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Periodic Solution of n-Species Gilpin-Ayala Competition System with Impulsive Perturbations

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ABSTRACT

The principle aim of this paper is to explore the existence of periodic solution of n-Species Gilpin-Ayala competition system with impulsive perturbations. Sufficient and realistic conditions are obtained by using Mawhin's continuation theorem of the coincidence degree. Further, some numerical simulations show that our model can occur in many forms of complexities including periodic oscillation and chaotic strange attractor.

Keywords: Periodic Solution; Impulsive Perturbations; Mawhin's Continuation Theorem

1. Introduction

The dynamics of Ayala-Gilpin competitive system, which was first introduced by Ayala et al. [1], has been widely studied by many authors [2-6]. However, the corresponding problems with periodic coefficients and impulsive perturbations were studied far less often [7]. In this paper, we will study the following impulsive Gilpin-Ayala system:

$$\begin{cases} \dot{z}_i(t) = r_i(t) z_i(t) \cdot \\ \qquad \left[1 - \left(\dfrac{z_i(t)}{K_i(t)} \right)^{\theta_i} - \displaystyle\sum_{j=1, j\neq i}^{n} \alpha_{ij}(t) \dfrac{z_j(t)}{K_j(t)} \right], \qquad (1.1) \\ \Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k^-) = p_k^i z_i(t_k^-), \ t = t_k, \end{cases}$$

where $z_i(t)$ represents the density of the ith species at time t; $r_i(t)$ denotes the intrinsic growth rate of the ith species; $K_i(t)$ means the environment carrying capacity of species i in the absence of competition; $\alpha_{ij}(t)$ ($i \neq j$) measures the amount of competition between the species x_i and x_j; θ_i is a positive constant and provide a nonlinear measure of intra-specific interference; p_k^i are constants.

In system (1.1), we give two hypotheses as follows.

(H1) $r_i(t), K_i(t)$ and $\alpha_{ij}(t)$ $(i, j = 1, \cdots, n, i \neq j)$ are all nonnegative T- periodic functions defined on R.

(H2) $1 + p_k^i > 0$ and there exists a positive integer q such that $t_{k+q} = t_k + T$, $p_{k+q}^i = p_k^i$.

2. Existence of Positive Solutions

To prove our results, we need the notion of the Mawhin's continuation theorem formulated in [8].

Lemma 1 ([8]) Let X and Y be two Banach spaces. Consider an operator equation $Lx = \lambda Nx$ where L: Dom $L \bigcap X \to Y$ is a Fredholm operator of index zero and $\lambda \in [0,1]$ is a parameter, then there exist two projectors $P: X \to X$ and $Q: Y \to Y$ such that $\operatorname{Im} P =$ Ker L and $\operatorname{Im} L = \operatorname{Ker} Q$. Assume that $N: \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$, where Ω is open bounded in X. Furthermore, assume that

a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \bigcap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;
b) for each $x \in \partial\Omega \bigcap \operatorname{Ker} L$, $QNx \neq 0$;
c) $\deg\{JQN, \Omega \bigcap \operatorname{Ker} L, 0\} \neq 0$, where $J: \operatorname{Im} Q \to$ Ker L is an isomorphism and deg{*} represents the Brouwer degree.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap \operatorname{Dom} L$.

For the sake of convenience, we shall make some preparation. Let $I \subset R$. Denote by $PC(I, R^n)$ the space of functions $x(t): I \to R^n$ which are continuous at $t \in I$, $t \neq t_k$, and are left continuous for

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$t = t_k \in I$. Let $u^L = \min_{0\le t\le T}\{u(t)\}$, $u^M = \max_{0\le t\le T}\{u(t)\}$,

$\overline{u} = \frac{1}{T}\int_0^T u(t)dt$, $\overline{uv} = \frac{1}{T}\int_0^T u(t)v(t)dt$,

where $u(t)$, $v(t)$ are T periodic functions.

Theorem 1. Suppose (H1) and (H2) hold, furthermore, the following conditions are satisfied.

(H1) $\sum_{k=1}^{q}\ln(1+p_k^i) + \overline{r}_i T > \sum_{j=1, j\neq i}^{n} \frac{1}{K_j^L}\overline{r_i\alpha_{ij}}Te^{C_j}$,

where

$$C_j = \frac{\ln\left[1+\frac{1}{\overline{r}_j T}\sum_{k=1}^{q}\ln(1+p_k^j)\right]+\ln K_j^M}{\theta_j} + 2\overline{r}_j T + \sum_{k=1}^{q}\ln(1+p_k^j) + \sum_{k=1}^{q}|\ln(1+p_k^j)|.$$

Then system (1.1) has at least one positive T- periodic solution.

