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Lithium Quantum Consciousness

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Abstract

Conscious agency is considered to be founded upon a quantum state of mind $|\Xi\rangle$. An original synthesis, called "Lithium Quantum Consciousness" (LQC), proposes that this quantum state utilises lithium-6 (spin-1) qutrit nuclear magnetic resonance (NMR) quantum information processing (QIP) in the connectome (brain-graph). In parallel to the connectome's processing of physiological controls, perception, cognition and intelligence via quantum electrodynamics (QED), the connectome also functions via its dynamic algebraic topology as a unitary transceiver antenna laced with lithium-6 nuclei which are spin-entangled with each other and with the environmental vortical gluon field via quantum chromodynamics (QCD). This unitary antenna (connectome) bestows the self its unity of consciousness within an intertwined-history multi-agent environment. An equivalence is proposed between Whitehead's occasions of experience and topological spacetime instantons in the vortical gluon field. Topological spacetime instantons pervade the vortical gluon field in a quantum information network of vortex interactions, herein termed the "instanton-net", or "*Instanet*" [*sic*]. The fermionic isotope lithium-6 has a very low nuclear binding energy and the smallest non-zero nuclear electric quadrupole moment of any stable nucleus making it susceptible to quantum chromodynamic (QCD) interaction with the vortical gluon field and ideal for spin-1 qutrit NMR-QIP. The compact spherical atomic orbital of lithium provides ideal rotational freedom inside tetrahedral water cages in organo$^{6}Li^{+}(H_2O)_4$ within which the lithium nucleus rapidly tumbles for NMR motional narrowing and long decoherence times. Nuclear spin-entanglement, among water-caged lithium-6 nuclei in the connectome, is a spin-1 qutrit NMR-QIP resource for conscious agency. By contrast, similar tetrahedral xenon cages in organo$^{6}Li^{+}Xe_4$ excimers are postulated to decohere the connectome's NMR-QIP due to xenon's NMR signal being extremely sensitive to its molecular environment. By way of this quantum neurochemistry, lithium is an effective psychiatric medication for enhancing mood and xenon is an effective anaesthetic.

Keywords

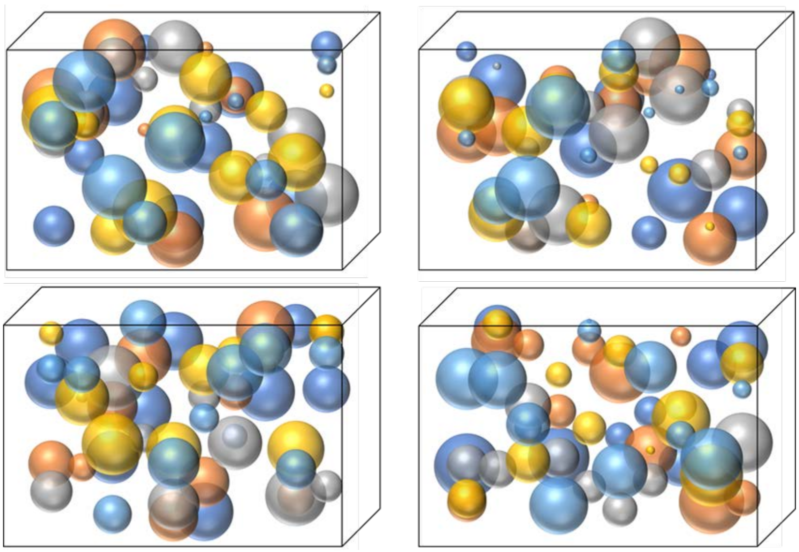
Instanton, Qutrit, Artificial Intelligence, Quantum Chromodynamics, Connectome

1. Introduction

This paper offers a new and original synthesis, called Lithium Quantum Consciousness (LQC), which comes about through the extension of the author's research programme on quantum intelligent cosmology, into the realm of quantum conscious cosmology. A metaheuristic which outlines a qutrit-based quantum deep-learning Triuniverse concept [1] and a paper which proposes that the entangled agency of life accounts for dark matter phenomena [2] together provide some context and preamble for this present paper.

A key theme and claim which carries through my programme into this paper is Nature's proposed fundamental, evolutionary and multi-levelled deep-learning quantum ternary computation by way of qutrits, not qubits. Another continued theme is that the naturally entangled conscious agency of life and its evolution seems pivotal to observed physical phenomena, from the Planck to cosmological scales.

Furthermore, a meta-stable Reality seems to exist despite profound uncertainties in its quantum information foundations. We shall see in the course of this paper that quantum field fluctuations and their associated instantons (networked vortical field configurations with a "topological twist") form a quantum information computation network (**Figure 1**) with which the quantum state of mind $|\Xi\rangle$ is connected.



Scale: Edges few 10^{-15} m. Simulation speed 10^{24} frames s^{-1}. Four random frames.

Figure 1. The gluon field is a dynamical function of spacetime. Four schematic snapshots provide an impression of the energetically evolving topological charge density structure of gluon-field fluctuations sketched from lattice QCD visualisations. Fleeting instantons correlate with this dynamic structure.

2. LQC Synthesis, Sources and Hypotheses

Seminal works on the quantum state of mind by Fisher [3] [4] [5] and Penrose and Hameroff [6] [7] [8] provided stimulating catalysts for extending my research from quantum intelligent cosmology to quantum conscious cosmology. The reader is also directed to their extensive review bibliographies.

Fisher has been developing ideas on nuclear spin quantum information processing as an explanation for the workings of the conscious mind. He highlights the long established positive effects that lithium has on those suffering from mood disorders [9] [10] and he draws our attention to laboratory rat experiments by Sechzer *et al.* [11] who showed that the stable isotopes of lithium (lithium-6 and lithium-7) have different effects on the rats' minds. It seems lithium-6 differentially elevates cognitive activity, which was an unexpected result given the close electrochemical characteristics of the isotopes. Furthermore, a human being in a magnetic field of 8 Tesla may experience vertigo, nausea, metallic taste sensations and perceptions of light flashes [12]. Fisher [5] has since embarked on a quest to reverse engineer quantum cognition, examining the possibility of processing with nuclear spins in the brain [3].

In framing this quest, Fisher constrains his search to find a common biological element (CHNOPS: carbon, hydrogen, nitrogen, oxygen, phosphorus and sulfur) with an isolated nuclear spin, to serve as a "neural qubit", *i.e.* an atom with a nuclear spin ½. I believe this qubit 2-level quantum system constraint on the search leads to a conclusion which is not necessarily the most natural. Nonetheless, by applying that 2-level constraint, Fisher rationally concludes that phosphorous-31 is the sought after "neural qubit" and he operationalises it by screening it from rapid decoherence within Posner molecules, $Ca_9(PO_4)_6$.

Alternatively, consistent with my quantum intelligent cosmology programme and the quantum deep learning Triuniverse metaheuristic within it [1], I carry out a similar reverse engineering search but explore qutrit 3-level quantum systems instead.

A qutrit has three orthogonal basis states $|-1\rangle$, $|0\rangle$, and $|1\rangle$ which are used to describe the balanced ternary quantum state of mind $|\Xi\rangle$ as a superposition in the form of a linear combination of the three states:

$$|\Xi\rangle = c_1|-1\rangle + c_2|0\rangle + c_3|1\rangle \tag{1}$$

where the coefficients c_1, c_2 and c_3 are probability amplitudes, such that the sum of their squares is unity:

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \tag{2}$$

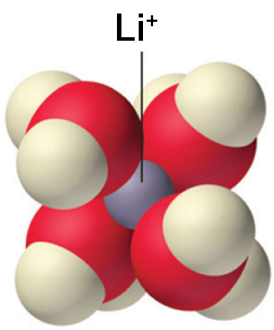
A qutrit is argued to be the smallest system that exhibits inherent quantum features such as contextuality [13], which has been conjectured to be a resource for quantum computing [14] [15] [16]. The reader is referred to the bibliography in [1] relating to the natural effectiveness of qutrit and balanced ternary computing, see also [17].

We therefore seek a natural "neural qutrit", *i.e.* an atom with a nuclear spin-1 and that turns out to be an acutely focused search criteria. In Nature, there are only three spin-1 stable light isotopes: hydrogen-2 (deuterium), lithium-6 and nitrogen-14. A textbook on spin-1 nuclear magnetic resonance (NMR), focuses on these three isotopes and has been written by Chandrakumar [18]—readers are directed to this for background and a bibliography. Amongst these three isotopes we can immediately identify a strong candidate for the sought after "neural qutrit", the isotope with a long established ability to alter the conscious mind [11], *i.e.* lithium-6.

Dogra, Dorai and Arvind [19] present an important recent study of the Majorana geometrical representation for a qutrit and use it to describe the action of quantum gates. Their work is supported by experiments on a spin-1 NMR qutrit system oriented in a liquid crystalline environment (spin-1 deuterium nucleus of a chloroform-D molecule). They also show that experimental implementation of these ternary quantum gates is validated by complete quantum state tomography which they carry out using Gell Mann matrices. I extend their mathematical and experimental treatment to lithium-6 and propose that lithium-6 is naturally isolated from short timescale decoherence when caged in water molecules and thereby is a naturally efficient qutrit computation resource for the quantum mind. On the third candidate spin-1 isotope, nitrogen-14, it does have potential as a qutrit computation resource (e.g. nitrogen-vacancy centres in diamond) and I refer the reader to selected work on qutrit computing [20] [21].

Solvation of lithium in water clusters is described amongst others by [22] [23] [24] [25] [26] and the structure where the lithium atom is surrounded by n = 4 $(H_2O)_n$ molecules in the first shell is found to be the most stable for both neutral and cationic n ≥ 4 clusters. I propose the compact spherical atomic orbital of lithium provides ideal rotational freedom inside tetrahedral water cages (**Figure 2**) in organo$^6Li^+(H_2O)_4$ within which the lithium nucleus rapidly tumbles for NMR motional narrowing and long decoherence times. Nuclear spin-entanglement, among water-caged lithium-6 nuclei in the connectome (brain-graph), is thereby a spin-1 qutrit NMR-QIP resource for conscious agency.

