Bell's Inequality Should Be Reconsidered in Quantum Language

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Abstract

Bell's inequality itself is usually considered to belong to mathematics and not quantum mechanics. We think that this is making our understanding of Bell' theory be confused. Thus in this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation. And we clarify that the violation of Bell's inequality (*i.e.*, whether or not Bell's inequality holds) does not depend on whether classical systems or quantum systems, but depend on whether a combined measurement exists or not. And further we conclude that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

Keywords

Bohr-Einstein Debates, Bell's Inequality, Combined Observable, Linguistic Copenhagen Interpretation, Quantum Language

1. Review: Quantum Language (=Measurement Theory (=MT))

1.1. Introduction

Recently (*cf.* refs. [1]-[10], also see (B_0) - (B_3) later), we proposed quantum language, which was not only characterized as the metaphysical and linguistic turn of quantum mechanics but also the linguistic turn of dualistic idealism. And further we believe that quantum language should be regarded as the foundations of quantum information science. Quantum language is formulated as follows.

(A) $\underbrace{\boxed{\underbrace{\boxed{\text{Quantumlanguage}}}_{\text{(language)}}}}^{\text{(=measurement theory)}} = \underbrace{\boxed{\text{Measurement}}}_{\text{(Axiom 1)}} + \underbrace{\boxed{\text{Causality}}}_{\text{(Axiom 2)}} + \underbrace{\boxed{\text{Linguistic}\left(\text{Copenhagen}\right)\text{interpretation}}}_{\text{(how to use Axioms 1 and 2)}}$

Note that this theory (A) is not physics but a kind of language based on the

quantum mechanical world view. That is, we think that the location of quantum language in the history of world-descriptions is as follows.

And in **Figure 1**, we think that the following four are equivalent (refs. [1] [8]):

(B_0) to propose quantum language (*cf.* ⑩ in **Figure 1**, ref. [1] [8]).

(B_1) to clarify the Copenhagen interpretation of quantum mechanics (*cf.* ⑦ in **Figure 1**, refs. [2] [7] [11]), that is, the linguistic Copenhagen interpretation is the true figure of so-called Copenhagen interpretation.

(B_2) to clarify the final goal of the dualistic idealism (*cf.* ⑧ in **Figure 1**, refs. [3] [9]).

(B_3) to reconstruct statistics in the dualistic idealism (*cf.* ⑨ in **Figure 1**, refs. [4] [5] [6] [12]).

In Bohr-Einstein debates (refs. [13] [14]), Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (*i.e.*, physics holds without observers)) is on the side of the realistic world view in **Figure 1**. On the other hand, we think that Bohr's standing point (that is, "*to be is to be perceived*" (*i.e.*, there is no science without measurements)) is on the side of the linguistic world view in **Figure 1** (though N. Bohr might believe that the Copenhagen interpretation (proposed by his school) belongs to physics).

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality (refs. [15] [16] [17] [18]) in quantum language (*i.e.*, quantum theory with the linguistic Copenhagen interpretation). And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether a combined measurement exists or not (in Section 2). And further we assert that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).
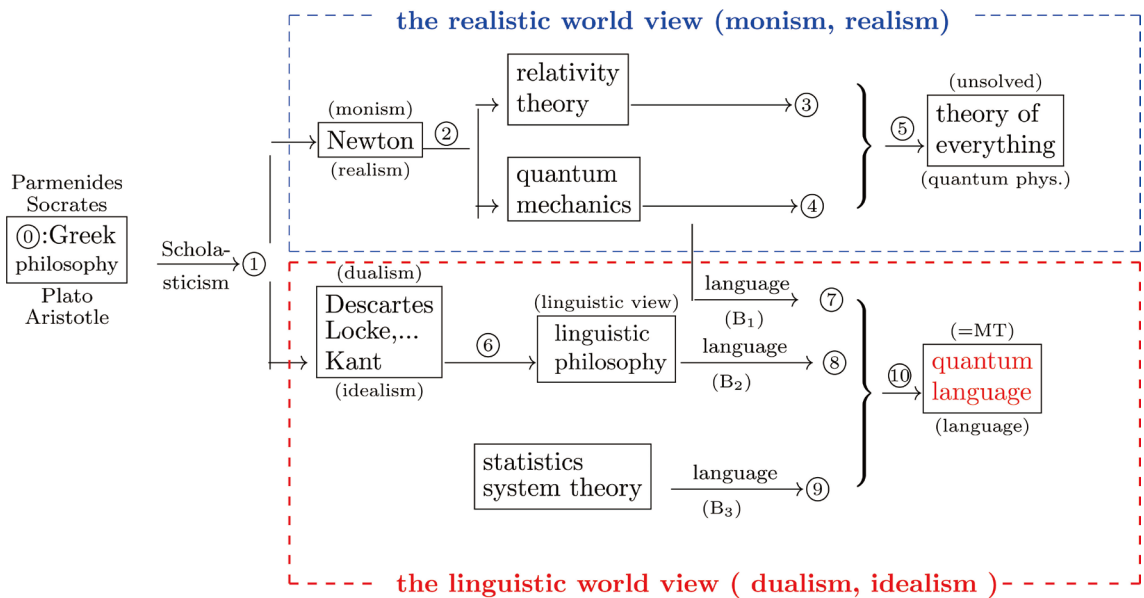

Figure 1. The history of the world-descriptions.

1.2. Quantum Language (=Measurement Theory); Mathematical Preparations

Now we shall explain the measurement theory (A).

Consider an operator algebra $B(H)$ (*i.e.*, an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H = 1} \|Fu\|_H$), and consider the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$, called a *basic structure*. Here, $\mathcal{A} (\subseteq B(H))$ is a C^*-algebra, and $\mathcal{N}$ ($\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)$) is a particular C^*-algebra (called a W^*-algebra) such that $\mathcal{N}$ is the weak closure of $\mathcal{A}$ in $B(H)$.

The measurement theory (=quantum language) is classified as follows.

$$\text{measurement theory} (A)$$

$$\text{(C)} = \begin{cases} (\text{C}_1): \text{quantum system theory} & \left(\text{when } \mathcal{A} = \mathcal{C}(H)\right) \\ (\text{C}_2): \text{classical system theory} & \left(\text{when } \mathcal{A} = C_0(\Omega)\right) \end{cases}$$

That is, when $\mathcal{A} = \mathcal{C}(H)$, the C^*-algebra composed of all compact operators on a Hilbert space H, the (C_1) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the C^*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* [19] [20])), the (C_2) is called classical measurement theory (or, classical system theory).

