuzzy sets). In this study, we present a systemic approach toward fuzzy logic formalization for approximate reasoning. We examine statistical characteristics of the proposed fuzzy logic. As the matter of practical interest, we construct a set of fuzzy conditional inference rules on the basis of the proposed fuzzy logic. Important features of these rules are investigated.

## Keywords

Fuzzy Logic, t-Norm, Implication, Antecedent, Consequent, Modus-Ponens, Fuzzy Conditional Inference Rule

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## 1. Introduction

In our daily life we often make inferences whose *antecedents* and *consequents* contain fuzzy concepts. Such an inference cannot be made adequately by the methods which are based either on classical two valued logic or on many valued logic. In order to make such an inference, Zadeh suggested an inference rule called “compositional rule of inference”. Using this inference rule, he, Mamdani, Mizumoto *et al.*, R. Aliev and A. Tserkovny suggested several methods for fuzzy reasoning in which the antecedent contain a conditional proposition with fuzzy concepts:

**Ant 1:** If *x* is *P* then *y* is *Q*

**Ant 2:** *x* is *P'*

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(1.1)

**Cons:**  $y$  is  $Q'$ .

Those methods are based on an implication operator in various fuzzy logics. This matter has been under discussion for the last couple decades [1]-[46].

In (1.1)  $x$  and  $y$  are the names of objects, and  $P, P', Q$  and  $Q'$  are fuzzy concepts represented by fuzzy sets in *universe of discourse*  $U, U, V$  and  $V$ , respectively. This form of inference may be viewed as a generalized *modus ponens* which reduces to modus ponens when  $P' = P$  and  $Q' = Q$ . Let  $P$  and  $Q$  be fuzzy sets in  $U$  and  $V$  respectively and correspondent fuzzy sets be represented as such  $P \subset U | \mu_p : U \rightarrow [0,1], Q \subset V | \mu_Q : V \rightarrow [0,1]$ , where

$$P = \int_U \mu_p(u)/u, \quad Q = \int_V \mu_Q(v)/v. \tag{1.2}$$

And let  $\times, \cup, \cap, \neg$  and  $\oplus$  be Cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively. Then the following fuzzy relations in  $U \times V$  can be derived from fuzzy conditional proposition "If  $x$  is  $P$  then  $y$  is  $Q$ " in Ant 1 of (1.1). The fuzzy relations  $R_m$  and  $R_a$  were proposed by Zadeh [42] [43] [44] [45],  $R_c$  by Mamdani [29],  $R_s, R_g$  are by Mizumoto [32] [33],  $R_{L1}, R_{L2}, R_{L3}$  and  $R_{L4}$  are by Aliev and Tserkovny [2] [3] [4] [5], which are

$$R_{L1} = (P \times V \Rightarrow_{L1} U \times Q) = \int_{U \times V} [\mu_p(u) \rightarrow_{L1} \mu_Q(v)] / (u, v) \tag{1.3}$$

where

$$\mu_p(u) \rightarrow_{L1} \mu_Q(v) = \begin{cases} 1 - \mu_p(u), & \mu_p(u) < \mu_Q(v), \\ 1, & \mu_p(u) = \mu_Q(v), \\ \mu_Q(v), & \mu_p(u) > \mu_Q(v). \end{cases}$$

$$R_{L2} = (P \times V \Rightarrow_{L2} U \times Q) = \int_{U \times V} [\mu_p(u) \rightarrow_{L2} \mu_Q(v)] / (u, v) \tag{1.4}$$

where

$$\mu_p(u) \rightarrow_{L2} \mu_Q(v) = \begin{cases} 1, & \mu_p(u) \leq \mu_Q(v), \\ (1 - \mu_p(u)) \wedge \mu_Q(v), & \mu_p(u) > \mu_Q(v). \end{cases}$$

$$R_{L3} = (P \times V \Rightarrow_{L3} U \times Q) = \int_{U \times V} [\mu_p(u) \rightarrow_{L3} \mu_Q(v)] / (u, v) \tag{1.5}$$

where

$$\mu_p(u) \rightarrow_{L3} \mu_Q(v) = \begin{cases} 1, & \mu_p(u) \leq \mu_Q(v), \\ \frac{\mu_Q(v)}{\mu_p(u) + (1 - \mu_Q(v))}, & \mu_p(u) > \mu_Q(v). \end{cases}$$

$$R_{L4} = (P \times V \Rightarrow_{L4} U \times Q) = \int_{U \times V} [\mu_p(u) \rightarrow_{L4} \mu_Q(v)] / (u, v) \tag{1.6}$$

where

$$\mu_p(u) \rightarrow_{L4} \mu_Q(v) = \begin{cases} \frac{1 - \mu_p(u) + \mu_Q(v)}{2}, & \mu_p(u) > \mu_Q(v), \\ 1, & \mu_p(u) \leq \mu_Q(v). \end{cases}$$

A necessary consideration for this discussion is that with the only few exceptions for *S-logic* (1.6) and *G-logic* (1.7), and L1-L4(1.3)-(1.6) all other known fuzzy logics don't satisfy either the classical "*modus-ponens*" principal, or other criteria which fit human intuition and first formulated in [32]. The proposed fuzzy logic has an implication operator, which satisfies the "*modus-ponens*" principal and criteria, which fit human intuition.

The second section of the article will cover some initial fuzzy logic creation considerations. In third section a set of operations in proposed fuzzy logic is presented. The fourth section is devoted to an introduction of a t-norm as a graduated conjunction operator in presented fuzzy logic. The Section five will cover a power sets based features of proposed fuzzy logic. The statistical analysis of the fuzzy logic is completed in Section six. Section Seven covers the issue of fuzzy conditional inference rules based on proposed fuzzy logic and extended investigation of its features.

## 2. Preliminary Considerations

In order to start formulating of a fuzzy logic major implication operator, we are proposing the following function as a part of it:

$$F_1(p, q) = \neg p \cdot q = (1 - p) \cdot q \quad (2.1)$$

### Definition 1.

An *implication* function is a continuous function  $I$  from  $[0, 1] \times [0, 1]$  into  $[0, 1]$  such that the following properties hold for  $\forall p, p', q, q', r \in [0, 1]$ :

(I1) If  $p \leq p'$  then  $I(p, q) \geq I(p', q)$ ;

(I2) If  $q \leq q'$  then  $I(p, q) \leq I(p, q')$ ;

(I3)  $I(0, q) = 1$ , (Falsity Principle);

(I4)  $I(1, q) \leq q$ , (Neutrality Principle);

(I5)  $I(p, I(q, r)) = I(q, I(p, r))$ , (Exchange Principle);

(I6)  $I(p, q) = I(n(q), n(p))$ . (Contra positive Symmetry Principle), where  $n(\cdot)$ -is a negation, which could be defined for  $n(q)$  as  $T(\neg Q) = 1 - T(Q)$ .

Before proving that  $I(p, q)$  defined as

$$I(p, q) = \begin{cases} F_1(p, q), & p > q; \\ 1, & p \leq q, \end{cases} \quad (2.2)$$

and  $F_1(p, q)$  is from (2.1) satisfies (I1)-(I6) axioms, let us show some basic operations in proposed fuzzy logic.

## 3. The Fuzzy Logic

Let us designate the truth values of logical *antecedent*  $P$  and *consequent*  $Q$  as  $T(P) = p$  and  $T(Q) = q$  respectively. Then relevant set of proposed fuzzy logic operators are shown in **Table 1**.

In other words we propose a new many-valued system, characterized by the set of base *union* ( $\cup$ ) and *intersection* ( $\cap$ ) operations with relevant *complement*, defined as  $T(\neg P) = 1 - T(P)$ . In addition, the operators  $\downarrow$  and  $\uparrow$  are

**Table 1.** Relevant set of proposed fuzzy logic operators.

Name	Designation	Value
Tautology	$P^1$	1
Controversy	$P^0$	0
Negation	$\neg P$	$1 - P$
Disjunction	$P \vee Q$	$\begin{cases} p \cdot q, p + q < 1, \\ 1, p + q \geq 1 \end{cases}$
Conjunction	$P \wedge Q$	$\begin{cases} p \cdot q, p + q > 1, \\ 0, p + q \leq 1 \end{cases}$
Implication	$P \rightarrow Q$	$\begin{cases} (1 - p) \cdot q, p > q, \\ 1, p \leq q \end{cases}$
Equivalence	$P \leftrightarrow Q$	$\begin{cases} (1 - p) \cdot q, p < q, \\ 1, p = q \\ (1 - q) \cdot p, q < p, \end{cases}$
Pierce Arrow	$P \downarrow Q$	$\begin{cases} (1 - p) \cdot (1 - q), p + q < 1, \\ 0, p + q \geq 1 \end{cases}$
Shaffer Stroke	$P \uparrow Q$	$\begin{cases} (1 - p) \cdot (1 - q), p + q > 1, \\ 1, p + q \leq 1 \end{cases}$

expressed as negations of the  $\cup$  and  $\cap$  correspondingly. It is a well-known fact that the operation *implication* in a fuzzy logic was the foundation of decision making procedure for numerous approximate reasoning tasks. Therefore let us prove that proposed *implication* operation from (2.2) satisfies axioms (I1)-(I6). For this matter let us pose the problem very explicitly. We are working in many-valued system, which for present purposes is all or some of the real interval  $\mathfrak{R} = [0, 1]$ . As was mentioned in [1], the rationales there are more than ample for our purposes in very much of practice, the following set  $\{0, 0.1, 0.2, \dots, 0.9, 1\}$  of 11 values is quite sufficient, and we shall use this set  $V_I$  in our illustration.