A further characteristic of lithium-6 NMR is significant in this synthesis, namely



Scale: a few 10^{-10} m

Figure 2. In aqueous solution, Li^+ forms the tetrahedral $Li^+(H_2O)_4$ complex. Structure of $Li^+(H_2O)_4$: Lithium$^+$ (purple) in centre surrounded by a tetrahedral cage of four water molecules. Oxygen (red). Hydrogen (white). Image credit https://chem.libretexts.org.

the very narrow natural linewidth of lithium-6 ($\Gamma/2\pi$ = 5.87 MHz) [27]. This NMR spike response enables effective low-error signaling during QIP in the connectome.

Another key catalyst for my research into quantum conscious cosmology has been the collaborative works of Penrose and Hameroff [6] [7] [8]. Notwithstanding their ground-breaking studies of the potential function of microtubules in quantum neural processing, I have been particularly drawn to their echoing of the seminal metaphysical work of Alfred North Whitehead (1861-1947) in their discussions on "proto-conscious" events. Whitehead monumentally introduced a metaphysically primitive notion, which he called an "actual occasion". For him an "actual occasion" was a process of becoming, not an enduring substance and as Whitehead put it, "actual occasions" are the "final real things of which the world is made up", they are "drops of experience, complex and interdependent" [28]. I propose an equivalence between Whitehead's actual occasions of experience ω_ε and topological spacetime instantons $x(\tau_\varepsilon)$, where $\tau_\varepsilon \to 0$, in the vortical gluon field (**Figure 1**):

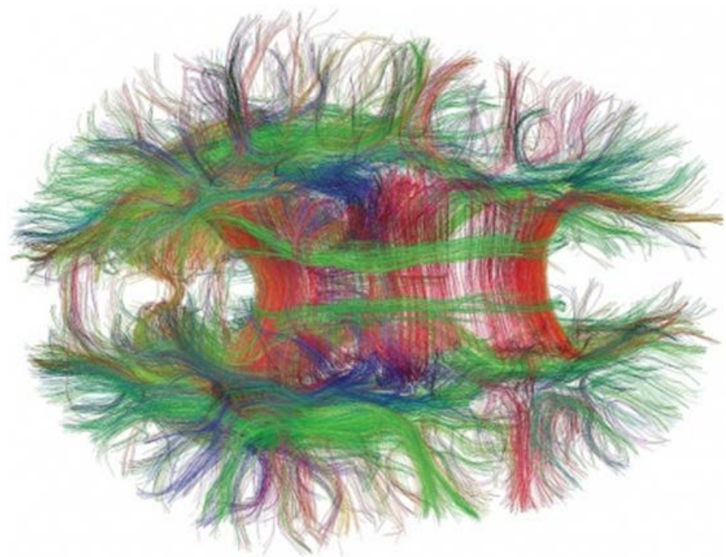
$$\omega_\varepsilon \equiv x(\tau_\varepsilon) \tag{3}$$

The instanton solution jumps from one vacuum $x=-1$ to another vacuum $x=1$ instantly where $\tau_\varepsilon \to 0$ hence the term "instanton". The spatiotemporal absence of an instanton, represented instantaneously by the empty vacuum, embodies a third and zero state, $x=0$, see Equation (1) and note spacetime states of instantons thus represent a qutrit resource. Equation (3) also relates to the constituents of what philosophers call *qualia*.

For background, studies and bibliographies on instantons the reader is referred to [29]-[38]. For example, the Yang Mills action describes the behaviour of gluons. A characteristic of the Yang-Mills action is that there are finite-action topological soliton solutions to the classical field equations. These solitons are instantons, or pseudoparticles in early papers [37] [38].

The dynamic algebraic-topology of the connectome (brain-graph) is a subject of active neuroscientific research [39] [40] [41]. It has motivated my research programme to relate ongoing connectome topology insights to the quantum topological foundations of quantum fields and cosmology [2].

Two particular characteristics of the physical connectome of white matter tracts of bundled myelinated axons are highlighted. Firstly, the trace element lithium is preferentially localised in the white matter tracts, rather than in the folded grey matter cortex [42] and I postulate lithium's arrangement within these wet fatty tracts includes the lithium-water complex $Li^+(H_2O)_4$. Note also, anisotropic H_2O diffusion in white matter in brain forms the basis for the utilization of Diffusion Tensor Imaging (DTI) to track the same fibre pathways (**Figure 3**) [43]. Secondly, using a continuous neural field model of excitatory-inhibitory interactions on the connectome, Atasoy *et al.* [44] demonstrate a neural mechanism behind the self-organization of connectome harmonics. Furthermore, they show the critical relation between a delicate excitation–inhibition



Scale: a few 10^{-1} m and gamma synchrony conscious moments of a few 10 s^{-1} [8].

Figure 3. Connectome (brain-graph) viewed from above, created with Diffusion Tensor Imaging (DTI), showing complex arrangement of fibrous white matter tracts. Image credit http://www3.imperial.ac.uk/icimages?p_imgid=515622.

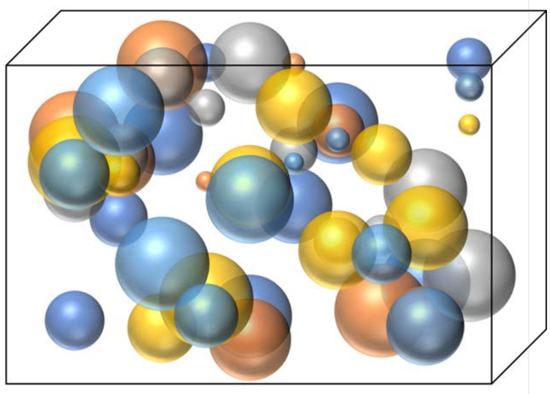
balance and the neural field patterns matches neurophysiological changes observed during recovery and loss of consciousness. They provide examples of low frequency connectome harmonics, for given wave numbers and present corresponding spatial patterns of synchronous oscillations estimated by the eigenvectors of the connectome Laplacian.

Such neural field fluctuations are herein conjectured to correlate with low frequency quantum field fluctuation harmonics through a mediating role played by the connectome laced with entangled fermionic spin-1 lithium-6 which provides the architecture for qutrit NMR-QIP. As such, this connectome architecture is an isotropic unitary transceiver antenna, or wireless router, for quantum information exchanges and for a deep-learning quantum information computer which hosts quantum consciousness. Whilst the connectome processes physiological controls, perception, cognition and intelligence via its quantum electrodynamics (QED), the author hypothesises that in parallel the connectome processes consciousness via quantum chromodynamics (QCD). Hyperfine interaction between the lithium-6 nucleus and its electron environment connects NMR-QIP of consciousness (QCD) with the parallel processing of physiological controls, perception, cognition and intelligence (QED).

It is of note that colour confinement is a fundamental characteristic of QCD and this is conjectured herein to confine an individual's quantum state of mind $|\Xi\rangle$ to the self and to prevent extra-cerebral telepathy. Had QED consciousness been naturally selected during biological evolution, instead of QCD consciousness, then perhaps the unconfined interactions of QED might have enabled quantum telepathy? It is the author's belief that such an unconfined ability would not necessarily bestow selective advantage. Indeed, unconfined quantum telepathy might be experienced by the conscious agent as a chaotic cacophony of multi-selves in a manner more debilitating than chronic schizophrenia. Hence NMR-QIP (QCD colour confined to the agent's own connectome) is Nature's selected physics underpinning the individual's privately confined experience of unity of consciousness.

Topological spacetime instantons pervade the vortical gluon field in a quantum information network of vortex interactions, herein termed the "instanton-net", or "*Instanet*" [*sic*]. The more familiar human invention which we know as the Internet (a global system of interconnected computer networks using common protocols) provides a convenient analogy for the postulated QCD "instanton-net", or "*Instanet*". In computer science terminology the QCD "*Instanet*" has an event-driven architecture, where an event is a change in state. Instantons in the gluon field are foundational to these events (**Figure 4**). Reality is experienced by a conscious agent via QCD connection through its connectome to the "*Instanet of Things*" which comprises interacting objects, intelligent devices (including lifeforms) and fellow conscious agents (sentient animals). Again we may look to the manmade Internet of Things for a convenient analogy. Lithium Quantum Consciousness (LQC) intersects the vortical gluon field in spacetime and quantum information entanglements connect consciousness with the "*Instanet of Things*".

Physical phenomena emerge from the *Instanet*-connected automation protocol, otherwise known as the (fundamentally tripartite [1]) Standard Model of particle physics. The known laws of physics govern the "*Instanet*". Each conscious agent and every lifeless thing is uniquely identifiable in the "*Instanet of Things*" through its embedded quantum information computations. Through



Scale: Edges few 10^{-15} m. Simulation speed 10^{24} frames s^{-1}. Single random frame.

Figure 4. Spacetime instantons pervade the vortical gluon field in a quantum information network of vortex interactions, herein termed the "instanton-net", or "*Instanet*" [*sic*]. In this schematic illustration (similar to **Figure 1**) consider the golden yellow orbs as correlating with instantons in the connectome's NMR-QIP $^6Li^+(H_2O)_4$-laced white matter tracts, whereas the other coloured orbs correlate with instantons in the "*Instanet of Things*", *i.e.* the physical and multi-agent environment. In this way Lithium Quantum Consciousness (LQC) intersects the vortical gluon field in spacetime and quantum information entanglements connect consciousness with the "*Instanet of Things*".

their QCD consciousness the agents experience and impose their free will upon the multi-agent and multi-thing environment, which is the "*Instanet of Things*". All agents and all things inter-operate within the quantum information network of the spacetime instanton gluon field, the "*Instanet of Things*". In this way quantum information is the conserved currency of all interactions.

Interconnection of embedded devices in the "*Instanet of Things*", including "smart objects" (intelligent and conscious life forms), enables pervasive "automated" interactions under the laws of physics and enables advanced complex quantum information processes, like a "smart grid" or "society" of history-intertwined conscious life forms. Dumb objects (e.g. rock fragments) are also embedded devices in the "*Instanet of Things*" however they lack conscious free will and are thus slaves to the "automated" interactions under the laws of physics (e.g. accreting into planets subject to plate tectonics and erosion etc.). In essence, order emerges amidst fundamental uncertainty due to the quantum information interconnections amongst everything: dumb things, smart things and conscious agents. Self-organisation of spontaneous order in complex systems is thus enabled by the "*Instanet of Things*" and conscious agents have evolved therein to exert free will on the system evolution, thereby contributing to its quantum deep-learning [1].