Also, note (*cf.* [19]) that, when $\mathcal{A} = \mathcal{C}(H)$,

1) $\mathcal{A}^* = Tr(H) (= \text{trace class})$, $\mathcal{N} = B(H)$, $\mathcal{N}_* = Tr(H)$ (*i.e.*, pre-dual space).

Also, when $\mathcal{A} = C_0(\Omega)$,

2) $\mathcal{A}^* =$ "the space of all signed measures on Ω",
$\mathcal{N} = L^\infty(\Omega, \nu) \left(\subseteq B(L^2(\Omega, \nu))\right)$, $\mathcal{N}_* = L^1(\Omega, \nu)$, where ν is some measure on Ω (*cf.* [19]). Also, the $L^\infty(\Omega, \nu)$ is usually denoted by $L^\infty(\Omega)$.

Let $\mathcal{A} (\subseteq B(H))$ be a C^*-algebra, and let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho \mid \rho$ is a continuous linear functional on $\mathcal{A}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\left\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}} \left(= \|F\|_{B(H)}\right) \leq 1\right\}$. Define the *mixed state* $\rho (\in \mathcal{A}^*)$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \geq 0$ for all $F \in \mathcal{A}$ such that $F \geq 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*) = \left\{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\right\}.$$

A mixed state $\rho \left(\in \mathfrak{S}^m(\mathcal{A}^*)\right)$ is called a *pure state* if it satisfies that $\rho = \theta \rho_1 + (1 - \theta) \rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$ implies $\rho = \rho_1 = \rho_2$. Put

$$\mathfrak{S}^p(\mathcal{A}^*) = \left\{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\right\},$$

which is called a *state space*. It is well known (*cf.* [19]) that $\mathfrak{S}^p(\mathcal{C}(H)^*) = \{|u\rangle\langle u|$

(*i.e.*, the Dirac notation) $| \ \|u\|_H = 1\}$, and $\mathfrak{S}^p \left(C_0 \left(\Omega \right)^* \right) = \{ \delta_{\omega_0} \ | \ \delta_{\omega_0}$ is a point measure at $\omega_0 \in \Omega \}$, where $\int_\Omega f \left(\omega \right) \delta_{\omega_0} \left(\mathrm{d}\omega \right) = f \left(\omega_0 \right)$ $\left(\forall f \in C_0 \left(\Omega \right) \right)$. The latter implies that $\mathfrak{S}^p \left(C_0 \left(\Omega \right)^* \right)$ can be also identified with Ω (called a *spectrum space* or simply *spectrum*) such as

$$\underset{\text{(states pace)}}{\mathfrak{S}^p \left(C_0 \left(\Omega \right)^* \right)} \ni \delta_\omega \leftrightarrow \omega \in \underset{\text{(spectrum)}}{\Omega} \tag{1}$$

For instance, in the above 2) we must clarify the meaning of the "value" of $F \left(\omega_0 \right)$ for $F \in L^\infty \left(\Omega, \nu \right)$ and $\omega_0 \in \Omega$. An element $F \left(\in \mathcal{N} \right)$ is said to be *essentially continuous at* $\rho_0 \left(\in \mathfrak{S}^p \left(\mathcal{A}^* \right) \right)$, if there uniquely exists a complex number α such that

- if $\rho \left(\in \mathcal{N}_*, \|\rho\|_{\mathcal{N}_*} = 1 \right)$ converges to $\rho_0 \left(\in \mathfrak{S}^p \left(\mathcal{A}^* \right) \right)$ in the sense of weak* topology of $\mathcal{A}^*$, that is,

$$\rho \left(G \right) \to \rho_0 \left(G \right) \ \left(\forall G \in \mathcal{A} \left(\subseteq \mathcal{N} \right) \right),$$

then $\rho \left(F \right)$ converges to α.

And the value of $\rho_0 \left(F \right)$ is defined by the α.

According to the noted idea (cf. [21]), an *observable* $\mathrm{O} := \left(X, \mathcal{F}, F \right)$ in $\mathcal{N}$ is defined as follows:

1) [σ-field] X is a set, $\mathcal{F}$ $\left(\subseteq 2^X = \mathcal{P} \left(X \right) \right)$, the power set of X) is a σ-field of X, that is, "$\Xi_1, \Xi_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^\infty \Xi_n \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$", "$X \in \mathcal{F}$".

2) [Countable additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{N}$ satisfying: a) for every $\Xi \in \mathcal{F}$, $F \left(\Xi \right)$ is a non-negative element in $\mathcal{N}$ such that $0 \leq F \left(\Xi \right) \leq I$, b) $F \left(\varnothing \right) = 0$ and $F \left(X \right) = I$, where 0 and I is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{ \Xi_1, \Xi_2, \cdots, \Xi_n, \cdots \}$ of Ξ (*i.e.*, $\Xi, \Xi_n \in \mathcal{F}$ $\left(n = 1, 2, 3, \cdots \right)$, $\bigcup_{n=1}^\infty \Xi_n = \Xi$, $\Xi_i \bigcap \Xi_j = \varnothing$ ($i \neq j$)), it holds that $F \left(\Xi \right) = \sum_{n=1}^\infty F \left(\Xi_n \right)$ in the sense of weak* topology in $\mathcal{N}$.

Remark 1. Quantum language has two formulations (*i.e.*, the C^*-algebraic formulation and the W^*-algebraic formulation). In this paper, we devote ourselves to the W^*-algebraic formulation, which may, from the mathematical point of view, be superiority to the C^*-algebraic formulation. That is, in the above 2), the countable additivity (*i.e.*, $F \left(\Xi \right) = \lim_{N \to \infty} \sum_{n=1}^N F \left(\Xi_n \right)$) is naturally discussed in the W^*-algebraic formulation. However, the C^*-algebraic formulation has a merit such that we can use it without sufficient mathematical preparation. For the C^*-algebraic version of this paper, see my preprint [10].