Proof. Let

$$z_i(t) = e^{x_i(t)} \quad (i = 1,\cdots,n) \tag{2.1}$$

then the system (1.1) becomes

$$\begin{cases} \dot{x}_i(t) = r_i(t)\left[1-\frac{e^{\theta_i x_i(t)}}{K_i^{\theta_i}(t)} - \sum_{j=1,j\neq i}^{n}\alpha_{ij}(t)\frac{e^{x_j(t)}}{K_j(t)}\right] t\neq t_k \\ \Delta x_i(t_k) = \ln(1+p_k^i) \quad t = t_k \end{cases} \tag{2.2}$$

In order to use Lemma 1, we set

$$x = (x_1(t),\cdots,x_n(t))^T,$$

$$X = \{x\in PC(R,R^n)\,|\,x(t+T) = x(t)\},$$

$$Y = X\times R^{nq},$$

then it is standard to show that both X and Y are Banach space when they are endowed with the norm $\|x\|_c = \sup_{t\in[0,T]}|x(t)|$ and

$\|(x,c_1,\ldots,c_q)\| = (\|x\|_c^2 + |c_1|^2+\cdots+|c_q|^2)^{1/2}$.

Set $L{:}\mathrm{Dom}\,L\subset X\to Y$ as

$(Lx)(t) = (\dot{x}(t),\Delta x(t_1),\cdots,\Delta x(t_q))$,

where $\mathrm{Dom}L = \{x\in X\,|\,x'(t)\in PC(R,R^n)\}$,

$$\mathrm{Im}\,L = \left\{(y,c_1,\cdots,c_q)\in Y\,|\int_0^T y(t)dt + \sum_{i=1}^{q}c_i = 0\right\}$$

and Ker $\mathrm{Ker}\,L = R^n$.

At the same time, we denote $N: X\to Y$ as

$$(Nx)(t) = (f(t,x(t)),\Phi_1(x(t_1)),\cdots,\Phi_q(x(t_q))),$$

where $f(t,x) = \left[r_i(t)\left(1-\frac{e^{\theta_i x_i(t)}}{K_i^{\theta_i}(t)} - \sum_{j=1,j\neq i}^{n}\alpha_{ij}(t)\frac{e^{x_j(t)}}{K_j(t)}\right)\right]_{n\times 1}$,

$$\Phi_k(x(t_k)) = \left(\ln(1+p_k^1),\cdots,\ln(1+p_k^n)\right)^T,$$

where $i = 1,\cdots,n$, $k = 1,2,\cdots,q$.

Define two projectors P and Q as

$$P: X\to \ker L, \quad Px = \frac{1}{T}\int_0^T x(t)dt; \quad Q: Y\to Y,$$

$$Q(y,c_1,\ldots,c_q) = \frac{1}{T}\left[\int_0^T y(s)ds + \sum_{k=1}^{q}c_k, 0,\cdots,0\right].$$

It can be easily proved that L is a Fredholm operator of index zero, that P, Q are projectors, and that N is L- compact on $\overline{\Omega}$ for any given open and bound subset Ω in X.

Now we are in a position to search for an appropriate open bounded subset Ω for the application of Lemma 1 corresponding to operator equation

$$Lx = \lambda Nx,\ \lambda\in(0,1) \tag{2.3}$$

Suppose that $x(t) = (x_1(t),\cdots,x_n(t))^T$ is a periodic solution of (2.3) for certain $\lambda\in(0,1)$. By integrating (2.3) over the interval $[0,T]$, we get

$$\overline{r}_i T = -\sum_{k=1}^{q}\ln(1+p_k^i) + \int_0^T \frac{r_i(t)}{K_i^{\theta_i}(t)}e^{\theta_i x_i(t)}dt + \sum_{j=1,j\neq i}^{n}\int_0^T r_i(t)\alpha_{ij}(t)\frac{e^{x_j(t)}}{K_j(t)}dt \tag{2.4}$$

From (2.3), (2.4), we can obtain

$$\int_0^T|\dot{x}_i(t)|dt \le 2\overline{r}_i T + \sum_{k=1}^{q}\ln(1+p_k^i) \equiv A_i \tag{2.5}$$