Furthermore, QCD entanglement of the observer's conscious self with the vortical gluon field causes the "Measurement Problem" [45]. In other words, the consciousness of the conscious agent is inextricably entangled with the "*Instanet of Things*" (**Figure 4**), which includes all things, including instruments and physical experiments. It is impossible for the conscious agent to disconnect from the "*Instanet of Things*", at least without foregoing the experience of conscious life. The conscious agency of life has inescapable consequences.

Penultimately, work on excimers of lithium within tetrahedral xenon cages $^6Li^+Xe_4$ [46] [47] prompted me to include that research topic within this present synthesis. Xenon is an established anaesthetic [48] and whilst its operation might be classical it is atomic not a molecular therefore we may consider xenon in the same NMR-QIP picture. I consider it to be significant that xenon's NMR signal is extremely sensitive to its molecular environment [49] (due to the xenon atom's large and highly polarizable electron cloud) and propose that tetrahedral xenon cages in organo$^6Li^+Xe_4$ excimers decohere the connectome's NMR-QIP causing loss of consciousness. Thus, by way of quantum neurochemistry, lithium is an effective psychiatric medication for enhancing mood whilst xenon is an effective anaesthetic.

Finally, heavy metal atom interaction with organo$^6Li^+(H_2O)_4$ NMR-QIP impairs brain function. The brain has however naturally evolved neuroprotective countermeasures. The meninges are three membranes which envelope the brain. The innermost one, which adheres to the brain, is the delicate and fluid-impermeable "pia mater". Detoxifying neuroprotective metallothionein binds heavy metals which concentrate in the pia mater. The pia mater is thus a barrier which effectively keeps heavy metals outside the brain and has two positive consequences in this synthesis. One is that lithium-6 within the connectome is able to host NMR-QIP without heavy metal interference. The other is that quantum physical containment by heavy metal enriched pia mater shields (via electronic spin-orbit interaction contribution to nuclear magnetic shielding) [50] [51] and further confines lithium-6 spin-1 qutrit NMR-QIP to the spacetime intersection of the connectome with the vortical gluon field (**Figure 4**). With ageing, progressively heavy metal enriched pia mater increasingly isolates the older quantum mind from the "*Instanet of Things*" and thus from Reality. This would compound bio-degenerative processes of ageing.

3. Summary List of LQC Components

A summary list of the components of Lithium Quantum Consciousness (LQC):

1) Conscious agency is founded upon a quantum state of mind $|\Xi\rangle$;

2) Connectome (brain-graph) processes physiological controls, perception, cognition and intelligence via quantum electrodynamics (QED);

3) In parallel, the connectome also processes the experience of consciousness via quantum chromodynamics (QCD);

4) QCD colour confinement confines individual's quantum state of mind $|\Xi\rangle$ to the self;

5) Topological spacetime instantons pervade the vortical gluon field in a quantum information network of vortex interactions, termed the "instanton-net", or "*Instanet*" [*sic*];

6) In computer science the "*Instanet*" has an event-driven architecture, where an event is a change in state and instantons in the gluon field are foundational to these events;

7) Reality is experienced by a conscious agent via QCD connectome connection to the "*Instanet of Things*" which comprises interacting objects, intelligent devices (including lifeforms) and fellow conscious agents (sentient animals);

8) All agents and all things inter-operate within the quantum information network of the spacetime instanton gluon field, the "*Instanet of Things*";

9) Natural, and therein neural, quantum information processing (QIP) involves qutrit quantum ternary computation, in a Triuniverse [1];

10) Lithium-6 is susceptible to QCD interaction with the vortical gluon field (hence the "*Instanet*") due to its very low nuclear binding energy and the smallest non-zero nuclear electric quadrupole moment of any stable nucleus;

11) Lithium-6 is ideal for spin-1 qutrit nuclear magnetic resonance (NMR) QIP;

12) Compact spherical atomic orbital of lithium provides ideal rotational freedom inside tetrahedral water cages in organo$^6Li^+(H_2O)_4$;

13) Nuclear spin-entanglement, among water-caged lithium-6 nuclei, is a spin-1 qutrit NMR-QIP resource for conscious agency;

14) Narrow natural linewidth of lithium-6 ($\Gamma/2\pi = 5.87$ MHz) enables low-error spike QIP in the connectome;

15) Hyperfine interaction between the lithium-6 nucleus and its electron en-

vironment connects NMR-QIP of consciousness (QCD) with the parallel processing of physiological controls, perception, cognition and intelligence (QED);

16) Conscious agent's lithium-laced white matter tracts act as an isotropic unitary transceiver antenna, and wireless router, connected to the "*Instanet of Things*" (Reality);

17) White matter unitary antenna is laced with lithium-6 nuclei which are spin-entangled with each other and the vortical gluon field;

18) White matter unitary transceiver antenna is a quantum computer, and wireless router, and bestows the self its unity of consciousness;

19) QCD entanglement of observer's self with the vortical gluon field leads to the "Measurement Problem" and the conscious agent is inextricably entangled with the "*Instanet of Things*" which includes all instruments and experiments;

20) Neural field fluctuations correlate with quantum field fluctuations and their harmonics;

21) Whitehead's actual occasions of experience are equivalent to topological spacetime instantons in the vortical gluon field (event-driven "*Instanet*" architecture);

22) Tetrahedral xenon cages in organo$^{6}Li^{+}Xe_{4}$ excimers decohere the connectome's NMR-QIP due to xenon's NMR signal being extremely sensitive to its molecular environment.

Considering this above synthesis holistically and being founded on quantum information, I propose the following synopsis. Meta-stable societal coexistence and coevolution, emerges with a semblance of order from fundamental uncertainty, as a complex quantum system where the conscious agent, other agents and the environment exert natural actions, reactions and free will (in the case of conscious agents) on each other, thus building correlations through qutrit quantum deep-learning, intertwined-histories and braided worldlines. This dynamic evolving ensemble manifests itself to a conscious agent as our Triuniverse [1] which is an "*Instanet of Things*". As Whitehead [28] said, actual occasions of experience are the "final real things of which the world is made up" and I propose these are equivalent to topological spacetime instantons in the vortical gluon field, with which our Lithium Quantum Consciousness (LQC) connects.

4. Discussion of Applications and Technologies

Biological receptors, processors and transceivers of physiological controls, perception, cognition, intelligence and consciousness are all subject to evolutionary natural selection, with advanced organisms having more capable and computationally advantageous sets of integrated sensory faculties. From this vantage point across the evolutionary fitness landscape, our species *Homo sapiens* is reasonably elevated, when compared to say *Caenorhabditis elegans* (roundworm). However, the author believes the present earthly biological level of evolution of quantum states of consciousness is unlikely to be at the maximum natural potential. This belief is an extension of universal non-anthropocentrism. More humbly, perhaps human consciousness is close to a local maximum on the evolutionary fitness landscape, but we may be currently limited to the biological "foothills" of a mountain range of conscious potential.

Indeed, the proposals herein can be technologically tested, harnessed, amplified and augmented, in manmade lithium-water synthetic brains for quantum deep-learning artificial consciousness. Such lithium-water synthetic brains could in principle surpass those which have evolved so far within the biology of our planet. Much as nuclear reactors are controlled by elements with high neutron capture cross-sections, e.g. silver, indium, boron and cadmium, the proposed level of consciousness of the lithium-water synthetic brain could be controlled by variably substituting water with anaesthetic xenon, which decoheres the connectome's NMR-QIP, with the creation of unconscious lithium-xenon excimers.

With appropriate *in vivo* clinical care and ethical controls, lithium-water synthetic brain material in impermeable vessels could be implanted within the brain, under the skull, under or exterior to the scalp, even applied topically to provide an additional proximal resource for one's own quantum mind. An ethical purpose would be to provide prosthetic medical care for patients with dissociative, bipolar and mood disorders, though for the mentally fit, augmented conscious enlightenment would also be accessible through such technologies. Lithium-water NMR-QIP technologies could also form the basis of a brain-computer interface.

Notwithstanding, lithium-water synthetic brain material could be assembled *ex vivo* into an artificial functionally dynamic algebraic topological connectome. That artificial connectome could exploit quantum deep-learning to evolve its design such that subsequently adapted versions, assembled robotically, could attain ever higher levels of consciousness. The levels to which such *ex vivo* synthetic quantum consciousness might rise are incomprehensible to present day humans. This therefore demands our utmost respect for unintended consequences and prior open debate of this technological opportunity and ethical (if not existential) threat.

5. Conclusions

An equivalence is proposed between Whitehead's occasions of experience and topological spacetime instantons in the vortical gluon field. The conscious agency of life is thereby considered to be founded upon a quantum state of mind $|\Xi\rangle$. Topological spacetime instantons pervade the vortical gluon field in a quantum information network of vortex interactions, termed the "instanton-net", or "*Instanet*" [*sic*].

The fermionic isotope lithium-6 has a very low nuclear binding energy and the smallest non-zero nuclear electric quadrupole moment of any stable nucleus making it susceptible to quantum chromodynamic (QCD) interaction with the vortical gluon field and ideal for spin-1 qutrit nuclear magnetic resonance (NMR) quantum information processing. The compact spherical atomic orbital of lithium provides ideal rotational freedom inside tetrahedral water cages in organo$^{6}Li^{+}(H_2O)_4$ within which the lithium nucleus rapidly tumbles for NMR motional narrowing and long decoherence times. The wet fatty white matter tracts of the brain's connectome are considered to be laced with trace amounts of organo$^6Li^+(H_2O)_4$.