1.3. Axiom 1 [Measurement] and Axiom 2 [Causality]

With any *system S*, a basic structure $\left[\mathcal{A}, \mathcal{N} \right]_{B(H)}$ can be associated in which the measurement theory (A) of that system can be formulated. A *state* of the system *S* is represented by an element $\rho \left(\in \mathfrak{S}^p \left(\mathcal{A}^* \right) \right)$ and an *observable* is represented by an observable $\mathrm{O} := \left(X, \mathcal{F}, F \right)$ in $\mathcal{N}$. Also, the *measurement of the observable* O *for the system S with the state* ρ is denoted by $\mathrm{M}_\mathcal{N} \left(\mathrm{O}, S_{[\rho]} \right)$ (or more precisely, $\mathrm{M}_\mathcal{N} \left(\mathrm{O} := \left(X, \mathcal{F}, F \right), S_{[\rho]} \right)$). An observer can obtain a measured value

$x\,(\in X\,)$ by the measurement $\mathrm{M}_{\mathcal{N}}\big(\mathrm{O},S_{[\rho]}\big)$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics (*cf.* ref. [22]). And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^*-algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value x $(\in X)$ obtained by the measurement $\mathrm{M}_{\mathcal{N}}\big(\mathrm{O}:=(X,\mathcal{F},F),S_{[\rho]}\big)$ belongs to a set $\Xi\,(\in\mathcal{F})$ is given by $\rho\big(F(\Xi)\big)$ if $F(\Xi)$ is essentially continuous at $\rho\big(\in\mathfrak{S}^p\big(\mathcal{A}^*\big)\big)$.*

Next, we explain Axiom 2. Let $[\mathcal{A}_1,\mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2,\mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2}:\mathcal{N}_2$ (with weak* topology) $\to\mathcal{N}_1$ (with weak* topology) is called a *Markov operator,* if it satisfies that 1) $\Phi_{1,2}(F_2)\geq 0$ for any non-negative element F_2 in $\mathcal{N}_2$, 2) $\Phi_{1,2}(I_2)=I_1$, where I_k is the identity in $\mathcal{N}_k$, $(k=1,2)$. In addition to the above 1) and 2), in this paper we assume that $\Phi_{1,2}(\mathcal{A}_2)\subseteq\mathcal{A}_1$ and $\sup\big\{\big\|\Phi_{1,2}(F_2)\big\|_{\mathcal{A}_1}\mid F_2\in\mathcal{A}_2 \text{ such that } \|F_2\|_{\mathcal{A}_2}\leq 1\big\}=1$.

It is clear that the dual operator $\Phi_{1,2}^*:\mathcal{A}_1^*\to\mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*\big(\mathfrak{S}^m\big(\mathcal{A}_1^*\big)\big)\subseteq\mathfrak{S}^m\big(\mathcal{A}_2^*\big)$. If it holds that $\Phi_{1,2}^*\big(\mathfrak{S}^p\big(\mathcal{A}_1^*\big)\big)\subseteq\mathfrak{S}^p\big(\mathcal{A}_2^*\big)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is said to be non-deterministic or decoherence. Here note that, for any observable $\mathrm{O}_2:=(X,\mathcal{F},F_2)$ in $\mathcal{N}_2$, the $(X,\mathcal{F},\Phi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$.

Now Axiom 2 in the measurement theory (A) is presented as follows:

Axiom 2 [Causality]. *Let $t_1\leq t_2$. The causality is represented by a Markov operator $\Phi_{t_1,t_2}:N_{t_2}\to N_{t_1}$.*

1.4. The Linguistic Interpretation (=The Manual to Use Axioms 1 and 2)

In the above, Axioms 1 and 2 are kinds of spells, (*i.e.*, incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation. However, it is better to know the linguistic interpretation (=the manual to use Axioms 1 and 2), if we would like to make progress quantum language early.

The essence of the manual is as follows:

(D) **Only one measurement is permitted.** And thus, the state after a measurement is meaningless since it cannot be measured any longer. Thus, the collapse of the wavefunction is prohibited (*cf.* [7]). We are not concerned with anything after measurement. That is, any statement including the phrase *after the measurement* is wrong. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited. Also, it is added that there is no probability without a measurement.

and so on. For details, see [8].

1.5. Generalized Simultaneous Measurement, Parallel Measurement

Definition 2. [Generalized simultaneous observable, Generalized simultaneous measurement] Let $[\mathcal{A}, \mathcal{N}]_{B(H)}$ be a basic structure. Consider observables $O_k = (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, \cdots, K)$ in $\mathcal{N}$. Let $\left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k \right)$ be the product measurable space, *i.e.*, the product space $\times_{k=1}^{K} X_k$ and the product σ-field $\boxtimes_{k=1}^{K} \mathcal{F}_k$, which is defined by the smallest σ-field that contains a family $\left\{ \times_{k=1}^{K} \Xi_k \mid \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K \right\}$. An observable $O = \overset{qp}{\underset{k=1,2,\cdots,K}{\times}} O_k = \left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k, F \right)$ in $\mathcal{N}$ is called the generalized simultaneous observable (or, quasi-product observable, combined observable, etc.) of O_k $(k = 1, 2, \cdots, K)$, if it holds that

$$
\begin{aligned}
&F \left(X_1 \times X_2 \times \cdots \times X_{k-1} \times \Xi_k \times X_{k+1} \times \cdots \times X_K \right) \\
&= F_k (\Xi_k), \left(\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K \right)
\end{aligned}
\tag{2}
$$

Also, the measurement $\mathsf{M}_{\mathcal{N}} \left(O, S_{[\rho_0]} \right)$ is called a generalized simultaneous measurement of measurements $\mathsf{M}_{\mathcal{N}} \left(O_k, S_{[\rho_0]} \right)$ $(k = 1, 2, \cdots, K)$. A generalized simultaneous observable is called a simultaneous observable, if it holds:

$$
\begin{aligned}
&F \left(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_{k-1} \times \Xi_k \times \Xi_{k+1} \times \cdots \times \Xi_K \right) \\
&= \overset{K}{\underset{k=1}{\times}} F_k (\Xi_k), \left(\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K \right)
\end{aligned}
$$

Note that the existence and the uniqueness of a generalized simultaneous observable $O = \left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k, F \right)$ in $\mathcal{N}$ are not assured in general, however the simultaneous observable always exists if observables O_k $(k = 1, 2, \cdots, K)$ commute, *i.e.*,

$$
F_k (\Xi_k) F_l (\Xi_l) = F_l (\Xi_l) F_k (\Xi_k), \left(\forall \Xi_k \in \mathcal{F}_k, \forall \Xi_l \in \mathcal{F}_l, k \neq l \right)
\tag{3}
$$