Since $x_i(t)\in PC([0,T],R)$, there exist $\xi_i,\eta_i\in[0,T]\bigcup\{t_1^+,t_2^+,\cdots,t_q^+\}$, such that

$$x_i(\xi_i) = \inf_{t\in[0,T]}x_i(t),\quad x_i(\eta_i) = \sup_{t\in[0,T]}x_i(t),$$

It follows from (2.4) that

$$\frac{1}{K_i^M}\overline{r}_i Te^{\theta_i x_i(\xi_i)} \le \int_0^T\frac{r_i(t)}{K_i^{\theta_i}(t)}e^{\theta_i x_i(t)}dt \le \overline{r}_i T + \sum_{k=1}^{q}\ln(1+p_k^i)$$

which implies

$$x_i(\xi_i) \le \frac{\ln\left[1+\frac{1}{\overline{r_i}T}\sum_{k=1}^{q}\ln(1+p_k^i)\right]+\ln K_i^M}{\theta_i} \equiv B_i$$

Thus we get

$$\begin{aligned} x_i(t) &\le x_i(\xi_i)+\int_0^T |\dot{x}_i(t)|dt+\sum_{k=1}^{q}|\ln(1+p_k^i)| \\ &\le B_i + A_i + \sum_{k=1}^{q}|\ln(1+p_k^i)| \equiv C_i \end{aligned} \quad (2.6)$$

In particular, we have $x_i(\eta_i) \le C_i$.

On the other hand, from (2.4), we have

$$\overline{r_i}T \le -\sum_{k=1}^{q}\ln(1+p_k^i)+\frac{\overline{r_i}T}{K_i^L}e^{\theta_i x_i(\eta_i)} + \sum_{j=1, j\ne i}^{n}\frac{1}{K_j^L}\overline{r_i\alpha_{ij}}Te^{x_j(\eta_j)}.$$

Then we get

$$\frac{\overline{r_i}T}{K_i^L}e^{\theta_i x_i(\eta_i)} \ge \sum_{k=1}^{q}\ln(1+p_k^i)+\overline{r_i}T-\sum_{j=1, j\ne i}^{n}\frac{1}{K_j^L}\overline{r_i\alpha_{ij}}Te^{C_j}$$

Because of (H3) we have

$$x_i(\eta_i) \ge \frac{\ln(K_i^L)-\ln(\overline{r_i}T)}{\theta_i} + \frac{\ln\left(\sum_{k=1}^{q}\ln(1+p_k^i)+\overline{r_i}T-\sum_{j=1, j\ne i}^{n}\frac{1}{K_j^L}\overline{r_i\alpha_{ij}}Te^{C_j}\right)}{\theta_i} \equiv D_i$$

Thus we get

$$\begin{aligned} x_i(t) &\ge x_i(\eta_i)-\int_0^T |\dot{x}_i(t)|dt+\sum_{k=1}^{q}|\ln(1+p_k^i)| \\ &\ge D_i - A_i - \sum_{k=1}^{q}|\ln(1+p_k^i)| \equiv E_i \end{aligned} \quad (2.7)$$

From (2.6) and (2.7), it follows that

$$|x_i(t)| \le F_i = \max\{|C_i|,|E_i|\}$$

Obviously, F_i ($i=1,\cdots,n$) are independent of λ. Thus, there exists a constant $F>0$, such that $\max\{|x_1|,\cdots,|x_n|\} \le F$. Let $r > F_1+\cdots+F_n+F$, $\Omega=\{x\in X : \|x\|_c < r\}$, then it is clear that Ω satisfies condition (a) of Lemma 1 and N is L-compact on $\overline{\Omega}$.

when $x=(x_1,\cdots,x_n)^T \in \partial\Omega\cap \mathrm{Ker}L=\partial\Omega\cap R^n$, x is a constant vector in R^n with $\|x\|=r$. Thus $QNx \ne 0$.

Let $J:\mathrm{Im}\,Q \to \mathrm{Ker}\,L$, $(d,0,\ldots,0)\to d$. A direct computation gives
$\deg\{JQN,\Omega\cap \ker L,0\} \ne 0$.

By now we have proved that Ω satisfies all the requirements in Mawhin's continuation theorem. Hence, (2.1) has at least one T - periodic solution. By of (2.1), we derive that (1.1) has at least one positive T - periodic solution. The proof is complete.