In parallel to the connectome's processing of physiological controls, perception, cognition and intelligence via quantum electrodynamics (QED), the connectome also functions via its dynamic algebraic topology as a unitary transceiver antenna laced with lithium-6 nuclei which are spin-entangled with each other and with the environmental vortical gluon field via quantum chromodynamics (QCD). Lithium Quantum Consciousness (LQC) in sentient agents thereby intersects the vortical gluon field in spacetime and quantum information entanglements connect consciousness with the "*Instanet of Things*". QCD entanglement of an observer's conscious self with the vortical gluon field thus leads to the "Measurement Problem".

Technologies which may be developed from these ideas include spin-1 qutrit organo$^6Li^+(H_2O)_4$ NMR quantum computers which 1) *ex vivo,* exhibit artificial intelligence and artificial consciousness, 2) *in vivo*, in an impermeable vessel placed in or proximal to the brain, augment human intelligence and human consciousness, and 3) provide a brain-computer interface.

Acknowledgements

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Bell's Inequality Should Be Reconsidered in Quantum Language

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Abstract

Bell's inequality itself is usually considered to belong to mathematics and not quantum mechanics. We think that this is making our understanding of Bell' theory be confused. Thus in this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in the framework of quantum theory with the linguistic Copenhagen interpretation. And we clarify that the violation of Bell's inequality (*i.e.*, whether or not Bell's inequality holds) does not depend on whether classical systems or quantum systems, but depend on whether a combined measurement exists or not. And further we conclude that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

Keywords

Bohr-Einstein Debates, Bell's Inequality, Combined Observable, Linguistic Copenhagen Interpretation, Quantum Language

1. Review: Quantum Language (=Measurement Theory (=MT))

1.1. Introduction

Recently (*cf.* refs. [1]-[10], also see (B_0) - (B_3) later), we proposed quantum language, which was not only characterized as the metaphysical and linguistic turn of quantum mechanics but also the linguistic turn of dualistic idealism. And further we believe that quantum language should be regarded as the foundations of quantum information science. Quantum language is formulated as follows.

$$\text{(A)}\quad \underbrace{\overbrace{\boxed{\boxed{\text{Quantumlanguage}}}}^{\text{(=measurement theory)}}}_{\text{(language)}} = \underbrace{\boxed{\text{Measurement}}}_{\text{(Axiom 1)}} + \underbrace{\boxed{\text{Causality}}}_{\text{(Axiom 2)}} + \underbrace{\boxed{\text{Linguistic}(\text{Copenhagen})\text{interpretation}}}_{\text{(how to use Axioms 1 and 2)}}$$

Note that this theory (A) is not physics but a kind of language based on the quantum mechanical world view. That is, we think that the location of quantum language in the history of world-descriptions is as follows.

And in **Figure 1**, we think that the following four are equivalent (refs. [1] [8]):

(B_0) to propose quantum language (*cf.* ⑩ in **Figure 1**, ref. [1] [8]).

(B_1) to clarify the Copenhagen interpretation of quantum mechanics (*cf.* ⑦ in **Figure 1**, refs. [2] [7] [11]), that is, the linguistic Copenhagen interpretation is the true figure of so-called Copenhagen interpretation.

(B_2) to clarify the final goal of the dualistic idealism (*cf.* ⑧ in **Figure 1**, refs. [3] [9]).

(B_3) to reconstruct statistics in the dualistic idealism (*cf.* ⑨ in **Figure 1**, refs. [4] [5] [6] [12]).

In Bohr-Einstein debates (refs. [13] [14]), Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (*i.e.*, physics holds without observers)) is on the side of the realistic world view in **Figure 1**. On the other hand, we think that Bohr's standing point (that is, "*to be is to be perceived*" (*i.e.*, there is no science without measurements)) is on the side of the linguistic world view in **Figure 1** (though N. Bohr might believe that the Copenhagen interpretation (proposed by his school) belongs to physics).

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality (refs. [15] [16] [17] [18]) in quantum language (*i.e.*, quantum theory with the linguistic Copenhagen interpretation). And we clarify that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether a combined measurement exists or not (in Section 2). And further we assert that our argument (based on the linguistic Copenhagen interpretation) should be regarded as a scientific representation of Bell's philosophical argument (based on Einstein's spirit).

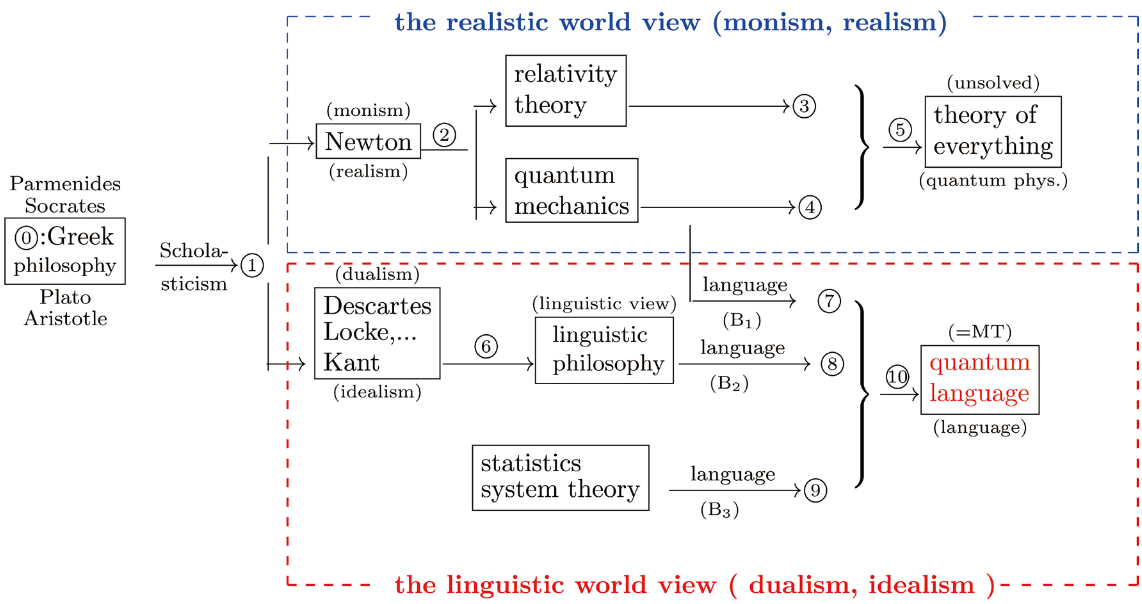

Figure 1. The history of the world-descriptions.

1.2. Quantum Language (=Measurement Theory); Mathematical Preparations

Now we shall explain the measurement theory (A).

Consider an operator algebra $B(H)$ (*i.e.*, an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$), and consider the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$, called a *basic structure*. Here, $\mathcal{A}(\subseteq B(H))$ is a C^*-algebra, and $\mathcal{N}$ ($\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)$) is a particular C^*-algebra (called a W^*-algebra) such that $\mathcal{N}$ is the weak closure of $\mathcal{A}$ in $B(H)$.

The measurement theory (=quantum language) is classified as follows.

$$
\text{(C)} \quad \begin{array}{l} \text{measurement theory}(\text{A}) \\ = \begin{cases} (\text{C}_1): \text{quantum system theory} & (\text{when } \mathcal{A} = \mathcal{C}(H)) \\ (\text{C}_2): \text{classical system theory} & (\text{when } \mathcal{A} = C_0(\Omega)) \end{cases} \end{array}
$$

That is, when $\mathcal{A} = \mathcal{C}(H)$, the C^*-algebra composed of all compact operators on a Hilbert space H, the (C_1) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the C^*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* [19] [20])), the (C_2) is called classical measurement theory (or, classical system theory).

Also, note (*cf.* [19]) that, when $\mathcal{A} = \mathcal{C}(H)$,

1) $\mathcal{A}^* = Tr(H)(=\text{trace class})$, $\mathcal{N} = B(H)$, $\mathcal{N}_* = Tr(H)$ (*i.e.*, pre-dual space).

Also, when $\mathcal{A} = C_0(\Omega)$,

2) $\mathcal{A}^* =$ "the space of all signed measures on Ω", $\mathcal{N} = L^\infty(\Omega,\nu)(\subseteq B(L^2(\Omega,\nu)))$, $\mathcal{N}_* = L^1(\Omega,\nu)$, where ν is some measure on Ω (*cf.* [19]). Also, the $L^\infty(\Omega,\nu)$ is usually denoted by $L^\infty(\Omega)$.

Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra, and let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^*$ = {ρ | ρ is a continuous linear functional on $\mathcal{A}$}, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}} (= \|F\|_{B(H)}) \le 1\}$. Define the *mixed state* $\rho(\in \mathcal{A}^*)$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \ge 0$ for all $F \in \mathcal{A}$ such that $F \ge 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$
\mathfrak{S}^m(\mathcal{A}^*) = \{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.
$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that $\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$ implies $\rho = \rho_1 = \rho_2$. Put

$$
\mathfrak{S}^p(\mathcal{A}^*) = \{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},
$$

which is called a *state space*. It is well known (*cf.* [19]) that $\mathfrak{S}^p(\mathcal{C}(H)^*) = \{|u\rangle\langle u|$

(*i.e.*, the Dirac notation) $\mid \; \|u\|_H = 1\}$, and $\mathfrak{S}^p\left(C_0(\Omega)^*\right) = \{\delta_{\omega_0} \mid \delta_{\omega_0}$ is a point measure at $\omega_0 \in \Omega\}$, where $\int_\Omega f(\omega)\delta_{\omega_0}(\mathrm{d}\omega) = f(\omega_0)$ $\left(\forall f \in C_0(\Omega)\right)$. The latter implies that $\mathfrak{S}^p\left(C_0(\Omega)^*\right)$ can be also identified with Ω (called a *spectrum space* or simply *spectrum*) such as

$$\underset{\text{(states pace)}}{\mathfrak{S}^p\left(C_0(\Omega)^*\right)} \ni \delta_\omega \leftrightarrow \omega \in \underset{\text{(spectrum)}}{\Omega} \tag{1}$$

For instance, in the above 2) we must clarify the meaning of the "value" of $F(\omega_0)$ for $F \in L^\infty(\Omega,\nu)$ and $\omega_0 \in \Omega$. An element $F(\in \mathcal{N})$ is said to be *essentially continuous at* $\rho_0\left(\in \mathfrak{S}^p\left(\mathcal{A}^*\right)\right)$, if there uniquely exists a complex number α such that

- if $\rho\left(\in \mathcal{N}_*, \|\rho\|_{\mathcal{N}_*} = 1\right)$ converges to $\rho_0\left(\in \mathfrak{S}^p\left(\mathcal{A}^*\right)\right)$ in the sense of weak* topology of $\mathcal{A}^*$, that is,

$$\rho(G) \to \rho_0(G) \; \left(\forall G \in \mathcal{A}(\subseteq \mathcal{N})\right),$$

then $\rho(F)$ converges to α.