Definition 3. [Parallel observable, Parallel measurement] For each $k = 1, 2, \cdots, K$, consider a basic structure $[\mathcal{A}_k, \mathcal{N}_k]_{B(H_k)}$ and a measurement $\mathsf{M}_{\mathcal{N}_k} \left(O_k := (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]} \right)$. We consider the spatial tensor W^*-algebra $\otimes_{k=1}^{K} \mathcal{N}_k \left(\subseteq B \left(\otimes_{k=1}^{K} H_k \right) \right)$, and consider the product measurable space $\left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k \right)$. Consider the observable $\otimes_{k=1}^{K} O_k = \left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k, \tilde{F} \right)$ in $\otimes_{k=1}^{K} \mathcal{N}_k$ such that

$$
\tilde{F} \left(\times_{k=1}^{K} \Xi_k \right) = \otimes_{k=1}^{K} F_k (\Xi_k) \ \left(\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K \right).
$$

which is called the parallel observable of $O_k := (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, \cdots, K)$. And let $\otimes_{k=1}^{K} \rho_k \in \mathfrak{S}^p \left(\left(\otimes_{k=1}^{K} \mathcal{A}_k \right)^* \right)$. Then the measurement $\mathsf{M}_{\otimes_{k=1}^{K} \mathcal{N}_k}$ $\left(\otimes_{k=1}^{K} O_k = \left(\times_{k=1}^{K} X_k, \boxtimes_{k=1}^{K} \mathcal{F}_k, \otimes_{k=1}^{K} F_k \right), S_{\left[\otimes_{k=1}^{K} \rho_k \right]} \right)$ (which is also denoted by $\otimes_{k=1}^{K} \mathsf{M}_{\mathcal{N}_k} \left(O_k, S_{[\rho_k]} \right)$) is called a parallel measurement of $\mathsf{M}_{\mathcal{N}_k} \left(O_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]} \right)$ $(k = 1, 2, \cdots, K)$. Note that the parallel measurement always exists uniquely.

2. Bell's Inequality Always Holds in Classical and Quantum Systems

Our Main Assertion about Bell's Inequality

In this paper, I assert that Bell's inequality should be studied in the framework of quantum language (*i.e.*, quantum theory with the linguistic Copenhagen interpretation). Let us start from the following definition, which is a slight modification of the generalized simultaneous observable in Definition 2. That is, Definitions 2 - 4 are due to the linguistic Copenhagen interpretation (D), "Only one measurement is permitted".

Definition 4 [Combined observable (*cf.* ref. [12])] *Let* $[\mathcal{A}, \mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X = \{-1,1\}$. *Consider four observables:* $\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13})$, $\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14})$, $\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23})$, $\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24})$ *in* $\mathcal{N}$. *The four observables are said to be combinable if there exists an observable* $\mathsf{O} = (X^4, \mathcal{P}(X^4), F)$ *in* $\mathcal{N}$ *such that*

$$F_{13}(\{(x_1, x_3)\}) = F(\{x_1\} \times X \times \{x_3\} \times X),$$
$$F_{14}(\{(x_1, x_4)\}) = F(\{x_1\} \times X \times X \times \{x_4\}),$$
$$F_{23}(\{(x_2, x_3)\}) = F(X \times \{x_2\} \times \{x_3\} \times X), \tag{4}$$
$$F_{24}(\{(x_2, x_4)\}) = F(X \times \{x_2\} \times X \times \{x_4\})$$

for any $(x_1, x_2, x_3, x_4) \in X^4$. *The observable* O *is said to be a combined observable of* O_{ij} $(i = 1, 2, j = 3, 4)$. *Also, the measurement* $\mathsf{M}_{\mathcal{N}}(\mathsf{O} = (X^4, \mathcal{P}(X^4), F), S_{[\rho_0]})$ *is called the combined measurement of* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{13}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{14}, S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{23}, S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{24}, S_{[\rho_0]})$.

Remark 5. 1) Note that the Formula (4) implies that

$$F_{13}(\{x\} \times X) = F_{14}(\{x\} \times X), \quad F_{23}(\{x\} \times X) = F_{24}(\{x\} \times X),$$
$$F_{13}(X \times \{x\}) = F_{23}(X \times \{x\}), \quad F_{14}(X \times \{x\}) = F_{24}(X \times \{x\}),$$

for all $x \in X$.

2) Syllogism (*i.e.*, $[[A \Rightarrow B] \wedge [B \Rightarrow C]] \Rightarrow [A \Rightarrow C]$) does not hold in quantum systems but in classical systems (*cf.* ref. [8]). A certain combined observable plays an important role in the proof of the classical syllogism (*cf.* ref. [12]).

The following theorem is all of our insistence concerning Bell's inequality. We assert that this is the true Bell's inequality.

Theorem 6. [Bell's inequality in quantum language] *Let* $[\mathcal{A}, \mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X = \{-1,1\}$. *Fix the pure state* $\rho_0 (\in \mathfrak{S}^p(\mathcal{A}^*))$. *And consider the four measurements* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{13} = (X^2, \mathcal{P}(X^2), F_{13}), S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{14} = (X^2, \mathcal{P}(X^2), F_{14}), S_{[\rho_0]})$, $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{23} = (X^2, \mathcal{P}(X^2), F_{23}), S_{[\rho_0]})$ *and* $\mathsf{M}_{\mathcal{N}}(\mathsf{O}_{24} = (X^2, \mathcal{P}(X^2), F_{24}), S_{[\rho_0]})$. *Or equivalently, consider the parallel measurement* $\otimes_{i=1,2, j=3,4} \mathsf{M}_{\mathcal{N}}(\mathsf{O}_{ij} = (X^2, \mathcal{P}(X^2), F_{ij}), S_{[\rho_0]})$. *Define four correlation functions* $(i = 1, 2, j = 3, 4)$ *such that*

$$R_{ij} = \sum_{(u,v) \in X \times X} u \cdot v \, \rho_0 \left(F_{ij} \left(\{ (u,v) \} \right) \right)$$

Assume that four observables $O_{13} = \left(X^2, \mathcal{P}(X^2), F_{13} \right)$, $O_{14} = \left(X^2, \mathcal{P}(X^2), F_{14} \right)$, $O_{23} = \left(X^2, \mathcal{P}(X^2), F_{23} \right)$ and $O_{24} = \left(X^2, \mathcal{P}(X^2), F_{24} \right)$ are combinable, that is, we have the combined observable $O = \left(X^4, \mathcal{P}(X^4), F \right)$ in $\mathcal{N}$ such that it satisfies (4). Then we have a combined measurement $M_{\mathcal{N}} \left(O = \left(X^4, \mathcal{P}(X^4), F \right), S_{[\rho_0]} \right)$ of $M_{\mathcal{N}} \left(O_{13}, S_{[\rho_0]} \right)$, $M_{\mathcal{N}} \left(O_{14}, S_{[\rho_0]} \right)$, $M_{\mathcal{N}} \left(O_{23}, S_{[\rho_0]} \right)$ and $M_{\mathcal{N}} \left(O_{24}, S_{[\rho_0]} \right)$. And further, we have Bell's inequality in quantum language as follows.