**Table 2** shows the operation *implication* in proposed fuzzy logic.

**Theorem 1.** Let a continuous function  $I(p, q)$  is defined in (2.3) and its values are from a **Table 2**, *i.e.*

$$I(p, q) = \begin{cases} F_1(p, q), p > q; \\ 1, p \leq q, \end{cases} = \begin{cases} (1 - p) \cdot q, p > q, \\ 1, p \leq q \end{cases} \tag{3.1}$$

Where function  $F(p, q)^{norm}$  is defined in (1), then axioms (I1)-(I6) are satisfied and, therefore it is an *implication* operation.

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**Proof:**

**(I1):**  $\forall p, p' \in [0, 1] \mid p' \geq p > q$

$$\Rightarrow I(p, q) - I(p', q) = (1 - p) \cdot q - (1 - p') \cdot q = (p' - p) \cdot q \mid p' - p \geq 0$$

**Table 2.** The operation implication in proposed fuzzy logic.

$p \rightarrow q$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1	1	1	1	1	1	1	1	1	1	1
0.1	0	1	1	1	1	1	1	1	1	1	1
0.2	0	0.08	1	1	1	1	1	1	1	1	1
0.3	0	0.07	0.14	1	1	1	1	1	1	1	1
0.4	0	0.06	0.12	0.18	1	1	1	1	1	1	1
0.5	0	0.05	0.1	0.15	0.2	1	1	1	1	1	1
0.6	0	0.04	0.08	0.12	0.16	0.2	1	1	1	1	1
0.7	0	0.03	0.06	0.09	0.12	0.15	0.18	1	1	1	1
0.8	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	1	1	1
0.9	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	1	1
1	0	0	0	0	0	0	0	0	0	0	1

$$\Rightarrow I(p, q) \geq I(p', q)$$

whereas

$$q \geq p' \geq p \Rightarrow I(p, q) - I(p', q) \equiv 0 \Rightarrow I(p, q) \equiv I(p', q)$$

**(I2):**  $\forall q, q' \in [0, 1] | q \leq q' < p$

$$\Rightarrow I(p, q) - I(p, q') = (1 - p) \cdot q - (1 - p) \cdot q' = (1 - p) \cdot (q - q') | q - q' \leq 0$$

$$\Rightarrow I(p, q) \leq I(p, q')$$

whereas

$$p \leq q \leq q' \Rightarrow I(p, q) - I(p, q') \equiv 0 \Rightarrow I(p, q) \equiv I(p, q')$$

**(I3):**  $I(0, q) \equiv 1, q \geq 0;$

**(I4):**  $I(1, q) \equiv 0 \Rightarrow I(1, q) \leq q;$

**(I5):** Since  $\forall p, q, r \in [0, 1] | p > q > r \Rightarrow q \rightarrow r = (1 - q) \cdot r$ , then

$$p \rightarrow (q \rightarrow r) = (1 - p) \cdot (1 - q) \cdot r \tag{3.2}$$

Whereas

$$p \rightarrow r = (1 - p) \cdot r \Rightarrow q \rightarrow (p \rightarrow r) = (1 - q) \cdot (1 - p) \cdot r \tag{3.3}$$

**(I6):**

$$I(n(q), n(p)) = (1 - q) \rightarrow (1 - p) = \begin{cases} q \cdot (1 - p), 1 - q > p, \\ 1, 1 - q \leq p \end{cases} = \begin{cases} (1 - p) \cdot q, p > q, \\ 1, p \leq q. \end{cases}$$

Q.E.D.

In addition proposed fuzzy logic is characterized by the following features:

*Commutativity* for both conjunction ( $\cap$ ) and disjunction ( $\cup$ ) operations, *i.e.:*

$$p \wedge q = q \wedge p \quad \text{and} \quad p \vee q = q \vee p;$$

*Assotiativity* for these operations:

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

and

$$p \vee (q \vee r) = (p \vee q) \vee r;$$

*Distributivity:*

$$(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r), \tag{3.4}$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r). \tag{3.5}$$

To prove the feature (3.4) note that

$$(p \vee q) \wedge r = \begin{cases} (p \cdot q) \wedge r, p + q < 1, \\ 1 \wedge r, p + q \geq 1 \end{cases} \tag{3.6}$$

$$(p \cdot q) \wedge r = \begin{cases} p \cdot q \cdot r, p \cdot q + r > 1, \\ 0, p \cdot q + r \leq 1 \end{cases}, 1 \wedge r = r$$

On the other hand

$$q \wedge r = \begin{cases} q \cdot r, q + r > 1, \\ 0, q + r \leq 1 \end{cases} \tag{3.7}$$

To prove the Feature (3.5) by using (3.7) we have

$$p \vee (q \wedge r) = p \vee \begin{cases} q \cdot r, q + r > 1, \\ 0, q + r \leq 1 \end{cases} = \begin{cases} p \cdot q \cdot r, p \cdot q + r > 1, \\ 0, p \cdot q + r \leq 1 \end{cases} \tag{3.8}$$

$$(p \vee q) \wedge (p \vee r) = p \vee (q \wedge r) \tag{3.9}$$

Therefore the Expression (3.8) equals (3.9) Q.E.D.

*DeMorgan* theorems, which are extrapolated over the  $M = [0, 1]$ :

$$\neg(p \wedge q) = \neg p \vee \neg q,$$

$$\neg(p \vee q) = \neg p \wedge \neg q;$$

To prove these theorems notice that

$$\neg(p \wedge q) = \begin{cases} \neg(p \cdot q), p + q > 1, \\ 1, p + q \leq 1 \end{cases} \tag{3.10}$$

On the other hand

$$\neg p \vee \neg q = \begin{cases} \neg p \cdot \neg q, \neg p + \neg q < 1, \\ 1, \neg p + \neg q \geq 1 \end{cases} = \begin{cases} \neg(p \cdot q), p + q > 1, \\ 1, p + q \leq 1 \end{cases} \tag{3.11}$$

Therefore the Expression (3.10) equals (3.11) Q.E.D. By analogous

$$\neg(p \vee q) = \begin{cases} \neg(p \cdot q), p + q < 1, \\ 0, p + q \geq 1 \end{cases} \tag{3.12}$$

On the other hand

$$\neg p \wedge \neg q = \begin{cases} \neg p \cdot \neg q, \neg p + \neg q > 1, \\ 0, \neg p + \neg q \leq 1 \end{cases} \tag{3.13}$$

Therefore the Expression (3.12) equals (3.13) Q.E.D. It should be mentioned that proposed fuzzy logic could also be characterized by yet another featured  $\neg\neg p = p$ . As a conclusion we should admit that all above features confirm that

resulting system can be applied to  $V_{II}$  for every finite and infinite  $n$  up to that  $(V_{II}, \neg, \wedge, \vee, \rightarrow)$  is then *closed* under all its operations.

#### 4. The t-Norm

##### Proposition.

In proposed fuzzy logic the operation of conjunction

$$\text{Conj}(p, q) = \begin{cases} p \cdot q, p + q > 1, \\ 0, p + q \leq 1 \end{cases} \quad (4.1)$$

is a t-norm.