And the value of $\rho_0(F)$ is defined by the α.

According to the noted idea (cf. [21]), an *observable* $\mathrm{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined as follows:

1) [σ-field] X is a set, $\mathcal{F}$ $(\subseteq 2^X = \mathcal{P}(X)$, the power set of X) is a σ-field of X, that is, "$\Xi_1, \Xi_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^\infty \Xi_n \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$", "$X \in \mathcal{F}$".

2) [Countable additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{N}$ satisfying: a) for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $\mathcal{N}$ such that $0 \le F(\Xi) \le I$, b) $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \cdots, \Xi_n, \cdots\}$ of Ξ (*i.e.*, $\Xi, \Xi_n \in \mathcal{F}$ $(n = 1,2,3,\cdots)$, $\bigcup_{n=1}^\infty \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ ($i \ne j$)), it holds that $F(\Xi) = \sum_{n=1}^\infty F(\Xi_n)$ in the sense of weak* topology in $\mathcal{N}$.

Remark 1. Quantum language has two formulations (*i.e.*, the C^*-algebraic formulation and the W^*-algebraic formulation). In this paper, we devote ourselves to the W^*-algebraic formulation, which may, from the mathematical point of view, be superiority to the C^*-algebraic formulation. That is, in the above 2), the countable additivity (*i.e.*, $F(\Xi) = \lim_{N\to\infty} \sum_{n=1}^N F(\Xi_n)$) is naturally discussed in the W^*-algebraic formulation. However, the C^*-algebraic formulation has a merit such that we can use it without sufficient mathematical preparation. For the C^*-algebraic version of this paper, see my preprint [10].

1.3. Axiom 1 [Measurement] and Axiom 2 [Causality]

With any *system S*, a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$ can be associated in which the measurement theory (A) of that system can be formulated. A *state* of the system S is represented by an element $\rho\left(\in \mathfrak{S}^p\left(\mathcal{A}^*\right)\right)$ and an *observable* is represented by an observable $\mathrm{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$. Also, the *measurement of the observable* O *for the system* S *with the state* ρ is denoted by $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}, S_{[\rho]}\right)$ (or more precisely, $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O} := (X, \mathcal{F}, F), S_{[\rho]}\right)$). An observer can obtain a measured value

$x\,(\in X)$ by the measurement $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}, S_{[\rho]}\right)$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics (*cf.* ref. [22]). And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^*-algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value* $x\ (\in X)$ *obtained by the measurement* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O} := (X, \mathcal{F}, F), S_{[\rho]}\right)$ *belongs to a set* $\Xi(\in \mathcal{F})$ *is given by* $\rho(F(\Xi))$ *if* $F(\Xi)$ *is essentially continuous at* $\rho\left(\in \mathfrak{S}^p\left(\mathcal{A}^*\right)\right)$.

Next, we explain Axiom 2. Let $[\mathcal{A}_1, \mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2, \mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : \mathcal{N}_2$ (with weak* topology) $\to \mathcal{N}_1$ (with weak* topology) is called a *Markov operator,* if it satisfies that 1) $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element F_2 in $\mathcal{N}_2$, 2) $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{N}_k$, $(k = 1, 2)$. In addition to the above 1) and 2), in this paper we assume that $\Phi_{1,2}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and $\sup\left\{\left\|\Phi_{1,2}(F_2)\right\|_{\mathcal{A}_1} \mid F_2 \in \mathcal{A}_2 \text{ such that } \|F_2\|_{\mathcal{A}_2} \leq 1\right\} = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*\left(\mathfrak{S}^m\left(\mathcal{A}_1^*\right)\right) \subseteq \mathfrak{S}^m\left(\mathcal{A}_2^*\right)$. If it holds that $\Phi_{1,2}^*\left(\mathfrak{S}^p\left(\mathcal{A}_1^*\right)\right) \subseteq \mathfrak{S}^p\left(\mathcal{A}_2^*\right)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is said to be non-deterministic or decoherence. Here note that, for any observable $\mathrm{O}_2 := (X, \mathcal{F}, F_2)$ in $\mathcal{N}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$.

Now Axiom 2 in the measurement theory (A) is presented as follows:

Axiom 2 [Causality]. *Let* $t_1 \leq t_2$. *The causality is represented by a Markov operator* $\Phi_{t_1,t_2} : N_{t_2} \to N_{t_1}$.

1.4. The Linguistic Interpretation (=The Manual to Use Axioms 1 and 2)

In the above, Axioms 1 and 2 are kinds of spells, (*i.e.*, incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation. However, it is better to know the linguistic interpretation (=the manual to use Axioms 1 and 2), if we would like to make progress quantum language early.

The essence of the manual is as follows:

(D) **Only one measurement is permitted.** And thus, the state after a measurement is meaningless since it cannot be measured any longer. Thus, the collapse of the wavefunction is prohibited (*cf.* [7]). We are not concerned with anything after measurement. That is, any statement including the phrase *after the measurement* is wrong. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited. Also, it is added that there is no probability without a measurement.

and so on. For details, see [8].

1.5. Generalized Simultaneous Measurement, Parallel Measurement

Definition 2. [Generalized simultaneous observable, Generalized simultaneous measurement] Let $[\mathcal{A}, \mathcal{N}]_{B(H)}$ be a basic structure. Consider observables $\mathrm{O}_k = (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, \cdots, K)$ in $\mathcal{N}$. Let $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$ be the product measurable space, *i.e.*, the product space $\times_{k=1}^K X_k$ and the product σ-field $\boxtimes_{k=1}^K \mathcal{F}_k$, which is defined by the smallest σ-field that contains a family $\{\times_{k=1}^K \Xi_k \mid \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K\}$. An observable $\mathrm{O} = \overset{\mathrm{qp}}{\times}_{k=1,2,\cdots,K} \mathrm{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ in $\mathcal{N}$ is called the generalized simultaneous observable (or, quasi-product observable, combined observable, etc.) of O_k $(k = 1, 2, \cdots, K)$, if it holds that

$$\begin{aligned} &F(X_1 \times X_2 \times \cdots \times X_{k-1} \times \Xi_k \times X_{k+1} \times \cdots \times X_K) \\ &= F_k(\Xi_k), (\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K) \end{aligned} \tag{2}$$

Also, the measurement $\mathrm{M}_{\mathcal{N}}(\mathrm{O}, S_{[\rho_0]})$ is called a generalized simultaneous measurement of measurements $\mathrm{M}_{\mathcal{N}}(\mathrm{O}_k, S_{[\rho_0]})$ $(k = 1, 2, \cdots, K)$. A generalized simultaneous observable is called a simultaneous observable, if it holds:

$$\begin{aligned} &F(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_{k-1} \times \Xi_k \times \Xi_{k+1} \times \cdots \times \Xi_K) \\ &= \underset{k=1}{\overset{K}{\times}} F_k(\Xi_k), (\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K) \end{aligned}$$

Note that the existence and the uniqueness of a generalized simultaneous observable $\mathrm{O} = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, F)$ in $\mathcal{N}$ are not assured in general, however the simultaneous observable always exists if observables O_k $(k = 1, 2, \cdots, K)$ commute, *i.e.*,

$$F_k(\Xi_k) F_l(\Xi_l) = F_l(\Xi_l) F_k(\Xi_k), (\forall \Xi_k \in \mathcal{F}_k, \forall \Xi_l \in \mathcal{F}_l, k \neq l) \tag{3}$$

Definition 3. [Parallel observable, Parallel measurement] For each $k = 1, 2, \cdots, K$, consider a basic structure $[\mathcal{A}_k, \mathcal{N}_k]_{B(H_k)}$ and a measurement $\mathrm{M}_{\mathcal{N}_k}(\mathrm{O}_k := (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$. We consider the spatial tensor W^*-algebra $\otimes_{k=1}^K \mathcal{N}_k (\subseteq B(\otimes_{k=1}^K H_k))$, and consider the product measurable space $(\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k)$. Consider the observable $\otimes_{k=1}^K \mathrm{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \tilde{F})$ in $\bigotimes_{k=1}^K \mathcal{N}_k$ such that

$$\tilde{F}(\times_{k=1}^K \Xi_k) = \otimes_{k=1}^K F_k(\Xi_k) \quad (\forall \Xi_k \in \mathcal{F}_k, k = 1, 2, \cdots, K).$$

which is called the parallel observable of $\mathrm{O}_k := (X_k, \mathcal{F}_k, F_k)$ $(k = 1, 2, \cdots, K)$. And let $\bigotimes_{k=1}^K \rho_k \in \mathfrak{S}^p\left(\left(\bigotimes_{k=1}^K \mathcal{A}_k\right)^*\right)$. Then the measurement $\mathrm{M}_{\otimes_{k=1}^K \mathcal{N}_k}\left(\bigotimes_{k=1}^K \mathrm{O}_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \bigotimes_{k=1}^K F_k), S_{[\otimes_{k=1}^K \rho_k]}\right)$ (which is also denoted by $\otimes_{k=1}^K \mathrm{M}_{\mathcal{N}_k}(\mathrm{O}_k, S_{[\rho_k]})$) is called a parallel measurement of $\mathrm{M}_{\mathcal{N}_k}(\mathrm{O}_k = (X_k, \mathcal{F}_k, F_k), S_{[\rho_k]})$ $(k = 1, 2, \cdots, K)$. Note that the parallel measurement always exists uniquely.

2. Bell's Inequality Always Holds in Classical and Quantum Systems

Our Main Assertion about Bell's Inequality

In this paper, I assert that Bell's inequality should be studied in the framework of quantum language (*i.e.*, quantum theory with the linguistic Copenhagen interpretation). Let us start from the following definition, which is a slight modification of the generalized simultaneous observable in Definition 2. That is, Definitions 2 - 4 are due to the linguistic Copenhagen interpretation (D), "Only one measurement is permitted".