$$\left| R_{13} - R_{14} \right| + \left| R_{23} + R_{24} \right| \le 2 \tag{5}$$

Proof. Clearly we see, $i = 1, 2, \; j = 3, 4$,

$$R_{ij} = \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} x_i \cdot x_j \, \rho_0 \left(F \left(\{ (x_1, x_2, x_3, x_4) \} \right) \right) \tag{6}$$

(for example, $R_{13} = \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} x_1 \cdot x_3 \, \rho_0 \left(F \left(\{ (x_1, x_2, x_3, x_4) \} \right) \right)$). Therefore, we see that

$$\begin{aligned}
&\left| R_{13} - R_{14} \right| + \left| R_{23} + R_{24} \right| \\
&= \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} \left[\left| x_1 \cdot x_3 - x_1 \cdot x_4 \right| + \left| x_2 \cdot x_3 + x_2 \cdot x_4 \right| \right] \rho_0 \left(F \left(\{ (x_1, x_2, x_3, x_4) \} \right) \right) \\
&= \sum_{(x_1, x_2, x_3, x_4) \in X \times X \times X \times X} \left[\left| x_3 - x_4 \right| + \left| x_3 + x_4 \right| \right] \rho_0 \left(F \left(\{ (x_1, x_2, x_3, x_4) \} \right) \right) \le 2
\end{aligned}$$

This completes the proof.

As the corollary of this theorem, we have the followings:

Corollary 7. *Consider the parallel measurement*

$\otimes_{i=1,2, j=3,4} M_{\mathcal{N}} \left(O_{ij} = \left(X^2, \mathcal{P}(X^2), F_{ij} \right), S_{[\rho_0]} \right)$ *as in Theorem* 6. *Let*

$$x = \left(\left(x_{13}^1, x_{13}^2 \right), \left(x_{14}^1, x_{14}^2 \right), \left(x_{23}^1, x_{23}^2 \right), \left(x_{24}^1, x_{24}^2 \right) \right) \in X^8 \left(\equiv \{-1, 1\}^8 \right)$$

be a measured value of the parallel measurement
$\otimes_{i=1,2, j=3,4} M_{\mathcal{N}} \left(O_{ij} = \left(X^2, \mathcal{P}(X^2), F_{ij} \right), S_{[\rho_0]} \right)$. Let N be sufficiently large natural number. Consider N-parallel measurement
$\otimes_{n=1}^{N} \left[\otimes_{i=1,2, j=2,3} M_{\mathcal{N}} \left(O_{ij} = \left(X^2, \mathcal{P}(X^2), F_{ij} \right), S_{[\rho_0]} \right) \right]$. Let $\{ x^n \}_{n=1}^{N}$ be the measured value. That is,

$$\{ x^n \}_{n=1}^{N} = \begin{bmatrix}
\left(\left(x_{13}^{1,1}, x_{13}^{2,1} \right), \left(x_{14}^{1,1}, x_{14}^{2,1} \right), \left(x_{23}^{1,1}, x_{23}^{2,1} \right), \left(x_{24}^{1,1}, x_{24}^{2,1} \right) \right) \\
\left(\left(x_{13}^{1,2}, x_{13}^{2,2} \right), \left(x_{14}^{1,2}, x_{14}^{2,2} \right), \left(x_{23}^{1,2}, x_{23}^{2,2} \right), \left(x_{24}^{1,2}, x_{24}^{2,2} \right) \right) \\
\vdots \\
\left(\left(x_{13}^{1,N}, x_{13}^{2,N} \right), \left(x_{14}^{1,N}, x_{14}^{2,N} \right), \left(x_{23}^{1,N}, x_{23}^{2,N} \right), \left(x_{24}^{1,N}, x_{24}^{2,N} \right) \right)
\end{bmatrix} \in \left(X^8 \right)^N$$

Here, note that the law of large numbers says: for sufficiently large N,

$$R_{ij} \approx \frac{1}{N} \sum_{n=1}^{N} x_{ij}^{1,n} x_{ij}^{2,n} \quad (i = 1, 2, \; j = 3, 4)$$

Then, it holds, by the Formula (5), that

$$\left| \sum_{n=1}^{N} \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^{N} \frac{x_{14}^{1,n} x_{14}^{2,n}}{N} \right| + \left| \sum_{n=1}^{N} \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^{N} \frac{x_{24}^{1,n} x_{24}^{2,n}}{N} \right| \leq 2, \tag{7}$$

which is also called Bell's inequality in quantum language.

Remark 8. [The conventional Bell's inequality (*cf.* refs. [17] [16] [18])] The mathematical Bell's inequality is as follows: Let $(\Theta, \mathcal{B}, P)$ be a probability space. Let $(f_1, f_2, f_3, f_4): \Theta \to X^4 \left(\equiv \{-1, 1\}^4 \right)$ be a measurable functions. Define the correlation functions $\hat{R}_{ij} (i = 1, 2, j = 3, 4)$ by $\int_{\Theta} f_i(\theta) f_j(\theta) P(\mathrm{d}\theta)$. Then, the following mathematical Bell's inequality (or precisely, CHSH inequality (*cf.* ref. [16])) holds:

$$\left| \tilde{R}_{13} - \tilde{R}_{14} \right| + \left| \tilde{R}_{23} + \tilde{R}_{24} \right| \leq 2 \tag{8}$$

(E) This is easily proved as follows.

"the left-hand side of the above (8)"

$$\leq \int_{\Theta} \left| f_3(\theta) - f_4(\theta) \right| P(\mathrm{d}\theta) + \int_{\Theta} \left| f_3(\theta) + f_4(\theta) \right| P(\mathrm{d}\theta) \leq 2$$

This completes the proof.

Recall Theorem 6 (Bell's inequality in quantum language), in which we have, by the combinable condition, the probability space $\left(X^4, \mathcal{P}(X^4), \rho_0(F(\cdot)) \right)$. Therefore the proof of Theorem 6 and the above proof (E) are, from the mathematical point of view, the same.