##### Proof:

The function  $\text{Conj}(p, q)$  is a t-norm if the following is true

- 1) Commutativity:  $p \wedge q = q \wedge p$
- 2) Associativity:  $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
- 3) Monotony:  $p \leq q, p \wedge r \leq q \wedge r$
- 4) Neutrality:  $1 \wedge p = p$
- 5) Absorption  $0 \wedge p = 0$

*Commutativity:*

$$\text{Conj}(p, q) = P \cap Q = \begin{cases} p \cdot q, p + q > 1, \\ 0, p + q \leq 1 \end{cases}$$

and

$$\text{Conj}(q, p) = Q \cap P = \begin{cases} q \cdot p, q + p > 1, \\ 0, q + p \leq 1 \end{cases}$$

therefore

$$\text{Conj}(p, q) = \text{Conj}(q, p) \quad (\text{Q.E.D.}).$$

*Associativity:*

Case:  $\text{Conj}(p, q) \wedge r$

$$\text{Conj}(p, q) = p \cdot q, p + q > 1$$

$$\Rightarrow \text{Conj}(p, q) \wedge r = \begin{cases} \text{Conj}(p, q) \cdot r, \text{Conj}(p, q) + r > 1 \\ 0, \text{Conj}(p, q) + r \leq 1 \end{cases} = \begin{cases} p \cdot q \cdot r, p + q + r > 1 \\ 0, p + q + r \leq 1 \end{cases}$$

From where we have that

$$\text{Conj}(p, r) = \begin{cases} p \cdot q \cdot r, p + q + r > 1 \\ 0, p + q + r \leq 1 \end{cases} \quad (4.2)$$

For the case  $p \wedge \text{Conj}(q, r)$ , where

$$\text{Conj}(q, r) = Q \cap R = \begin{cases} q \cdot r, q + r > 1, \\ 0, q + r \leq 1 \end{cases} \quad (4.3)$$

Using (4.3) we are getting similar to (4.2) results

$$p \wedge \text{Conj}(q, r) = \begin{cases} p \cdot q \cdot r, p + q + r > 1 \\ 0, p + q + r \leq 1 \end{cases}$$

$$\text{Conj}(p, q) = p \wedge \text{Conj}(q, r) \quad (\text{Q.E.D.}).$$

*Monotony:*

If  $p \leq q \Rightarrow p \wedge r \leq q \wedge r$  then given

$$p \wedge r = \begin{cases} p \cdot r, p + r > 1, \\ 0, p + r \leq 1 \end{cases} \quad \text{and} \quad q \wedge r = \begin{cases} q \cdot r, q + r > 1, \\ 0, q + r \leq 1 \end{cases}$$

we are getting the following

$$p \cdot r \leq q \cdot r \mid p \leq q \tag{Q.E.D.}$$

*Neutrality:*

$$1 \wedge p = \begin{cases} 1 \cdot p, 1 + p > 1, \\ 0, 1 + p \leq 1 \end{cases} = 1 \tag{Q.E.D.}$$

*Absorption:*

$$0 \wedge p = \begin{cases} 0 \cdot p, p > 1, \\ 0, p \leq 1 \end{cases} = 0 \tag{Q.E.D.}$$

### 5. Fuzzy Power Sets and the Fuzzy Logic

**Definition 2.** [9]

Given a fuzzy implication operator  $\rightarrow$  and a fuzzy subset  $Q$  of a crisp universe  $U$ , the *fuzzy power set*  $PQ$  of  $Q$  is given by the membership function  $\mu_{PQ}$ , with

$$\mu_{PQ}(x) = \bigcap_{x \in U} (\mu_P x \rightarrow \mu_Q x) \tag{5.1}$$

The degree to which  $P$  is subset of  $Q$  is

$$\pi(P \subseteq Q) = \mu_{PQ}(x)$$

**Definition 3.** [9]

Where conditions are as in Definition 2, the degree to which the fuzzy sets  $P$  and  $Q$  is the same, or their *degree of sameness*, is

$$\pi(P \equiv Q) = \pi(P \subseteq Q) \wedge \pi(P \supseteq Q) \tag{5.2}$$

The following is then immediate.

**Proposition 1.** [9]

$$\pi(P \equiv Q) = \bigcap_{x \in U} (\mu_P x \leftrightarrow \mu_Q x) \tag{5.3}$$

Based on (5.1)-(5.3) and taken into account (3.1) we can formulate the following.

**Proposition 2.** [9] (Degree of possibility of *set-inclusion*)

$$\pi(P \subseteq Q) = \begin{cases} (1 - \mu_P x) \cdot \mu_Q x, \mu_P x > \mu_Q x, \\ 1, \mu_P x \leq \mu_Q x, \end{cases}$$

**Proposition 3.** (Degree of possibility of *set-equality*)

$$\pi(P \equiv Q) = \begin{cases} (1 - \mu_P x) \cdot \mu_Q x, \mu_P x > \mu_Q x, \\ 1, \mu_P x = \mu_Q x, \\ (1 - \mu_Q x) \cdot \mu_P x, \mu_Q x < \mu_P x, \end{cases} \tag{5.4}$$



From (4-4) is clear that  $\pi(P \equiv Q) = 1$  in case when  $\mu_P x = \mu_Q x$ . As it was mentioned in [9] there seem to be two plausible ways to define the degree to which sets  $P$  and  $Q$  may be said to be *disjointed*. One is the degree to which each is a subset of the other's complement. The second is the degree to which their intersection is empty.

**Definition 4.** [9] The degree of disjointness of  $P$  and  $Q$ , or degree to which  $P$  and  $Q$  are disjointed, in the first and second sense, are

$$(1) \pi(Pdisj_1Q) = \pi(P \subseteq Q^c) \wedge \pi(Q \supseteq B^c)$$

$$(2) \pi(Pdisj_2Q) = \pi(P \cap Q \equiv \emptyset).$$

For the case (1)

$$\pi(P \subseteq Q^c) = \mu_P x \rightarrow (1 - \mu_Q x) = \begin{cases} (1 - \mu_P x) \cdot (1 - \mu_Q x), & \mu_P x > (1 - \mu_Q x), \\ 1, & \mu_P x \leq (1 - \mu_Q x), \end{cases} \quad (5.5)$$

$$\pi(Q \supseteq P^c) = \mu_Q x \rightarrow (1 - \mu_P x) = \begin{cases} (1 - \mu_Q x) \cdot (1 - \mu_P x), & \mu_Q x > (1 - \mu_P x), \\ 1, & \mu_Q x \leq (1 - \mu_P x), \end{cases} \quad (5.6)$$

Therefore from (5.5) and (5.6) definition (1) looks like

$$\pi(Pdisj_1Q) = \begin{cases} (1 - \mu_P x) \cdot (1 - \mu_Q x), & \mu_P x + \mu_Q x > 1, \\ 1, & \mu_P x + \mu_Q x \leq 1 \end{cases} \quad (5.7)$$

For the case (2)

$$\pi(Pdisj_2Q) = \mu_P x \wedge \mu_Q x = \begin{cases} \mu_P x \cdot \mu_Q x, & \mu_P x + \mu_Q x > 1, \\ 0, & \mu_P x + \mu_Q x \leq 1 \end{cases} \quad (5.8)$$

**Definition 5.** [9] (Degree to which a set is a subset of its complement). The expression

$$\pi(P \subseteq P^c) = \mu_P x \rightarrow (1 - \mu_P x) = \begin{cases} (1 - \mu_P x)^2, & \mu_P x > 0.5, \\ 1, & \mu_P x \leq 0.5, \end{cases}$$

**Definition 6.** [9] (Degree to which a set is disjointed from its complement, in the two senses). From (5.7), (5.8) the following is taking place

The value of  $\pi(Pdisj_1P^c) \equiv 1$ , whereas  $\pi(Pdisj_2P^c) \equiv 0$ .

## 6. Statistical Property of the Fuzzy Logic

In this chapter we discuss some properties of proposed fuzzy implication operator (3.1), assuming that the two propositions (*antecedent/consequent*) in a given compound proposition are independent of each other and the truth values of the propositions are uniformly distributed [20] on the interval [0,1]. In other words we assume that the propositions  $P$  and  $Q$  are independent of each other and the truth values  $v(P)$  and  $v(Q)$  are uniformly distributed across the interval [0, 1]. Let  $p = v(P)$ ,  $q = v(Q)$ . Then the value of the implication  $I = v(p \rightarrow q)$  is some function  $I = I(p, q)$ .