3. An Illustrative Example

To easy to call functions, let $x_i(t)=z_i(t)$. In (1.1), we take $n=2$, $t_k=kT$,

$$r_1(t)=5+0.6\sin t,\quad r_2(t)=4-0.4\cos t,$$
$$K_1(t)=2+0.3\sin t,\quad K_2(t)=2+0.1\sin t,$$
$$\theta_1=1.5,\quad \theta_2=1.6,$$
$$\alpha_{12}(t)=0.8+0.1\cos t,\quad \alpha_{21}(t)=0.9+0.2\sin t.$$

Obviously, $r_1(t)$, $r_2(t)$, $K_1(t)$, $K_2(t)$, α_{12}, α_{21}, satisfy (H1). $p_k^1=0.3$, $p_k^2=0.2$.

If $T=\pi/2$, then system (1.1) under the conditions (H5) has a unique 2π -periodic solution (In **Figures 1-3**, we take $[x_1(0),x_2(0)]^T=[0.5,0.5]^T$). Because of the influence of the period pulses, the influence of pulse is obvious.

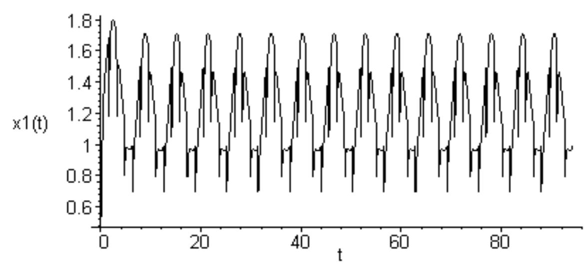

Figure 1. Time-series of $x_1(t)$ evolved in system (1) with $T=\pi/2$.

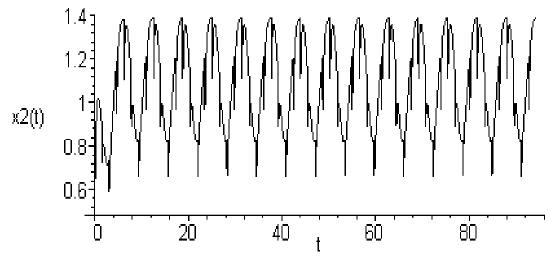

Figure 2. Time-series of $x_2(t)$ evolved in system (1) with $T=\pi/2$.

But if $T = 2$, then (H2) is not satisfied. Periodic oscillation of system (1.1) under the conditions (H5) will be destroyed by impulsive effect. Numeric results show that system (1.1) under the conditions (H5) has gui chaotic strange attractor (see **Figure 4**) [9]. In **Figure 4**, we take $[x_1(0), x_2(0)]^T = [0.5, 0.5]^T$.

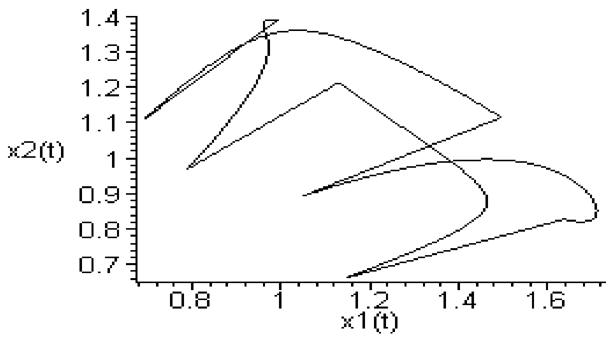

Figure 3. Phase portrait of periodic solutions of system (1) with $T = \pi/2$.

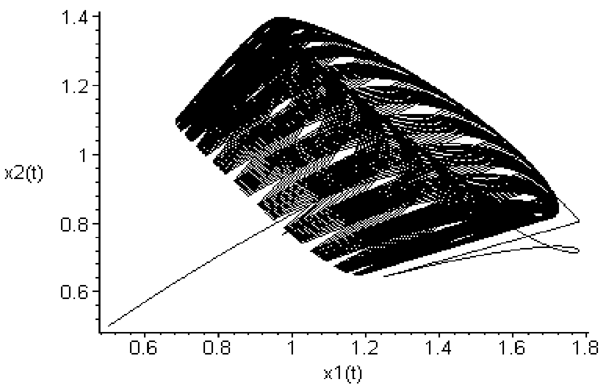

Figure 4. Phase portrait of chaotic strange attractor of system (1) with $T = 2$.

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