3. "Bell's Inequality" Is Violated in Classical Systems as Well as Quantum Systems

In the previous section, we show that Theorem 6 (or Corollary 7) says

(F$_1$) Under the combinable condition (*cf.* Definition 4), Bell's Inequality (5) (or, (7)) holds in both classical systems and quantum systems.

Or, equivalently,

(F$_2$) If Bell's Inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the combined measurement).

This is similar to the following elementary statement in quantum mechanics:

(F$_2'$) We have no (generalized) simultaneous measurement of the position observable Q and the momentum observable P, and thus we cannot obtain the *measured value* (by the generalized simultaneous measurement),

which may be, from Einstein's point of view, represented that "*true value* (or, *hidden variable*) of the position and momentum" does not exist. Since the error Δ is usually defined by $\Delta = \left| \text{rough measured value} - \text{true value} \right|$, it is not easy to define the errors Δ_Q and Δ_P in Heisenberg's uncertainty principle $\Delta_Q \cdot \Delta_P \geq \hbar/2$.

This definition was completed and Heisenberg's uncertainty principle was proved in ref. [11]. Also, according to the maxim of dualism: "To be is to be perceived" due to G. Berkeley, we think that it is not necessary to name that does not exist (or equivalently, that is not measured).

The above statement (F$_2$) makes us expect that

(G) Bell's inequality (5) (or (7)) is violated in classical systems as well as quan-

tum systems without the combinable condition.

This (G) was already shown in my previous paper [2]. However, I received a lot of questions concerning (G) from the readers. Thus, in this section, we again explain the (G) precisely.

Bell Test Experiment

In order to show the (G), three steps ([Step: I] - [Step: III]) are prepared in what follows.

[Step: I].

Put $X = \{-1, 1\}$. Define complex numbers a_k $(= \alpha_k + \beta_k \sqrt{-1} \in \mathbb{C}$: the complex field) $(k = 1, 2, 3, 4)$ such that $|a_k| = 1$. Define the probability space $\left(X^2, \mathcal{P}(X^2), \nu_{a_i a_j}\right)$ such that $(i = 1, 2, j = 3, 4)$

$$\nu_{a_i a_j}\left(\{(1,1)\}\right) = \nu_{a_i a_j}\left(\{(-1,-1)\}\right) = \left(1 - \alpha_i \alpha_j - \beta_i \beta_j\right)/4 \tag{9}$$

$$\nu_{a_i a_j}\left(\{(-1,1)\}\right) = \nu_{a_i a_j}\left(\{(1,-1)\}\right) = \left(1 + \alpha_i \alpha_j + \beta_i \beta_j\right)/4$$

The correlation $R(a_i, a_j)$ $(i = 1, 2, j = 3, 4)$ is defined as follows:

$$R(a_i, a_j) \equiv \sum_{(x_1, x_2) \in X \times X} x_1 \cdot x_2 \nu_{a_i a_j}\left(\{(x_1, x_2)\}\right) = -\alpha_i \alpha_j - \beta_i \beta_j \tag{10}$$

Now we have the following problem:

(H) Find a measurement $\mathsf{M}_{\mathcal{N}}\left(\mathsf{O}_{a_i a_j} = \left(X^2, \mathcal{P}(X^2), F_{a_i a_j}\right), S_{[\rho_0]}\right)$ $(i = 1, 2, j = 3, 4)$ such that

$$\rho_0\left(F_{a_i a_j}(\Xi)\right) = \nu_{a_i a_j}(\Xi) \quad \left(\forall \Xi \in \mathcal{P}(X^2)\right) \tag{11}$$

and

$$F_{a_1 a_3}\left(\{x_1\} \times X\right) = F_{a_1 a_4}\left(\{x_1\} \times X\right), \quad F_{a_2 a_3}\left(\{x_2\} \times X\right) = F_{a_2 a_4}\left(\{x_2\} \times X\right)$$

$$F_{a_1 a_3}\left(X \times \{x_3\}\right) = F_{a_2 a_3}\left(X \times \{x_3\}\right), \quad F_{a_1 a_4}\left(X \times \{x_4\}\right) = F_{a_2 a_4}\left(X \times \{x_4\}\right)$$

$$\left(\forall x_k \in X \left(\equiv \{-1, 1\}\right), k = 1, 2, 3, 4\right)$$

which is the same as the condition in Remark 5.

[Step: II].

Let us answer this problem (H) in the two cases (*i.e.*, classical case and quantum case), that is,

$$\bullet \begin{cases} 1) \text{ the case of quantum systems :} \\ \left[\mathcal{A} = B\left(\mathbb{C}^2\right) \otimes B\left(\mathbb{C}^2\right)\left(\equiv B\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)\right), \mathcal{N} = B\left(\mathbb{C}^2\right) \otimes B\left(\mathbb{C}^2\right)\right] \\ 2) \text{ the case of classical systems :} \\ \left[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega)\left(\equiv C_0(\Omega \times \Omega)\right), \mathcal{N} = L^\infty(\Omega) \otimes L^\infty(\Omega)\right] \end{cases}$$

1) the case of quantum system: $\left[\mathcal{A} = B\left(\mathbb{C}^2\right) \otimes B\left(\mathbb{C}^2\right)\right]$

Put

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \left(\in \mathbb{C}^2\right)$$

For each a_k $(k=1,2,3,4)$, define the observable $O_{a_k} \equiv (X, \mathcal{P}(X), G_{a_k})$ in $B(\mathbb{C}^2)$ such that

$$G_{a_k}(\{1\}) = \frac{1}{2}\begin{bmatrix} 1 & \overline{a}_k \\ a_k & 1 \end{bmatrix}, \quad G_{a_k}(\{-1\}) = \frac{1}{2}\begin{bmatrix} 1 & -\overline{a}_k \\ -a_k & 1 \end{bmatrix}$$

where $\overline{a}_k = \alpha_k - \beta_k\sqrt{-1}$. Then, we have four observable:

$$\hat{O}_{a_i} = (X, \mathcal{P}(X), G_{a_i} \otimes I), \quad \hat{O}_{a_j} = (X, \mathcal{P}(X), I \otimes G_{a_j}) \quad (i=1,2, j=3,4) \, (12)$$

and further,

$$O_{a_i a_j} = (X^2, \mathcal{P}(X^2), F_{a_i a_j} := G_{a_i} \otimes G_{a_j}) \quad (i=1,2, j=3,4) \tag{13}$$

in $B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)$, where it should be noted that $F_{a_i a_j}$ is separated by G_{a_i} and G_{a_j}.