Because  $p$  and  $q$  are assumed to be uniformly and independently distributed across [0, 1], the expected value of the implication is

$$E(I) = \iint_{\mathfrak{R}} I(p, q) dpdq \tag{6.1}$$

And its variance is

$$Var(I) = E\left[ (I - E(I))^2 \right] = \iint_{\mathfrak{R}} (I(p, q) - E(I))^2 dpdq = E[I^2] - E[I]^2 \tag{6.2}$$

where  $\mathfrak{R} = \{(p, q) : 0 \leq p \leq 1, 0 \leq q \leq 1\}$ . From (6.1) and given Expression (3.1) and the fact that

$$I(p, q) = \begin{cases} I_1(p, q), & p > q, \\ I_2(p, q), & p \leq q \end{cases}$$

we have the following

But from **Table 2** it is clear that for  $I_1(p, q)$  we have

$$\mathfrak{R}_1 = \{(p, q) : 0.2 \leq p \leq 0.9, 0.1 \leq q \leq 0.8\}$$

because of the following

$I_1(1, q) \equiv 0$  and  $I_1(p, 0) \equiv 0$ , therefore

$$E(I_1) = \iint_{\mathfrak{R}_1} I_1(p, q) dpdq = \int_{0.1}^{0.8} \int_{0.2}^{0.9} [(1-p) \cdot q] dpdq = \int_{0.1}^{0.8} q \int_{0.2}^{0.9} (1-p) dpdq. \tag{6.3}$$

Given (6.3) the following is true

$$\int_{0.2}^{0.9} (1-p) dp = (p - p^2 \cdot 1/2) \Big|_{p=0.2}^{p=0.9} = 0.325 \tag{6.4}$$

Whereas

$$\int_{0.1}^{0.8} q dq = (q^2 \cdot 1/2) \Big|_{q=0.1}^{q=0.8} = 0.325 \tag{6.5}$$

From (6.4) and (6.5)  $E(I_1) = 0.325^2 = 0.105625$ . Whereas  $E(I_2) = 1$ . Let us notice that for the most implications we have the following

$$\frac{Card(I_1)}{Card(I_2)} \approx 0.8 \Rightarrow E(I) = 0.55 \cdot E(I_2) + 0.45 \cdot E(I_1) = 0.598$$

From (6.2) we have

$$I_1^2(p, q) = [(1-p) \cdot q]^2 = [q - p \cdot q]^2 = q^2 - 2pq^2 + p^2q^2 \tag{6.6}$$

From (6.6) finally we have

$$\begin{aligned} I_1^2(p, q) &= q^2 \cdot (1 - 2p + p^2) \\ \Rightarrow E(I_1^2) &= \iint_{\mathfrak{R}_1} I_1^2(p, q) dpdq = \frac{q^3}{3} \Big|_{q=0.1}^{q=0.8} \cdot \left( p - \frac{p^2}{2} + \frac{p^3}{3} \right) \Big|_{p=0.2}^{p=0.9} = 0.056898. \end{aligned}$$

Whereas  $E(I_2^2) = 1$ .

Therefore  $E(I^2) = (E(I_1^2) + E(I_2^2)) / 2 = 0.528449$  From (6.2) we have

$$Var(I) = 0.170845$$

Both values of  $E(I)$  and  $Var(I)$  demonstrate that the proposed fuzzy implication operator could be considered as a second of the fuzziest implication

from the list [34] of known so far. In addition to that feature it satisfies the set of important Criteria I-IV, which is not the case for the most above mentioned implication operators.

## 7. The Fuzzy Logic and Fuzzy Conditional Inference

As it was mentioned in [32] in the semantics of natural language there exist vast amounts of concepts and we humans very often make inferences antecedents and consequences of which contain fuzzy concepts. Therefore, from the standpoint of artificial intelligence, it seems that formalization of inference methods for such inferences is very important. Following a well-known pattern, established a couple of decades ago and the standard approaches toward such formalization, let  $U$  and  $V$  (from now on) be two *universes of discourses* and correspondent fuzzy sets be represented as such

$$P \subset U \mid \mu_P : U \rightarrow [0,1], \quad Q \subset V \mid \mu_Q : V \rightarrow [0,1] = 0.170845$$

where

$$P = \int_U \mu_P(u)/u \quad (7.1)$$

and

$$Q = \int_V \mu_Q(v)/v \quad (7.2)$$

Whereas given (7.1) and (7.2) a *binary relationship* for the fuzzy conditional proposition of the type: "If  $x$  is  $P$  then  $y$  is  $Q$ " for proposed fuzzy logic is defined as

$$\begin{aligned} R(A_1(x), A_2(y)) &= P \times V \rightarrow U \times Q = \int_{U \times V} \mu_P(u)/(u,v) \rightarrow \int_{U \times V} \mu_Q(v)/(u,v) \\ &= \int_{U \times V} (\mu_P(u) \rightarrow \mu_Q(v))/(u,v). \end{aligned} \quad (7.3)$$

Given (3.1) expression (7.3) looks like

$$\mu_P(u) \rightarrow \mu_Q(v) = \begin{cases} (1 - \mu_P(u)) \cdot \mu_Q(v), & \mu_P(u) > \mu_Q(v), \\ 1, & \mu_P(u) \leq \mu_Q(v). \end{cases} \quad (7.4)$$

It is well known that given a *unary relationship*  $R(A_1(x)) = P$  one can obtain the consequence  $R(A_2(y))$  by applying compositional rule of inference (CRI) to  $R(A_1(x))$  and  $R(A_1(x), A_2(y))$  of Type (7.3):

$$\begin{aligned} R(A_2(y)) &= P \circ R(A_1(x), A_2(y)) \\ &= \int_U \mu_P(u)/u \circ \int_{U \times V} \mu_P(u) \rightarrow \mu_Q(v)/(u,v) \\ &= \int_V \bigcup_{u \in U} [\mu_P(u) \wedge (\mu_P(u) \rightarrow \mu_Q(v))]/v. \end{aligned} \quad (7.5)$$

In order that Criterion I is satisfied, that is  $R(A_2(y)) = Q$  from (7.5) the equality

$$\bigcup_{u \in U} [\mu_P(u) \wedge (\mu_P(u) \rightarrow \mu_Q(v))] = \mu_Q(v) \quad (7.6)$$

must be satisfied for arbitrary  $v$  in  $V$  and in order that the equality (7.6) is satisfied, it is necessary that the inequality

$$\mu_P(u) \wedge (\mu_P(u) \rightarrow \mu_Q(v)) \leq \mu_Q(v) \tag{7.7}$$

holds for arbitrary  $u \in U$  and  $v \in V$ . Let us define new methods of fuzzy conditional inference of the following type:

**Ant 1:** If  $x$  is  $P$  then  $y$  is  $Q$

**Ant2:**  $x$  is  $P'$

----- (7.8)

**Cons:**  $y$  is  $Q'$

Where  $P, P' \subseteq U$  and  $Q, Q' \subseteq V$ , which requires the satisfaction of Criteria I-IV from Appendix. It is clear that (6.8) is translated by Expression (7.5), and  $R(A_2(y)) = Q'$  into (7.8).

**Theorem 2.**

If fuzzy sets  $P \subseteq U$  and  $Q \subseteq V$  are defined as (7.1) and (7.2) respectively and  $R(A_1(x), A_2(y))$  is defined by the following

$$\begin{aligned} R(A_1(x), A_2(y)) &= P \times V \xrightarrow{L_5} U \times Q = \int_{U \times V} \mu_P(u) / (u, v) \xrightarrow{L_5} \int_{U \times V} \mu_Q(v) / (u, v) \\ &= \int_{U \times V} (\mu_P(u) \xrightarrow{L_5} \mu_Q(v)) / (u, v), \end{aligned}$$

where

$$\mu_P(u) \xrightarrow{L_5} \mu_Q(v) = \begin{cases} (1 - \mu_P(u)) \cdot \mu_Q(v), & \mu_P(u) > \mu_Q(v), \\ 1, & \mu_P(u) \leq \mu_Q(v). \end{cases} \tag{7.9}$$

then Criteria I, II, III and IV-1 [32] are satisfied.

**Proof:**

For Criteria I-III let  $R(A_1(x)) = P^\alpha$  ( $\alpha > 0$ ) then

$$\begin{aligned} R(A_2(y)) &= P^\alpha \circ R(A_1(x), A_2(y)) \\ &= \int_U \mu_P^\alpha(u) / u \circ \int_{U \times V} \mu_P(u) \xrightarrow{L_5} \mu_Q(v) / (u, v) \\ &= \int \bigcup_{u \in U} [\mu_P^\alpha(u) \wedge (\mu_P(u) \xrightarrow{L_5} \mu_Q(v))] / v. \end{aligned} \tag{7.10}$$

$$\exists U_1, U_2 \subset U \mid U_1 \cup U_2 = U; \mid U_1 \cap U_2 = \emptyset$$

$$\Rightarrow \forall u \in U_1 \mid \mu_P(u) > \mu_Q(v); \forall u \in U_2 \mid \mu_P(u) \leq \mu_Q(v). \tag{7.11}$$

From (7.10) and given subsets from (7.11) we have

$$\begin{aligned} R(A_2(y)) &= \left[ \int \bigcup_{u \in U_1} [\mu_P^\alpha(u) \wedge (1 - \mu_P(u)) \cdot \mu_Q(v)] / v \right] \vee \left[ \int \bigcup_{u \in U_2} [\mu_P^\alpha(u) \wedge 1] / v \right] \end{aligned} \tag{7.12}$$