Definition 4 [Combined observable (*cf.* ref. [12])] *Let* $[\mathcal{A},\mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X=\{-1,1\}$. *Consider four observables:* $\mathrm{O}_{13}=\left(X^2,\mathcal{P}(X^2),F_{13}\right)$, $\mathrm{O}_{14}=\left(X^2,\mathcal{P}(X^2),F_{14}\right)$, $\mathrm{O}_{23}=\left(X^2,\mathcal{P}(X^2),F_{23}\right)$, $\mathrm{O}_{24}=\left(X^2,\mathcal{P}(X^2),F_{24}\right)$ *in* $\mathcal{N}$. *The four observables are said to be combinable if there exists an observable* $\mathrm{O}=\left(X^4,\mathcal{P}(X^4),F\right)$ *in* $\mathcal{N}$ *such that*

$$
\begin{aligned}
F_{13}\left(\{(x_1,x_3)\}\right)&=F\left(\{x_1\}\times X\times\{x_3\}\times X\right),\\
F_{14}\left(\{(x_1,x_4)\}\right)&=F\left(\{x_1\}\times X\times X\times\{x_4\}\right),\\
F_{23}\left(\{(x_2,x_3)\}\right)&=F\left(X\times\{x_2\}\times\{x_3\}\times X\right),\\
F_{24}\left(\{(x_2,x_4)\}\right)&=F\left(X\times\{x_2\}\times X\times\{x_4\}\right)
\end{aligned}
\tag{4}
$$

for any $(x_1,x_2,x_3,x_4)\in X^4$. *The observable* O *is said to be a combined observable of* O_{ij} $(i=1,2,\ j=3,4)$. *Also, the measurement* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}=\left(X^4,\mathcal{P}(X^4),F\right),S_{[\rho_0]}\right)$ *is called the combined measurement of* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{13},S_{[\rho_0]}\right)$, $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{14},S_{[\rho_0]}\right)$, $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{23},S_{[\rho_0]}\right)$ *and* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{24},S_{[\rho_0]}\right)$.

Remark 5. 1) Note that the Formula (4) implies that

$$
F_{13}\left(\{x\}\times X\right)=F_{14}\left(\{x\}\times X\right),\quad F_{23}\left(\{x\}\times X\right)=F_{24}\left(\{x\}\times X\right),
$$

$$
F_{13}\left(X\times\{x\}\right)=F_{23}\left(X\times\{x\}\right),\quad F_{14}\left(X\times\{x\}\right)=F_{24}\left(X\times\{x\}\right),
$$

for all $x\in X$.

2) Syllogism (*i.e.*, $[[A\Rightarrow B]\wedge[B\Rightarrow C]]\Rightarrow[A\Rightarrow C]$) does not hold in quantum systems but in classical systems (*cf.* ref. [8]). A certain combined observable plays an important role in the proof of the classical syllogism (*cf.* ref. [12]).

The following theorem is all of our insistence concerning Bell's inequality. We assert that this is the true Bell's inequality.

Theorem 6. [Bell's inequality in quantum language] *Let* $[\mathcal{A},\mathcal{N}]_{B(H)}$ *be a basic structure. Put* $X=\{-1,1\}$. *Fix the pure state* $\rho_0\left(\in\mathfrak{S}^p(\mathcal{A}^*)\right)$. *And consider the four measurements* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{13}=\left(X^2,\mathcal{P}(X^2),F_{13}\right),S_{[\rho_0]}\right)$, $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{14}=\left(X^2,\mathcal{P}(X^2),F_{14}\right),S_{[\rho_0]}\right)$, $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{23}=\left(X^2,\mathcal{P}(X^2),F_{23}\right),S_{[\rho_0]}\right)$ *and* $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{24}=\left(X^2,\mathcal{P}(X^2),F_{24}\right),S_{[\rho_0]}\right)$. *Or equivalently, consider the parallel measurement* $\otimes_{i=1,2,j=3,4}\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{ij}=\left(X^2,\mathcal{P}(X^2),F_{ij}\right),S_{[\rho_0]}\right)$. *Define four correlation functions* $(i=1,2,\ j=3,4)$ *such that*

$$R_{ij} = \sum_{(u,v)\in X\times X} u\cdot v\ \rho_0\left(F_{ij}\left(\{(u,v)\}\right)\right)$$

Assume that four observables $O_{13}=\left(X^2,\mathcal{P}\left(X^2\right),F_{13}\right)$, $O_{14}=\left(X^2,\mathcal{P}\left(X^2\right),F_{14}\right)$, $O_{23}=\left(X^2,\mathcal{P}\left(X^2\right),F_{23}\right)$ and $O_{24}=\left(X^2,\mathcal{P}\left(X^2\right),F_{24}\right)$ are combinable, that is, we have the combined observable $O=\left(X^4,\mathcal{P}\left(X^4\right),F\right)$ in $\mathcal{N}$ such that it satisfies (4). Then we have a combined measurement $M_{\mathcal{N}}\left(O=\left(X^4,\mathcal{P}\left(X^4\right),F\right),S_{[\rho_0]}\right)$ of $M_{\mathcal{N}}\left(O_{13},S_{[\rho_0]}\right)$, $M_{\mathcal{N}}\left(O_{14},S_{[\rho_0]}\right)$, $M_{\mathcal{N}}\left(O_{23},S_{[\rho_0]}\right)$ and $M_{\mathcal{N}}\left(O_{24},S_{[\rho_0]}\right)$. And further, we have Bell's inequality in quantum language as follows.

$$\left|R_{13}-R_{14}\right|+\left|R_{23}+R_{24}\right|\le 2 \tag{5}$$

Proof. Clearly we see, $i=1,2,\ j=3,4$,

$$R_{ij} = \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_i\cdot x_j\ \rho_0\left(F\left(\{(x_1,x_2,x_3,x_4)\}\right)\right) \tag{6}$$

(for example, $R_{13}=\sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X} x_1\cdot x_3\ \rho_0\left(F\left(\{(x_1,x_2,x_3,x_4)\}\right)\right)$). Therefore, we see that

$$\begin{aligned}
&\left|R_{13}-R_{14}\right|+\left|R_{23}+R_{24}\right| \\
&= \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X}\left[\left|x_1\cdot x_3-x_1\cdot x_4\right|+\left|x_2\cdot x_3+x_2\cdot x_4\right|\right]\rho_0\left(F\left(\{(x_1,x_2,x_3,x_4)\}\right)\right) \\
&= \sum_{(x_1,x_2,x_3,x_4)\in X\times X\times X\times X}\left[\left|x_3-x_4\right|+\left|x_3+x_4\right|\right]\rho_0\left(F\left(\{(x_1,x_2,x_3,x_4)\}\right)\right)\le 2
\end{aligned}$$

This completes the proof.

As the corollary of this theorem, we have the followings:

Corollary 7. *Consider the parallel measurement* $\otimes_{i=1,2,j=3,4} M_{\mathcal{N}}\left(O_{ij}=\left(X^2,\mathcal{P}\left(X^2\right),F_{ij}\right),S_{[\rho_0]}\right)$ *as in Theorem* 6. *Let*

$$x=\left(\left(x_{13}^1,x_{13}^2\right),\left(x_{14}^1,x_{14}^2\right),\left(x_{23}^1,x_{23}^2\right),\left(x_{24}^1,x_{24}^2\right)\right)\in X^8\left(\equiv\{-1,1\}^8\right)$$

be a measured value of the parallel measurement $\otimes_{i=1,2,j=3,4} M_{\mathcal{N}}\left(O_{ij}=\left(X^2,\mathcal{P}\left(X^2\right),F_{ij}\right),S_{[\rho_0]}\right)$. Let N be sufficiently large natural number. Consider N-parallel measurement $\bigotimes_{n=1}^{N}\left[\otimes_{i=1,2,j=2,3} M_{\mathcal{N}}\left(O_{ij}=\left(X^2,\mathcal{P}\left(X^2\right),F_{ij}\right),S_{[\rho_0]}\right)\right]$. Let $\{x^n\}_{n=1}^N$ be the measured value. That is,

$$\{x^n\}_{n=1}^N=\begin{bmatrix}
\left(\left(x_{13}^{1,1},x_{13}^{2,1}\right),\left(x_{14}^{1,1},x_{14}^{2,1}\right),\left(x_{23}^{1,1},x_{23}^{2,1}\right),\left(x_{24}^{1,1},x_{24}^{2,1}\right)\right)\\
\left(\left(x_{13}^{1,2},x_{13}^{2,2}\right),\left(x_{14}^{1,2},x_{14}^{2,2}\right),\left(x_{23}^{1,2},x_{23}^{2,2}\right),\left(x_{24}^{1,2},x_{24}^{2,2}\right)\right)\\
\vdots\\
\left(\left(x_{13}^{1,N},x_{13}^{2,N}\right),\left(x_{14}^{1,N},x_{14}^{2,N}\right),\left(x_{23}^{1,N},x_{23}^{2,N}\right),\left(x_{24}^{1,N},x_{24}^{2,N}\right)\right)
\end{bmatrix}\in\left(X^8\right)^N$$

Here, note that the law of large numbers says: for sufficiently large N,

$$R_{ij}\approx\frac{1}{N}\sum_{n=1}^{N}x_{ij}^{1,n}x_{ij}^{2,n}\quad\left(i=1,2,\ j=3,4\right)$$

Then, it holds, by the Formula (5), that

$$\left| \sum_{n=1}^{N} \frac{x_{13}^{1,n} x_{13}^{2,n}}{N} - \sum_{n=1}^{N} \frac{x_{14}^{1,n} x_{14}^{2,n}}{N} \right| + \left| \sum_{n=1}^{N} \frac{x_{23}^{1,n} x_{23}^{2,n}}{N} + \sum_{n=1}^{N} \frac{x_{24}^{1,n} x_{24}^{2,n}}{N} \right| \leq 2, \tag{7}$$

which is also called Bell's inequality in quantum language.