Further define the singlet state $\rho_0 = |\psi_s\rangle\langle\psi_s| \quad \left(\in S^p\left(B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*\right)\right)$, where

$$\psi_s = (e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2}$$

Thus we have the measurement $M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*}\left(O_{a_i a_j}, S_{[\rho_0]}\right)$ in $B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)$ $(i=1,2, j=3,4)$. The followings are clear: for each $(x_1, x_2) \in X^2 \left(\equiv \{-1,1\}^2\right)$,

$$\rho_0\left(F_{a_i a_j}(\{(x_1, x_2)\})\right) = \left\langle \psi_s, \left(G_{a_i}(\{x_1\}) \otimes G_{a_j}(\{x_2\})\right)\psi_s \right\rangle$$
$$= v_{a_i a_j}(\{(x_1, x_2)\}) \quad (i=1,2, j=3,4) \tag{14}$$

For example, we easily see:

$$\rho_0\left(F_{a_i b_j}(\{(1,1)\})\right) = \left\langle \psi_s, \left(G_{a_i}(\{1\}) \otimes G_{a_j}(\{1\})\right)\psi_s \right\rangle$$
$$= \frac{1}{8}\left\langle (e_1 \otimes e_2 - e_2 \otimes e_1), \left(\begin{bmatrix} 1 & \overline{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \overline{a}_j \\ a_j & 1 \end{bmatrix}\right)(e_1 \otimes e_2 - e_2 \otimes e_1) \right\rangle$$
$$= \frac{1}{8}\left\langle \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 1 & \overline{a}_i \\ a_i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \overline{a}_j \\ a_j & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \right\rangle$$
$$= \frac{1}{8}\left\langle \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 1 \\ a_i \end{bmatrix} \otimes \begin{bmatrix} \overline{a}_j \\ 1 \end{bmatrix} - \begin{bmatrix} \overline{a}_i \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ a_j \end{bmatrix}\right) \right\rangle$$
$$= \frac{1}{8}\left(2 - \alpha_i\overline{a}_j - \overline{a}_i a_j\right) = \left(1 - \alpha_i\alpha_j - \beta_i\beta_j\right)/4 = v_{a_i a_j}(\{(1,1)\})$$

Therefore, the measurement $M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}\left(O_{a_i a_j}, S_{[\rho_0]}\right)$ satisfies the condition (H).

2) the case of classical systems: $\left[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega)\right]$

Put $\omega_0 \left(= (\omega_0', \omega_0'')\right) \in \Omega \times \Omega$, $\rho_0 = \delta_{\omega_0} \left(\in \mathfrak{S}^p\left(C_0(\Omega \times \Omega)^*\right)\right)$, i.e., the point measure at ω_0)). Define the observable $O_{a_i a_j} := (X^2, \mathcal{P}(X^2), F_{a_i a_j})$ in $L^\infty(\Omega \times \Omega)$ such that

$$\left[F_{a_i a_j}(\{(x_1, x_2)\})\right](\omega) = v_{a_i a_j}(\{(x_1, x_2)\})$$
$$\left(\forall (x_1, x_2) \in X^2, i=1,2, j=3,4, \forall \omega \in \Omega \times \Omega\right)$$

Thus, we have four observables

$$O_{a_i a_j} = \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_i a_j} \right) \quad (i = 1, 2, j = 3, 4) \tag{15}$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the formula (13)). Then, it is clear that the measurement $M_{C_0(\Omega \times \Omega)}\left(O_{a_i a_j}, S_{\left[\delta_{\omega_0} \right]} \right)$ satisfies the condition (H).

2)' the case of classical systems: $\left[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega) \right]$

It is easy to show a lot of different answers from the above 2). For example, as a slight generalization of (9), define the probability measure $v_{a_i a_j}^t$ $(0 \le t \le 1)$ such that

$$v_{a_i a_j}^t \left(\{ (1,1) \} \right) = v_{a_i a_j}^t \left(\{ (-1,-1) \} \right) = \left(1 - t \left(\alpha_i \alpha_j + \beta_i \beta_j \right) \right) / 4$$
$$v_{a_i a_j}^t \left(\{ (-1,1) \} \right) = v_{a_i a_j}^t \left(\{ (1,-1) \} \right) = \left(1 + t \left(\alpha_i \alpha_j + \beta_i \beta_j \right) \right) / 4 \tag{16}$$

And consider the real-valued continuous function $t \left(\in C_0(\Omega \times \Omega) \right)$ such that $0 \le t(\omega', \omega'') \le 1$ $\left(\forall \omega = (\omega', \omega'') \in \Omega \times \Omega \right)$. And assume that $t(\omega_0) = 1$ for some $\omega_0 \left(= (\omega'_0, \omega''_0) \right) \in \Omega \times \Omega$, $\rho_0 = \delta_{\omega_0}$ $\left(\in \mathfrak{S}^p \left(C_0(\Omega \times \Omega)^* \right) \right.$, *i.e.*, the point measure at ω_0)). Define the observable $O_{a_i a_j} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_i a_j} \right)$ in $L^\infty(\Omega \times \Omega)$ such that

$$\left[F_{a_i a_j} \left(\{ (x_1, x_2) \} \right) \right](\omega) = v_{a_i a_j}^{t(\omega)} \left(\{ (x_1, x_2) \} \right)$$
$$\left(\forall (x_1, x_2) \in X^2, i = 1, 2, j = 3, 4, \forall \omega \in \Omega \times \Omega \right) \tag{17}$$

Thus, we have four observables

$$O_{a_i a_j} = \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_i a_j} \right) \quad (i = 1, 2, j = 3, 4)$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the Formula (13)). Then, it is clear that the measurement $M_{L^\infty(\Omega \times \Omega)}\left(O_{a_i a_j}, S_{\left[\delta_{\omega_0} \right]} \right)$ satisfies the condition (H).

[Step: III].

As defined by (9), consider four complex numbers $a_k \left(= \alpha_k + \beta_k \sqrt{-1}; k = 1, 2, 3, 4 \right)$ such that $|a_k| = 1$. Thus we have four observables

$$O_{a_1 a_3} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_1 a_3} \right), \quad O_{a_1 a_4} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_1 a_4} \right),$$
$$O_{a_2 a_3} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_2 a_3} \right), \quad O_{a_2 a_4} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_2 a_4} \right),$$

in $\mathcal{N}$. Thus, we have the parallel measurement
$\otimes_{i=1,2, j=3,4} M_{\mathcal{N}} \left(O_{a_i a_j} := \left(X^2, \mathcal{P}\left(X^2 \right), F_{a_i a_j} \right), S_{\left[\rho_0 \right]} \right)$ in $\otimes_{i=1,2, j=3,4} \mathcal{N}$.