Let us introduce the following function (as a part of implication operation)

$$f(u, v) = (1 - \mu_P(u)) \cdot \mu_Q(v) \tag{7.13}$$

Then the following is taking place:

$$\forall u \in U_1 | \mu_p^\alpha(u) \wedge f(u, v) = \begin{cases} \mu_p^\alpha(u), \mu_p^\alpha(u) \leq f(u, v), \\ f(u, v), \mu_p^\alpha(u) > f(u, v), \end{cases} \quad (7.14)$$

$$\forall u \in U_2 | \mu_p^\alpha(u) \wedge 1 = \mu_p^\alpha(u), \quad (7.15)$$

From (7.14) and (7.15) we have

$$(7.12) = \left[ \int \bigcup_{u \in U_2} \mu_p^\alpha(u) / v \right] = \int \mu_Q^\alpha(v) / v = Q^\alpha \quad (\text{Q.E.D.})$$

For Criteria IV-2 [19] let  $R(A_1(x)) = \neg P = 1 - P$  then

$$\begin{aligned} R(A_2(y)) &= \neg P \circ R(A_1(x), A_2(y)) \\ &= \int_U (1 - \mu_p(u)) / u \circ \int_{U \times V} \mu_p(u) \xrightarrow{L5} \mu_Q(v) / (u, v) \\ &= \int \bigcup_{u \in U} \left[ (1 - \mu_p(u)) \wedge \left( \mu_p(u) \xrightarrow{L5} \mu_Q(v) \right) \right] / v. \end{aligned} \quad (7.16)$$

From (7.16) and given subsets from (7.11) we have

$$\begin{aligned} R(A_2(y)) &= \left[ \int \bigcup_{u \in U_1} \left[ (1 - \mu_p(u)) \wedge (1 - \mu_p(u)) \cdot \mu_Q(v) \right] / v \right] \vee \left[ \int \bigcup_{u \in U_2} \left[ (1 - \mu_p(u)) \wedge 1 \right] / v \right] \\ &= \left[ \int \bigcup_{u \in U_1} \left[ (1 - \mu_p(u)) \wedge f(u, v) \right] / v \right] \vee \left[ \int \bigcup_{u \in U_2} \left[ (1 - \mu_p(u)) \wedge 1 \right] / v \right]. \end{aligned} \quad (7.17)$$

Apparently the following is taking place

$$\left[ (1 - \mu_p(u)) \wedge f(u, v) \right] \leq \left[ (1 - \mu_p(u)) \wedge 1 \right]$$

therefore

$$(5.17) = \left[ \int \bigcup_{u \in U_2} \left[ (1 - \mu_p(u)) \wedge 1 \right] / v \right] = \left[ \int_V (1 - \mu_Q(v)) / v \right] = \neg Q \quad (\text{Q.E.D.})$$

### Theorem 3.

If fuzzy sets  $P \subseteq U$  and  $Q \subseteq V$  are defined as (7.1) and (7.2) respectively and  $R(A_1(x), A_2(y))$  is defined by the following

$$\begin{aligned} R(A_1(x), A_2(y)) &= \left( P \times V \xrightarrow{L5} U \times Q \right) \cap \left( \neg P \times V \xrightarrow{L5} U \times \neg Q \right) \\ &= \int_{U \times V} \left( \mu_p(u) \xrightarrow{L5} \mu_Q(v) \right) \wedge \left( (1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v)) \right) / (u, v). \end{aligned} \quad (7.18)$$

where

$$\begin{aligned} &\left( \mu_p(u) \xrightarrow{L5} \mu_Q(v) \right) \wedge \left( (1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v)) \right) \\ &= \begin{cases} (1 - \mu_p(u)) \cdot \mu_Q(v), \mu_p(u) > \mu_Q(v), \\ 1, \mu_p(u) = \mu_Q(v), \\ (1 - \mu_Q(v)) \cdot \mu_p(u), \mu_p(u) < \mu_Q(v). \end{cases} \end{aligned}$$

Then Criteria I, II, III and IV-2 [32] are satisfied.

**Proof:**

$$\exists U_1, U_2, U_3 \subset U | U_1 \cup U_2 \cup U_3 = U; | U_1 \cap U_2 \cap U_3 = \emptyset$$

$$\begin{aligned} &\Rightarrow \forall u \in U_1 \mid \mu_p(u) > \mu_Q(v); \\ &\forall u \in U_2 \mid \mu_p(u) = \mu_Q(v), \forall u \in U_3 \mid \mu_p(u) < \mu_Q(v). \end{aligned} \tag{7.19}$$

Let us introduce the following functions

$$h_1(u, v) = (1 - \mu_p(u)) \cdot \mu_Q(v), h_2(u, v) = (1 - \mu_Q(u)) \cdot \mu_p(v). \tag{7.20}$$

Therefore from (7.18)-(7.20) for Criteria I-III let  $R(A_1(x)) = P^\alpha (\alpha > 0)$  then

$$\begin{aligned} R(A_2(y)) &= P^\alpha \circ R(A_1(x), A_2(y)) \\ &= \int_U \mu_p^\alpha(u) / u \circ \int_{U \times V} (\mu_p(u) \xrightarrow{L5} \mu_Q(v)) \wedge ((1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v))) / (u, v) \tag{7.21} \\ &= \int_V \bigcup_{u \in U} [\mu_p^\alpha(u) \wedge (\mu_p(u) \xrightarrow{L5} \mu_Q(v)) \wedge ((1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v)))] / v. \end{aligned}$$

From (7.20), (7.21) and given subsets from (7.19) we have

$$\begin{aligned} R(A_2(y)) &= \left[ \int_V \bigcup_{u \in U_1} [\mu_p^\alpha(u) \wedge h_1(u, v)] / v \right] \vee \left[ \int_V \bigcup_{u \in U_2} [\mu_p^\alpha(u) \wedge 1] / v \right] \\ &\quad \vee \left[ \int_V \bigcup_{u \in U_3} [\mu_p^\alpha(u) \wedge h_2(u, v)] / v \right]. \end{aligned} \tag{7.22}$$

Then again the following is taking place:

$$\forall u \in U_1 \mid \mu_p^\alpha(u) \wedge h_1(u, v) = \begin{cases} \mu_p^\alpha(u), \mu_p^\alpha(u) \leq h_1(u, v), \\ h_1(u, v), \mu_p^\alpha(u) > h_1(u, v), \end{cases} \tag{7.23}$$

$$\forall u \in U_2 \mid \mu_p^\alpha(u) \wedge 1 = \mu_p^\alpha(u), \tag{7.24}$$

$$\forall u \in U_3 \mid \mu_p^\alpha(u) \wedge h_2(u, v) = \begin{cases} \mu_p^\alpha(u), \mu_p^\alpha(u) \leq h_2(u, v), \\ h_2(u, v), \mu_p^\alpha(u) > h_2(u, v), \end{cases} \tag{7.25}$$

From (7.23) - (7.25) we have

$$(7.21) = \left[ \int_V \bigcup_{u \in U_2} \mu_p^\alpha(u) / v \right] = \int_V \mu_Q^\alpha(v) / v = Q^\alpha \tag{Q.E.D.}.$$

For Criteria IV-2 let  $R(A_1(x)) = \neg P = 1 - P$  then

$$\begin{aligned} R(A_2(y)) &= \neg P \circ R(A_1(x), A_2(y)) \\ &= \int_U (1 - \mu_p(u)) / u \circ \int_{U \times V} (\mu_p(u) \xrightarrow{L5} \mu_Q(v)) \wedge ((1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v))) / (u, v) \tag{7.26} \\ &= \int_V \bigcup_{u \in U} [(\mu_p(u) \xrightarrow{L5} \mu_Q(v)) \wedge ((1 - \mu_p(u)) \xrightarrow{L5} (1 - \mu_Q(v)))] / v. \end{aligned}$$

From (7.22), (7.26) and given subsets from (7.19) we have

$$\begin{aligned} R(A_2(y)) &= \left[ \int_V \bigcup_{u \in U_1} [(1 - \mu_p(u)) \wedge h_1(u, v)] / v \right] \vee \left[ \int_V \bigcup_{u \in U_2} [(1 - \mu_p(u)) \wedge 1] / v \right] \\ &\quad \vee \left[ \int_V \bigcup_{u \in U_3} [(1 - \mu_p(u)) \wedge h_2(u, v)] / v \right]. \end{aligned} \tag{7.27}$$