Remark 8. [The conventional Bell's inequality (*cf.* refs. [17] [16] [18])] The mathematical Bell's inequality is as follows: Let $(\Theta, \mathcal{B}, P)$ be a probability space. Let $(f_1, f_2, f_3, f_4): \Theta \to X^4 \left(\equiv \{-1,1\}^4\right)$ be a measurable functions. Define the correlation functions $\hat{R}_{ij}\,(i=1,2,\ j=3,4)$ by $\int_\Theta f_i(\theta) f_j(\theta) P(\mathrm{d}\theta)$. Then, the following mathematical Bell's inequality (or precisely, CHSH inequality (*cf.* ref. [16])) holds:

$$\left| \tilde{R}_{13} - \tilde{R}_{14} \right| + \left| \tilde{R}_{23} + \tilde{R}_{24} \right| \leq 2 \tag{8}$$

(E) This is easily proved as follows.

$$\begin{aligned} &\text{“the left-hand side of the above (8)”} \\ &\leq \int_\Theta \left| f_3(\theta) - f_4(\theta) \right| P(\mathrm{d}\theta) + \int_\Theta \left| f_3(\theta) + f_4(\theta) \right| P(\mathrm{d}\theta) \leq 2 \end{aligned}$$

This completes the proof.

Recall Theorem 6 (Bell's inequality in quantum language), in which we have, by the combinable condition, the probability space $\left(X^4, \mathcal{P}\left(X^4\right), \rho_0\left(F(\cdot)\right)\right)$. Therefore the proof of Theorem 6 and the above proof (E) are, from the mathematical point of view, the same.

3. "Bell's Inequality" Is Violated in Classical Systems as Well as Quantum Systems

In the previous section, we show that Theorem 6 (or Corollary 7) says

(F_1) Under the combinable condition (*cf.* Definition 4), Bell's Inequality (5) (or, (7)) holds in both classical systems and quantum systems.

Or, equivalently,

(F_2) If Bell's Inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the measured value (by the combined measurement).

This is similar to the following elementary statement in quantum mechanics:

(F_2') We have no (generalized) simultaneous measurement of the position observable Q and the momentum observable P, and thus we cannot obtain the *measured value* (by the generalized simultaneous measurement),

which may be, from Einstein's point of view, represented that "*true value* (or, *hidden variable*) of the position and momentum" does not exist. Since the error Δ is usually defined by $\Delta = \left|\text{rough measured value} - \text{true value}\right|$, it is not easy to define the errors Δ_Q and Δ_P in Heisenberg's uncertainty principle $\Delta_Q \cdot \Delta_P \geq \hbar/2$.

This definition was completed and Heisenberg's uncertainty principle was proved in ref. [11]. Also, according to the maxim of dualism: "To be is to be perceived" due to G. Berkeley, we think that it is not necessary to name that does not exist (or equivalently, that is not measured).

The above statement (F_2) makes us expect that

(G) Bell's inequality (5) (or (7)) is violated in classical systems as well as quantum systems without the combinable condition.

This (G) was already shown in my previous paper [2]. However, I received a lot of questions concerning (G) from the readers. Thus, in this section, we again explain the (G) precisely.

Bell Test Experiment

In order to show the (G), three steps ([Step: I] - [Step: III]) are prepared in what follows.

[Step: I].

Put $X = \{-1,1\}$. Define complex numbers a_k $(= \alpha_k + \beta_k \sqrt{-1} \in \mathbb{C}$: the complex field) $(k = 1,2,3,4)$ such that $|a_k| = 1$. Define the probability space $\left(X^2, \mathcal{P}\left(X^2\right), \nu_{a_i a_j}\right)$ such that $(i = 1,2,\ j = 3,4)$

$$\begin{aligned} \nu_{a_i a_j}\left(\{(1,1)\}\right) = \nu_{a_i a_j}\left(\{(-1,-1)\}\right) = \left(1 - \alpha_i\alpha_j - \beta_i\beta_j\right)/4 \\ \nu_{a_i a_j}\left(\{(-1,1)\}\right) = \nu_{a_i a_j}\left(\{(1,-1)\}\right) = \left(1 + \alpha_i\alpha_j + \beta_i\beta_j\right)/4 \end{aligned} \tag{9}$$

The correlation $R\left(a_i, a_j\right)$ $(i = 1,2,\ j = 3,4)$ is defined as follows:

$$R\left(a_i, a_j\right) \equiv \sum_{(x_1,x_2)\in X\times X} x_1 \cdot x_2 \nu_{a_i a_j}\left(\{(x_1,x_2)\}\right) = -\alpha_i\alpha_j - \beta_i\beta_j \tag{10}$$

Now we have the following problem:

(H) Find a measurement $\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{a_i a_j} = \left(X^2, \mathcal{P}\left(X^2\right), F_{a_i a_j}\right), S_{[\rho_0]}\right)$ $(i = 1,2,\ j = 3,4)$ such that

$$\rho_0\left(F_{a_i a_j}(\Xi)\right) = \nu_{a_i a_j}(\Xi) \quad \left(\forall \Xi \in \mathcal{P}\left(X^2\right)\right) \tag{11}$$

and

$$F_{a_1 a_3}\left(\{x_1\}\times X\right) = F_{a_1 a_4}\left(\{x_1\}\times X\right), \quad F_{a_2 a_3}\left(\{x_2\}\times X\right) = F_{a_2 a_4}\left(\{x_2\}\times X\right)$$

$$F_{a_1 a_3}\left(X\times\{x_3\}\right) = F_{a_2 a_3}\left(X\times\{x_3\}\right), \quad F_{a_1 a_4}\left(X\times\{x_4\}\right) = F_{a_2 a_4}\left(X\times\{x_4\}\right)$$

$$\left(\forall x_k \in X\left(\equiv \{-1,1\}\right), k = 1,2,3,4\right)$$

which is the same as the condition in Remark 5.

[Step: II].

Let us answer this problem (H) in the two cases (*i.e.*, classical case and quantum case), that is,

$$\bullet \begin{cases} \text{1) the case of quantum systems :} \\ \left[\mathcal{A} = B\left(\mathbb{C}^2\right)\otimes B\left(\mathbb{C}^2\right)\left(\equiv B\left(\mathbb{C}^2\otimes\mathbb{C}^2\right)\right), \mathcal{N} = B\left(\mathbb{C}^2\right)\otimes B\left(\mathbb{C}^2\right)\right] \\ \text{2) the case of classical systems :} \\ \left[\mathcal{A} = C_0(\Omega)\otimes C_0(\Omega)\left(\equiv C_0(\Omega\times\Omega)\right), \mathcal{N} = L^\infty(\Omega)\otimes L^\infty(\Omega)\right] \end{cases}$$

1) the case of quantum system: $\left[\mathcal{A} = B\left(\mathbb{C}^2\right)\otimes B\left(\mathbb{C}^2\right)\right]$

Put

$$e_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\end{bmatrix} \quad \left(\in \mathbb{C}^2\right)$$

For each $a_k\ (k=1,2,3,4)$, define the observable $\mathrm{O}_{a_k} \equiv \left(X, \mathcal{P}(X), G_{a_k}\right)$ in $B(\mathbb{C}^2)$ such that

$$G_{a_k}(\{1\}) = \frac{1}{2}\begin{bmatrix} 1 & \bar{a}_k \\ a_k & 1 \end{bmatrix},\ G_{a_k}(\{-1\}) = \frac{1}{2}\begin{bmatrix} 1 & -\bar{a}_k \\ -a_k & 1 \end{bmatrix}$$

where $\bar{a}_k = \alpha_k - \beta_k\sqrt{-1}$. Then, we have four observable:

$$\hat{\mathrm{O}}_{a_i} = \left(X, \mathcal{P}(X), G_{a_i} \otimes I\right),\quad \hat{\mathrm{O}}_{a_j} = \left(X, \mathcal{P}(X), I \otimes G_{a_j}\right)\quad (i=1,2,\ j=3,4) \tag{12}$$

and further,

$$\mathrm{O}_{a_i a_j} = \left(X^2, \mathcal{P}(X^2), F_{a_i a_j} := G_{a_i} \otimes G_{a_j}\right)\quad (i=1,2,\ j=3,4) \tag{13}$$

in $B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)$, where it should be noted that $F_{a_i a_j}$ is separated by G_{a_i} and G_{a_j}.

Further define the singlet state $\rho_0 = |\psi_s\rangle\langle\psi_s|\ \left(\in S^p\left(B(\mathbb{C}^2 \otimes \mathbb{C}^2)^*\right)\right)$, where

$$\psi_s = (e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2}$$

Thus we have the measurement $\mathrm{M}_{B(\mathbb{C}^2\otimes\mathbb{C}^2)^*}\left(\mathrm{O}_{a_i a_j}, S_{[\rho_0]}\right)$ in $B(\mathbb{C}^2)\otimes B(\mathbb{C}^2)$ $(i=1,2,\ j=3,4)$. The followings are clear: for each $(x_1,x_2) \in X^2\left(\equiv \{-1,1\}^2\right)$,

$$\begin{aligned}\rho_0\left(F_{a_i a_j}(\{(x_1,x_2)\})\right) &= \left\langle \psi_s, \left(G_{a_i}(\{x_1\}) \otimes G_{a_j}(\{x_2\})\right)\psi_s\right\rangle \\ &= \nu_{a_i a_j}(\{(x_1,x_2)\})\quad (i=1,2,\ j=3,4)\end{aligned} \tag{14}$$

For example, we easily see:

$$\rho_0\left(F_{a_i b_j}(\{(1,1)\})\right) = \left\langle \psi_s, \left(G_{a_i}(\{1\}) \otimes G_{a_j}(\{1\})\right)\psi_s\right\rangle$$

$$= \frac{1}{8}\left\langle (e_1\otimes e_2 - e_2 \otimes e_1), \left(\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1\end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1\end{bmatrix}\right)(e_1\otimes e_2 - e_2\otimes e_1)\right\rangle$$

$$= \frac{1}{8}\left\langle \left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right), \left(\begin{bmatrix} 1 & \bar{a}_i \\ a_i & 1\end{bmatrix} \otimes \begin{bmatrix} 1 & \bar{a}_j \\ a_j & 1\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right)\right\rangle$$

$$= \frac{1}{8}\left\langle \left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right), \left(\begin{bmatrix}1\\a_i\end{bmatrix}\otimes\begin{bmatrix}\bar{a}_j\\1\end{bmatrix} - \begin{bmatrix}\bar{a}_i\\1\end{bmatrix}\otimes\begin{bmatrix}1\\a_j\end{bmatrix}\right)\right\rangle$$

$$= \frac{1}{8}\left(2 - \alpha_i\bar{a}_j - \bar{a}_i a_j\right) = \left(1 - \alpha_i\alpha_j - \beta_i\beta_j\right)/4 = \nu_{a_i a_j}(\{(1,1)\})$$

Therefore, the measurement $\mathrm{M}_{B(\mathbb{C}^2\otimes\mathbb{C}^2)}\left(\mathrm{O}_{a_i a_j}, S_{[\rho_0]}\right)$ satisfies the condition (H).