Thus, putting

$$a_1 = \sqrt{-1}, a_2 = 1, a_3 = \frac{1 + \sqrt{-1}}{\sqrt{2}}, a_4 = \frac{1 - \sqrt{-1}}{\sqrt{2}},$$

we see, by (10), that

$$\left| R(a_1, a_3) - R(a_1, a_4) \right| + \left| R(a_2, a_3) + R(a_2, a_4) \right| = 2\sqrt{2} \tag{18}$$

Further, assume that the measured value is $x \left(\in X^8 \right)$. That is,

$$x = \left(\left(x_{13}^1, x_{13}^2 \right), \left(x_{14}^1, x_{14}^2 \right), \left(x_{23}^1, x_{23}^2 \right), \left(x_{24}^1, x_{24}^2 \right) \right) \in \underset{i,j=1,2}{\times} X^2 \left(\equiv \{-1, 1\}^8 \right)$$

Let N be sufficiently large natural number. Consider N-parallel measurement $\otimes_{n=1}^{N} \left[\otimes_{i=1,2, j=3,4} \mathsf{M}_{\mathcal{N}} \left(\mathsf{O}_{a_i a_j} := \left(X^2, \mathcal{P}(X^2), F_{a_i a_j} \right), S_{[\rho_0]} \right) \right]$. Assume that its measured value is $\left\{ x^n \right\}_{n=1}^{N}$. That is,

$$\left\{ x^n \right\}_{n=1}^{N} = \begin{bmatrix} \left(\left(x_{13}^{1,1}, x_{13}^{2,1} \right), \left(x_{14}^{1,1}, x_{14}^{2,1} \right), \left(x_{23}^{1,1}, x_{23}^{2,1} \right), \left(x_{24}^{1,1}, x_{24}^{2,1} \right) \right) \\ \left(\left(x_{13}^{1,2}, x_{13}^{2,2} \right), \left(x_{14}^{1,2}, x_{14}^{2,2} \right), \left(x_{23}^{1,2}, x_{23}^{2,2} \right), \left(x_{24}^{1,2}, x_{24}^{2,2} \right) \right) \\ \vdots \\ \left(\left(x_{13}^{1,N}, x_{13}^{2,N} \right), \left(x_{14}^{1,N}, x_{14}^{2,N} \right), \left(x_{23}^{1,N}, x_{23}^{2,N} \right), \left(x_{24}^{1,N}, x_{24}^{2,N} \right) \right) \end{bmatrix}$$

$$\in \left(\underset{i=1,2, j=3,4}{\times} X^2 \right)^N \left(\equiv \{-1,1\}^{8N} \right)$$

Then, the law of large numbers says that

$$R(a_i, a_j) \approx \frac{1}{N} \sum_{n=1}^{N} x_{ij}^{1,n} x_{ij}^{2,n} \quad (i=1,2, j=3,4)$$

This and the Formula (18) say that

$$\left| \sum_{n=1}^{N} \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^{N} \frac{x_{14}^{1,n} x_{14}^{2,n}}{N} \right| + \left| \sum_{n=1}^{N} \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^{N} \frac{x_{24}^{1,n} x_{24}^{2,n}}{N} \right| \approx 2\sqrt{2} \tag{19}$$

Therefore, Bell's Inequality (5) (or (7)) is violated in classical systems as well as quantum systems.

Remark 9. For completeness, note that the observables $\mathsf{O}_{a_i a_j}$ $(i=1,2, j=3,4)$ in the classical $L^\infty(\Omega \times \Omega)$ are not combinable in spite that these commute. Also, note that the Formulas (16) and (17) imply that

$$\left[F_{a_1 a_3} \left(\{x\} \times X \right) \right](\omega) = \left[F_{a_1 a_4} \left(\{x\} \times X \right) \right](\omega) = 1/2,$$

$$\left[F_{a_2 a_3} \left(\{x\} \times X \right) \right](\omega) = \left[F_{a_2 a_4} \left(\{x\} \times X \right) \right](\omega) = 1/2,$$

$$\left[F_{a_1 a_3} \left(X \times \{x\} \right) \right](\omega) = \left[F_{a_2 a_3} \left(X \times \{x\} \right) \right](\omega) = 1/2,$$

$$\left[F_{a_1 a_4} \left(X \times \{x\} \right) \right](\omega) = \left[F_{a_2 a_4} \left(X \times \{x\} \right) \right](\omega) = 1/2$$

$$(\forall x \in X, \forall \omega \in \Omega \times \Omega),$$

which is similar as in Remark 5; 1) or in (H).

4. Conclusions

In Bohr-Einstein debates (refs. [13] [14]), Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (*i.e.*, physics holds without observers)) is on the side of the realistic world view in **Figure 1**. On the other hand, we think that Bohr's standing point (that is, "*to be is to be perceived*" (*i.e.*, there is no science without measurements)) is on the side of the linguistic world view in **Figure 1**.

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in Bohr's spirit (*i.e.*, in the framework of quantum lan-

guage). And we show Theorem 6 (Bell's inequality in quantum language), which says the statement (F_2), that is,

(I_1) ($\equiv$(F_2)): If Bell's Inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the *measured value* (by the measurement of the combined observable).

Also, recall that Bell's original argument says, roughly speaking, that

(I_2) If the mathematical Bell's Inequality (8) is violated in Bell test experiment (the quantum case of Section 3.1), then *hidden variables* do not exist.

It should be note that the concept of "hidden variable" is independent of measurements, thus, the (I_2) is a philosophical statement in Einstein's spirit, on the other hand, the (I_1) is a statement in Bohr's spirit (*i.e.*, there is no science without measurements). It is sure that Bell's answer (I_2) is attractive philosophically, however, we believe in the scientific superiority of our answer (I_1). That is, we think that our (I_1) is a scientific representation of the philosophical (I_2). If so, we can, for the first time, understand Bell's inequality in science. That is, Theorem 6 is the true Bell's inequality. And we conclude that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether the combined measurement exists or not (in Section 2).

We hope that our proposal will be examined from various points of view[1].

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