Apparently the following is taking place

$$\left[ (1 - \mu_p(u)) \wedge h_1(u, v) \right] \leq \left[ (1 - \mu_p(u)) \wedge 1 \right]$$

and

$$\left[ (1 - \mu_p(u)) \wedge h_2(u, v) \right] \leq \left[ (1 - \mu_p(u)) \wedge 1 \right]$$

therefore

$$(7.27) = \left[ \int_V \bigcup_{u \in U_2} \left[ (1 - \mu_p(u)) \wedge 1 \right] / v \right] = \left[ \int_V (1 - \mu_Q(u)) / v \right] = \neg Q \quad (\text{Q.E.D.})$$

For many real practical applications a decision making apparatus could be based not on a fuzzy conditional proposition of the type: “If  $x$  is  $P$  then  $y$  is  $Q$ ”, but rather on a rule of the following type

$$\text{“If } x_1 \text{ is } P_1 \text{ and } x_2 \text{ is } P_2 \text{ and } \dots \text{ and } x_n \text{ is } P_n \text{ then } y \text{ is } Q\text{”} \quad (7.28)$$

And correspondent fuzzy sets are represented as such that

$$\forall i = \overline{1, n} \Rightarrow P_i \subset U \mid \mu_{P_i} : U \rightarrow [0, 1], Q \subset V \mid \mu_Q : V \rightarrow [0, 1] \quad (7.29)$$

Given (7.1) and (7.2) a *binary relationship* for a fuzzy conditional proposition of the Type (7.28) for proposed fuzzy logic is defined as

$$D = R(A_1(x), A_2(y)) = \left[ \bigcap_i P_i \right] \times V \xrightarrow{L^5} U \times Q \quad (7.30)$$

**Theorem 4.** If a fuzzy conditional proposition is defined as (7.28), correspondent fuzzy sets of *antecedents* and *consequent* are presented as (7.29) and a *binary relationship* for a fuzzy conditional proposition is from (7.30), and “elementary” *binary relationships* are defined as following

$$\forall i = \overline{1, n} \mid D_i = R_i(A_1(x), A_2(y)) = P_i \times V \xrightarrow{L^5} U \times Q \quad (7.31)$$

then the following expression is taking place

$$D = D_1 \cup D_2 \cup \dots \cup D_n = \bigcup_i D_i \quad (7.32)$$

and as a result the following is also true.

$$\left[ \bigcap_i P_i \right] \times V \xrightarrow{L^5} U \times Q = \bigcup_i \left( P_i \times V \xrightarrow{L^5} U \times Q \right) \quad (7.33)$$

**Proof:**

Let  $\exists U_1, U_2 \subset U \mid U_1 \cup U_2 = U; U_1 \cap U_2 = \emptyset, \forall u \in U_1 \mid \bigwedge_i \mu_{P_i}(u) \leq \mu_Q(v);$   
 $\forall u \in U_2 \mid \bigwedge_i \mu_{P_i}(u) > \mu_Q(v)$  and let us denote each

$$D_i = \begin{cases} D_i', \forall u \in U_1 \\ D_i'', \forall u \in U_2 \end{cases} \Rightarrow D_i' = 1 \mid \mu_{P_i}(u) \leq \mu_Q(v)$$

whereas

$$D' = 1 \mid \bigwedge_i \mu_{P_i}(u) \leq \mu_Q(v) \quad (7.34)$$

From (7.34)

$$D' = \bigcup_i D_i' \quad (7.35)$$

Since

$$\begin{aligned}
 D_i'' &= (1 - \wedge_i \mu_p(u)) \cdot \mu_Q(v) \Big|_{\wedge_i \mu_{P_i}(u) > \mu_Q(v)} \\
 &= \vee_i \mu_{P_i}(u) \cdot \mu_Q(v) \Big|_{\wedge_i \mu_{P_i}(u) > \mu_Q(v)}
 \end{aligned}
 \tag{7.36}$$

From (7.35)

$$D'' = \bigcup_i D_i'' \tag{7.37}$$

Both (7.35) and (7.37) prove (7.33) (Q.E.D.)

Let us consider more complex fuzzy conditional proposition of the type:

$$\begin{aligned}
 R_i &= \text{“If } x_1 \text{ is } P_{ik_1} \text{ and } x_2 \text{ is } P_{ik_2} \text{ and } \dots \text{ and } x_n \text{ is } P_{ikn} \\
 &\text{ then } y \text{ is } Q_{il}, \text{ else } y \text{ is not } Q_{il}\text{”}
 \end{aligned}
 \tag{7.38}$$

For

$$i = \overline{1, N}, k = \overline{1, CardU}, l = \overline{1, CardV}$$

where  $P_{ikj} \subseteq U, j = \overline{1, n}, Q_{il} \subseteq V$  are normal fuzzy sets of a type (7.1) and (7.2)

$$P_{ikj} = \int_U \mu_{P_{ikj}}(u)/u, \quad Q_{il} = \int_V \mu_{Q_{il}}(v)/v \tag{7.39}$$

with unimodal membership functions  $\mu_{P_{ikj}}(u): U \rightarrow [0, 1], \mu_{Q_{il}}(v): V \rightarrow [0, 1]$  and  $U, V \subset \mathfrak{R}^+$  are countable, finite *universes of discourses*, i.e.

$CardU = CardV = M$ . Note that a unimodality of (7.39) means that for all singletons  $\mu_{P_{ikj}}(u)/u, \mu_{Q_{il}}(v)/v$  the following is taking place

$$\exists! u' \in U \mid \mu_{P_{ikj}}(u') = 1; \forall u \neq u' \in U \mid \mu_{P_{ikj}}(u) < 1,$$

and

$$\exists! v' \in V \mid \mu_{Q_{il}}(v') = 1; \forall v \neq v' \in V \mid \mu_{Q_{il}}(v) < 1$$

Note that each rule of type (7.38) looks like that

$$\begin{aligned}
 &R_i(A_1(x), A_2(y)) \\
 &= \left[ \left( \bigcap_j P_{ikj} \right) \times V \xrightarrow{L5} U \times Q_{il} \right] \cap \left[ \neg \left( \bigcap_j P_{ikj} \right) \times V \xrightarrow{L5} U \times \neg Q_{il} \right], \\
 &j = \overline{1, n}; k, l = \overline{1, M}; i = \overline{1, N}
 \end{aligned}
 \tag{7.40}$$

Let us point to the fact that (7.40) is a mathematical representation of the following fuzzy conditional rule:

**Ant 1:**

If  $x_1$  is  $P_{ik_1}$  and  $x_2$  is  $P_{ik_2}$  and  $\dots$  and  $x_n$  is  $P_{ikn}$  then  $y$  is  $Q_{il}$ , else  $y$  is not  $Q_{il}$

**Ant2:**  $x_1$  is  $P'_{ik_1}$  and  $x_2$  is  $P'_{ik_2}$  and  $\dots$  and  $x_n$  is  $P'_{ikn}$

----- (7.41)

**Cons:**  $y$  is  $Q'_{il}$ ,

In traditional way (7.41) looks like:

$$Q'_{il} = \left[ \bigcap_j P'_{ikj} \right] \circ R_i(A_1(x), A_2(y)), j = \overline{1, n}; k, l = \overline{1, M}; i = \overline{1, N}, \tag{7.42}$$



But for  $j = \overline{1, n}$  we have

$$\begin{aligned} & \int_V \mu_{Q_{il}}(v)/v \\ &= \int_U \bigcap_j \mu_{P_{ikj}}(u)/u \circ \int_{U \times V} \left( \bigcap_j \mu_{P_{ikj}}(u) \xrightarrow{L_5} \mu_{Q_{il}}(v) \right) \wedge \left( \left( 1 - \bigcap_j \mu_{P_{ikj}}(u) \right) \xrightarrow{L_5} (1 - \mu_{Q_{il}}(v)) \right) / (u, v) \quad (7.43) \\ &= \bigcup_j \int_V \bigcup_{u \in U} \left[ \mu_{P_{ikj}}(u) \wedge \left( \mu_{P_{ikj}}(u) \xrightarrow{L_5} \mu_{Q_{il}}(v) \right) \wedge \left( \left( 1 - \mu_{P_{ikj}}(u) \right) \xrightarrow{L_5} (1 - \mu_{Q_{il}}(v)) \right) \right] / v. \end{aligned}$$

Let us define

$$\mu_{R_i}(u, v) = \left[ \left( \mu_{P_{ik}}(u) \xrightarrow{L_5} \mu_{Q_{il}}(v) \right) \wedge \left( \left( 1 - \mu_{P_{ik}}(u) \right) \xrightarrow{L_5} (1 - \mu_{Q_{il}}(v)) \right) \right] / v$$

from (7.43) we get

$$\int_V \mu_{Q_{il}}(v)/v = \int_V \bigcup_{u \in U} \mu_{P_{ikj}}(u)/u \wedge \mu_{R_i}(u, v)/v \quad (7.44)$$

Suppose

$$U = U_1 \cup U_2 \mid U_1 \cap U_2 = \emptyset, \quad \text{and} \quad \left. \begin{aligned} & \forall u \in U_1 \mid \mu_{P_{ik}}(u) \neq \mu_{Q_{il}}(v) \\ & \forall u \in U_2 \mid \mu_{P_{ik}}(u) = \mu_{Q_{il}}(v) \end{aligned} \right\} \quad (7.45)$$

**Proposition 4.** For discussed fuzzy logic a fuzzy conditional rule of Type (7.41) satisfies the following feature

$$\forall u \in U_2 \mid \left[ \mu_{P_{ik}}(u) = \mu_{Q_{il}}(v) \right] \Rightarrow \mu_{R_i}^*(u, v) = \sup_{U \times V} \mu_{R_i}(u, v) = 1; \quad (7.46)$$

Let us call the binary relationship matrix  $R_i(A_1(x), A_2(y)), i = \overline{1, N}$ , from (7.42) by “elementary knowledge” (EK). Each EK is characterized by the following features

$$U^{\#} \subseteq U = \left\{ \forall r, p \in K = \{k\}; u_r, u_p \in U \mid \mu_{P_{ikj}}(u_k) = \mu_{P_{ikj}}(u_p) \right\}^*$$

Same is also true for  $V^{\#} \subseteq V$ . Since both membership functions  $\mu_{P_{ikj}}(u): U \rightarrow [0, 1], \mu_{Q_{il}}(v): V \rightarrow [0, 1]$  are unimodal, then

$$Card U^{\#} \Big|_{\max} = \frac{M-1}{2} \quad \text{and} \quad Card U^{\#} \Big|_{\min} = 0. \quad (\text{The same is true for } Card V^{\#}).$$

It is apparent that both  $U^{\#} = \emptyset, V^{\#} = \emptyset$ . In practical terms it means that if  $U^{\#} = \emptyset$ , then either

$$\mu_{P_{ikj}}(u_1) = \sup_{\forall u_k \in U} \mu_{P_{ikj}}(u_k); \mu_{P_{ikj}}(u_M) = \inf_{\forall u_k \in U} \mu_{P_{ikj}}(u_k)$$

or

$$\mu_{P_{ikj}}(u_1) = \inf_{\forall u_k \in U} \mu_{P_{ikj}}(u_k); \mu_{P_{ikj}}(u_M) = \sup_{\forall u_k \in U} \mu_{P_{ikj}}(u_k) \quad (7.47)$$

whereas if  $V^{\#} = \emptyset$ , then either

$$\mu_{Q_{il}}(v_1) = \sup_{\forall v_l \in V} \mu_{Q_{il}}(v_l); \mu_{Q_{il}}(v_M) = \inf_{\forall v_l \in V} \mu_{Q_{il}}(v_l)$$

or

$$\mu_{Q_{il}}(v_1) = \inf_{\forall v_l \in V} \mu_{Q_{il}}(v_l); \mu_{Q_{il}}(v_M) = \sup_{\forall v_l \in V} \mu_{Q_{il}}(v_l) \quad (7.48)$$

**Definition 7.** The *EK* of the type  $R_i(A_1(x), A_2(y)), i = \overline{1, N}$ , from (7.42) is called *logically contradictive* or *fruitless* if a membership function of a *consequent* in fuzzy conditional rule (7.41) is non-unimodal or when two different *antecedents* induct the same *consequent*.

Given (7.46)-(7.48), based on defined earlier  $U^II, V^II$  sets and taking into account that  $R_i(A_1(x), A_2(y)) = \|\mu_{R_i}(u_k, v_l)\|$ ,  $k, l = \overline{1, M}; i = \overline{1, N}$  suppose that

$$U^II \neq \emptyset, V^II \neq \emptyset$$

then

$$\begin{aligned} \exists u^* \in U; \exists v^*, v^{**} \in V; v^* \neq v^{**} & \left[ \mu_{P_{kj}}(u^*) = \mu_{Q_{il}}(v^*) = \mu_{Q_{il}}(v^{**}) \right] \\ \Rightarrow \mu_{R_i}(u^*, v^*) = \mu_{R_i}(u^*, v^{**}) & = \sup_{\forall u_k \in U} \mu_{R_i}(u, v) = 1 \\ \exists v^l \in V; \exists u^l, u^{II} \in U; u^l \neq u^{II} & \left[ \mu_{Q_{il}}(v^l) = \mu_{P_{kj}}(u^l) = \mu_{P_{kj}}(u^{II}) \right] \\ \Rightarrow \mu_{R_i}(u^l, v^l) = \mu_{R_i}(u^{II}, v^l) & = \sup_{\forall v_l \in V} \mu_{R_i}(u, v) = 1 \end{aligned} \tag{7.49}$$

\*From now on upper and lower indices of based variables  $u$  and  $v$  denote not a membership functions, but their correspondent singletons

It is clear, that for  $U^2$  from (7.45)

$$\begin{aligned} CardU^II |^{\max} = CardV^II |^{\max} & = \frac{M-1}{2} \\ \Rightarrow CardU_2 |^{\max} = 2M-1; Tr[R_i] & = Tr[R_i^l] = M \end{aligned} \tag{7.50}$$

where  $Tr[R_i], Tr[R_i^l]$ -are traces of matrixes  $R_i, R_i^l$  and  $r_{ij}^l = r_{i, N+1-j}; j = \overline{1, N}$ .

1. Let  $U^II = \emptyset, V^II \neq \emptyset$ , then

$$\begin{aligned} \exists u^* \in U; \exists v^*, v^{**} \in V; v^* \neq v^{**} & \left[ \mu_{P_{kj}}(u^*) = \mu_{Q_{il}}(v^*) = \mu_{Q_{il}}(v^{**}) \right] \\ \Rightarrow \mu_{R_i}(u^*, v^*) = \mu_{R_i}(u^*, v^{**}) & = \sup_{\forall u_k \in U} \mu_{R_i}(u, v) = 1 \end{aligned} \tag{7.51}$$

$$CardU^II |^{\max} = \frac{M-1}{2} \Rightarrow CardU_2 |^{\max} = M; Tr[R_i] = Tr[R_i^l] \tag{7.52}$$

2. Let  $U^II \neq \emptyset, V^II = \emptyset$ , then

$$\begin{aligned} \exists v^l \in V; \exists u^l, u^{II} \in U; u^l \neq u^{II} & \left[ \mu_{Q_{il}}(v^l) = \mu_{P_{kj}}(u^l) = \mu_{P_{kj}}(u^{II}) \right] \\ \Rightarrow \mu_{R_i}(u^l, v^l) = \mu_{R_i}(u^{II}, v^l) & = \sup_{\forall v_l \in V} \mu_{R_i}(u, v) = 1 \end{aligned} \tag{7.53}$$

$$CardU^II |^{\max} = \frac{M-1}{2} \Rightarrow CardU_2 |^{\max} = M; Tr[R_i] = Tr[R_i^l] \tag{7.54}$$

3. Let  $U^II = \emptyset, V^II = \emptyset$ , then either  $\forall l = \overline{1, M}; \exists! r = l \in K | \mu_{P_{kj}}(u_r) = \mu_{Q_{il}}(v_l)$

$$\Rightarrow \mu_{R_i}(u_r, v_l) = \sup_{(u,v) \in U \times V} \mu_{R_i}(u, v) = 1$$

and  $Tr[R_i(\cdot)] = M$ , which means that there is a major diagonal of a matrix  $R_i(\cdot)$  consists of singles (1), or

$$\begin{aligned} \forall p = \overline{1, M}; \exists ! l = M + 1 - p \in K \mid \mu_{P_{ikj}}(u_p) &= \mu_{Q_{il}}(v_{M+1-p}) \\ \Rightarrow \mu_{R_i}(u_p, v_l) &= \sup_{(u,v) \in U \times V} \mu_{R_i}(u, v) = 1 \end{aligned}$$

which means that there is a peripheral diagonal of a matrix  $R_i(\cdot)$  consists of singles. Taking into account (7.49)-(7.54) we put together the following

**Definition 8.**

If  $P_{ikj} \subseteq U, j = \overline{1, n}, Q_{il} \subseteq V, i = \overline{1, N}, k = 1, \overline{CardU}, l = 1, \overline{CardV}$  are normal fuzzy sets of a type (7.39) and the sets are defined as the follows  $U'' \subseteq U, V'' \subseteq V \mid U'' = \emptyset, V'' = \emptyset$ , then binary relationship matrix  $R_i(A_1(x), A_2(y)), i = \overline{1, N}$  of type (7.40) or its counterpart  $R'_i(\cdot)$ , have major diagonal, which consists of singles.

Based on this definition and (7.49)-(7.53) let us formulate the following:

**Theorem 5.**

An EK of the type  $R_i(A_1(x), A_2(y)), i = \overline{1, N}$ , from (7.42) is *logically contradictory* or *fruitless* if the following inequality is taking place

$$\left| Tr[R_i(\cdot)] - Tr[R'_i(\cdot)] \right| \leq \varepsilon, i = \overline{1, n}$$

where  $\varepsilon$  should be defined based on certain practical considerations, *i.e.*  $\varepsilon = f(CardU)$ .

**Proof:**

Let is

$$\left| Tr[R_i(\cdot)] - Tr[R'_i(\cdot)] \right| = 0, i = \overline{1, n} \quad (7.55)$$

then based on (7.42) we have

$$\begin{aligned} & \int_V \mu_{Q_{il}}(v) / v \\ &= \int_V \bigcup_{u \in U} \left[ \mu_{P_{ikj}}(u) \wedge \left( \mu_{P_{ikj}}(u) \xrightarrow{L5} \mu_{Q_{il}}(v) \right) \wedge \left( (1 - \mu_{P_{ikj}}(u)) \xrightarrow{L5} (1 - \mu_{Q_{il}}(v)) \right) \right] / v \quad (7.56) \\ &= \int_V \bigcup_{u \in U} \left[ \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v) \right] / v. \end{aligned}$$

Let us introduce the following sets  $U_1, U_2 \subseteq U \mid U_1 \cap U_2 = \emptyset, U_1 \cup U_2 = U$ , which are

$$\begin{aligned} & \forall u \in U_1 \mid \mu_{P_{ikj}}(u) \neq \mu_{Q_{il}}(v) \Rightarrow \mu_{R_i}(u, v) < 1, \\ & \forall u \in U_2 \mid \mu_{P_{ikj}}(u) = \mu_{Q_{il}}(v) \Rightarrow \mu_{R_i}(u, v) = 1 \end{aligned} \quad (7.57)$$

From (7.55) and (7.56) we are getting the following

$$(7.56) = \int_V \left\{ \bigcup_{u \in U_1} \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v) \right\} \vee \left\{ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v) \right\} / v. \quad (7.58)$$

Taking into account (7.56) we get the following:

$$\bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v) > \bigcup_{u \in U_1} \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v)$$

therefore we are getting

$$(7.56) = \left\{ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \wedge \mu_{R_i}(u, v) \right\} / v. \quad (7.59)$$

Note that the set  $U_2$  is defined by both  $U''$  and  $V''$ . We are considering the following case first:

1)  $U'' \neq \emptyset, V'' \neq \emptyset$ . Since  $P_{ikj}^I \subseteq U$  is normal fuzzy set, then

$$\exists u^* \in U \mid \mu'_{P_{ikj}}(u^*) = 1 \text{ and given (7.49) and (7.50) we get}$$

$$\forall u \in U_2; \exists v^*, v^{**} \in V; v^* \neq v^{**} \mid \mu_{R_i}(u, v^*) = \mu_{R_i}(u, v^{**}) = 1$$

therefore from (7.59) we get the following  $\exists v_1, v_2 \in V; v_1 \neq v_2 \mid \mu_{Q_{il}^I}(v_1) = \mu_{Q_{il}^I}(v_2)$  (here we consider not a membership functions (7.39), but their correspondent singletons) and therefore

$$(7.59) = \left\{ \left[ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \right] \wedge \left[ \bigcup_{u \in U_2} \mu_{R_i}(u, v) \right] \right\}$$

$$= \mu_{P_{ikj}}(u^*) \wedge \mu_{R_i}(u, v^*) = \mu_{P_{ikj}}(u^*) \wedge \mu_{R_i}(u, v^{**}) = 1.$$

In other words from (7.59) we see that the membership function of a *Consequent* from fuzzy conditional inference rule (7.41)  $Q_{il}^I = \int_V \mu_{Q_{il}^I}(v)/v$  is poly modal one.

2)  $U'' = \emptyset, V'' \neq \emptyset$ . Since  $P_{ikj}^I \subseteq U$  is normal fuzzy set, then

$$\exists u^1 \in U_2 \mid \mu'_{P_{ikj}}(u^1) = 1 \text{ and given (7.51) and (7.52) we get}$$

$$\exists u^* \in U_2; \exists v^*, v^{**} \in V; v^* \neq v^{**} \mid \mu_{R_i}(u^*, v^*) = \mu_{R_i}(u^*, v^{**}) = 1$$

therefore when  $u^1 = u^*$  from (7.59) we get the following

$$\exists v_1, v_2 \in V; v_1 \neq v_2 \mid \mu_{Q_{il}^I}(v_1) = \mu_{Q_{il}^I}(v_2) = \left\{ \left[ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \right] \wedge \left[ \bigcup_{u \in U_2} \mu_{R_i}(u, v) \right] \right\}$$

$$= \mu_{P_{ikj}}(u^*) \wedge \mu_{R_i}(u^*, v^*) = \mu_{P_{ikj}}(u^*) \wedge \mu_{R_i}(u^*, v^{**}) = 1.$$

Here we also see that the membership function of a *Consequent* from fuzzy conditional inference rule (7.41)  $Q_{il}^I = \int_V \mu_{Q_{il}^I}(v)/v$  is poly modal one.

3)  $U'' \neq \emptyset, V'' = \emptyset$ . Since  $P_{ikj}^I \subseteq U$  is normal fuzzy set, then  $\exists u^1 \mid \mu'_{P_{ikj}}(u^1) = 1$  and given (7.53) and (7.54) we get

$$\exists v^I \in V; \exists u^I, u^{II} \in U; u^I \neq u^{II} \mid \mu_{Q_{il}^I}(v^I) = \mu_{P_{ikj}}(u^I) = \mu_{P_{ikj}}(u^{II})$$

therefore from (7.59) we get the following

$$\exists v^I \in V \mid \mu_{Q_{il}^I}(v) = \left\{ \left[ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \right] \wedge \left[ \bigcup_{u \in U_2} \mu_{R_i}(u, v) \right] \right\} = \mu_{P_{ikj}}(u^I) \wedge \mu_{R_i}(u^I, v^I) = 1$$

In a meantime one can see that

$$\exists u^{II} \mid \mu_{P_{ikj}}(u^{II}) = 1 \Rightarrow \mu_{Q_{il}^I}(v^I) = \left\{ \left[ \bigcup_{u \in U_2} \mu_{P_{ikj}}(u) \right] \wedge \left[ \bigcup_{u \in U_2} \mu_{R_i}(u, v) \right] \right\}$$

$$= \mu_{P_{ikj}}(u^{II}) \wedge \mu_{R_i}(u^{II}, v^I) = 1$$

In other words we see that there are two fuzzy sets  $P_{ikj}^I, P_{ikj}^{II} \subseteq U \mid P_{ikj}^I \neq P_{ikj}^{II}$ , when used as an *Antecedents* in fuzzy conditional inference rule (7.41), induce the same *Consequent*, in other words, based on the **Definition 7** we have *logically contradictive EK* from (7.42). (Q.E.D.)

Based on the results of this *Theorem* we have to present the following

### Corollary.

In order to make  $EK$  from (7.42) *logically non-contradictive*, both membership functions of a fuzzy sets  $P_{ikj} \subseteq U, j = \overline{1, n}, Q_{il} \subseteq V$  from (7.39) be unimodal and  $U'', V''$ , defined in (7.47) and (7.48) correspondingly, would be empty, i.e.  $U'' = \emptyset, V'' = \emptyset$ .

## 8. Concluding Remarks

In this paper we proposed new t-norm fuzzy logic in which: 1) Truth values of an implication operator are based on truth values of both *antecedent* and *consequent*; 2) Implication operator could be considered as one of the fuzziest implication from the list [34] of known so far; 3) The suggested implication operator is a base for fuzzy conditional inference rules and satisfies the set of important human intuition Criteria; 4) Important features of these rules are investigated.

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## Appendix

### Criterion I

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is  $P$

-----  
Cons:  $y$  is  $Q$ .

### Criterion II-1

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is *very*  $P$

-----  
Cons:  $y$  is *very*  $Q$ .

### Criterion II-2

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is *very*  $P$

-----  
Cons:  $y$  is  $Q$ .

### Criterion III

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is *more or less*  $P$

-----  
Cons:  $y$  is *more or less*  $Q$ .

### Criterion IV-1

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is *not*  $P$

-----  
Cons:  $y$  is *unknown*

### Criterion IV-2

Ant 1: If  $x$  is  $P$  then  $y$  is  $Q$

Ant 2:  $x$  is *not*  $P$

-----  
Cons:  $y$  is *not*  $Q$ .



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