2) the case of classical systems: $\left[\mathcal{A} = C_0(\Omega)\otimes C_0(\Omega) = C_0(\Omega\times\Omega)\right]$

Put $\omega_0\left(=(\omega_0', \omega_0'')\right) \in \Omega\times\Omega$, $\rho_0 = \delta_{\omega_0}\ (\in \mathfrak{S}^p\left(C_0(\Omega\times\Omega)^*\right)$, *i.e.*, the point measure at ω_0)). Define the observable $\mathrm{O}_{a_i a_j} := \left(X^2, \mathcal{P}(X^2), F_{a_i a_j}\right)$ in $L^\infty(\Omega\times\Omega)$ such that

$$\left[F_{a_i a_j}(\{(x_1,x_2)\})\right](\omega) = \nu_{a_i a_j}(\{(x_1,x_2)\})$$
$$\left(\forall (x_1,x_2) \in X^2, i=1,2,\ j=3,4, \forall\omega \in \Omega\times\Omega\right)$$

Thus, we have four observables

$$\mathrm{O}_{a_i a_j} = \left(X^2, \mathcal{P}\left(X^2\right), F_{a_i a_j} \right) \quad (i = 1,2,\ j = 3,4) \tag{15}$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the formula (13)). Then, it is clear that the measurement $\mathrm{M}_{C_0(\Omega\times\Omega)}\left(\mathrm{O}_{a_i a_j}, S_{\left[\delta_{\omega_0}\right]}\right)$ satisfies the condition (H).

2)' the case of classical systems: $\left[\mathcal{A} = C_0(\Omega) \otimes C_0(\Omega) = C_0(\Omega \times \Omega)\right]$

It is easy to show a lot of different answers from the above 2). For example, as a slight generalization of (9), define the probability measure $\nu^t_{a_i a_j}\ (0 \le t \le 1)$ such that

$$\begin{aligned} \nu^t_{a_i a_j}\left(\{(1,1)\}\right) = \nu^t_{a_i a_j}\left(\{(-1,-1)\}\right) = \left(1 - t\left(\alpha_i \alpha_j + \beta_i \beta_j\right)\right)/4 \\ \nu^t_{a_i a_j}\left(\{(-1,1)\}\right) = \nu^t_{a_i a_j}\left(\{(1,-1)\}\right) = \left(1 + t\left(\alpha_i \alpha_j + \beta_i \beta_j\right)\right)/4 \end{aligned} \tag{16}$$

And consider the real-valued continuous function $t\left(\in C_0(\Omega\times\Omega)\right)$ such that $0 \le t(\omega', \omega'') \le 1$ $\left(\forall \omega = (\omega', \omega'') \in \Omega \times \Omega\right)$. And assume that $t(\omega_0) = 1$ for some $\omega_0 \left(= (\omega_0', \omega_0'')\right) \in \Omega \times \Omega$, $\rho_0 = \delta_{\omega_0}$ $\left(\in \mathfrak{S}^p\left(C_0(\Omega\times\Omega)^*\right)\right.$, *i.e.*, the point measure at ω_0)). Define the observable $\mathrm{O}_{a_i a_j} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_i a_j}\right)$ in $L^\infty(\Omega\times\Omega)$ such that

$$\begin{aligned} &\left[F_{a_i a_j}\left(\{(x_1, x_2)\}\right)\right](\omega) = \nu^{t(\omega)}_{a_i a_j}\left(\{(x_1, x_2)\}\right) \\ &\left(\forall (x_1, x_2) \in X^2, i = 1,2,\ j = 3,4, \forall \omega \in \Omega \times \Omega\right) \end{aligned} \tag{17}$$

Thus, we have four observables

$$\mathrm{O}_{a_i a_j} = \left(X^2, \mathcal{P}\left(X^2\right), F_{a_i a_j} \right) \quad (i = 1,2,\ j = 3,4)$$

in $L^\infty(\Omega \times \Omega)$ (though the variables are not separable (*cf.* the Formula (13)). Then, it is clear that the measurement $\mathrm{M}_{L^\infty(\Omega\times\Omega)}\left(\mathrm{O}_{a_i a_j}, S_{\left[\delta_{\omega_0}\right]}\right)$ satisfies the condition (H).

[Step: III].

As defined by (9), consider four complex numbers $a_k \left(= \alpha_k + \beta_k \sqrt{-1}; k = 1,2,3,4\right)$ such that $|a_k| = 1$. Thus we have four observables

$$\mathrm{O}_{a_1 a_3} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_1 a_3}\right), \quad \mathrm{O}_{a_1 a_4} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_1 a_4}\right),$$

$$\mathrm{O}_{a_2 a_3} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_2 a_3}\right), \quad \mathrm{O}_{a_2 a_4} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_2 a_4}\right),$$

in $\mathcal{N}$. Thus, we have the parallel measurement

$\otimes_{i=1,2,j=3,4} \mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{a_i a_j} := \left(X^2, \mathcal{P}\left(X^2\right), F_{a_i a_j}\right), S_{[\rho_0]}\right)$ in $\otimes_{i=1,2,j=3,4} \mathcal{N}$.

Thus, putting

$$a_1 = \sqrt{-1},\ a_2 = 1,\ a_3 = \frac{1 + \sqrt{-1}}{\sqrt{2}},\ a_4 = \frac{1 - \sqrt{-1}}{\sqrt{2}},$$

we see, by (10), that

$$\left|R(a_1, a_3) - R(a_1, a_4)\right| + \left|R(a_2, a_3) + R(a_2, a_4)\right| = 2\sqrt{2} \tag{18}$$

Further, assume that the measured value is $x\left(\in X^8\right)$. That is,

$$x = \left(\left(x^1_{13}, x^2_{13}\right), \left(x^1_{14}, x^2_{14}\right), \left(x^1_{23}, x^2_{23}\right), \left(x^1_{24}, x^2_{24}\right)\right) \in \times_{i,j=1,2} X^2 \left(\equiv \{-1,1\}^8\right)$$

Let N be sufficiently large natural number. Consider N-parallel measurement $\bigotimes_{n=1}^{N}\left[\otimes_{i=1,2,j=3,4}\mathrm{M}_{\mathcal{N}}\left(\mathrm{O}_{a_i a_j}:=\left(X^2,\mathcal{P}\left(X^2\right),F_{a_i a_j}\right),S_{[\rho_0]}\right)\right]$. Assume that its measured value is $\left\{x^n\right\}_{n=1}^{N}$. That is,

$$\left\{x^n\right\}_{n=1}^{N}=\begin{bmatrix} \left(\left(x_{13}^{1,1},x_{13}^{2,1}\right),\left(x_{14}^{1,1},x_{14}^{2,1}\right),\left(x_{23}^{1,1},x_{23}^{2,1}\right),\left(x_{24}^{1,1},x_{24}^{2,1}\right)\right) \\ \left(\left(x_{13}^{1,2},x_{13}^{2,2}\right),\left(x_{14}^{1,2},x_{14}^{2,2}\right),\left(x_{23}^{1,2},x_{23}^{2,2}\right),\left(x_{24}^{1,2},x_{24}^{2,2}\right)\right) \\ \vdots \\ \left(\left(x_{13}^{1,N},x_{13}^{2,N}\right),\left(x_{14}^{1,N},x_{14}^{2,N}\right),\left(x_{23}^{1,N},x_{23}^{2,N}\right),\left(x_{24}^{1,N},x_{24}^{2,N}\right)\right) \end{bmatrix}$$

$$\in\left(\underset{i=1,2,j=3,4}{\times}X^2\right)^N\left(\equiv\{-1,1\}^{8N}\right)$$

Then, the law of large numbers says that

$$R\left(a_i,a_j\right)\approx\frac{1}{N}\sum_{n=1}^{N}x_{ij}^{1,n}x_{ij}^{2,n}\quad\left(i=1,2,\ j=3,4\right)$$

This and the Formula (18) say that

$$\left|\sum_{n=1}^{N}\frac{x_{13}^{1,n}x_{13}^{2,n}}{N}-\sum_{n=1}^{N}\frac{x_{14}^{1,n}x_{14}^{2,n}}{N}\right|+\left|\sum_{n=1}^{N}\frac{x_{23}^{1,n}x_{23}^{2,n}}{N}+\sum_{n=1}^{N}\frac{x_{24}^{1,n}x_{24}^{2,n}}{N}\right|\approx 2\sqrt{2}\tag{19}$$

Therefore, Bell's Inequality (5) (or (7)) is violated in classical systems as well as quantum systems.

Remark 9. For completeness, note that the observables $\mathrm{O}_{a_i a_j}$ $\left(i=1,2,\ j=3,4\right)$ in the classical $L^\infty\left(\Omega\times\Omega\right)$ are not combinable in spite that these commute. Also, note that the Formulas (16) and (17) imply that

$$\left[F_{a_1a_3}\left(\{x\}\times X\right)\right](\omega)=\left[F_{a_1a_4}\left(\{x\}\times X\right)\right](\omega)=1/2,$$

$$\left[F_{a_2a_3}\left(\{x\}\times X\right)\right](\omega)=\left[F_{a_2a_4}\left(\{x\}\times X\right)\right](\omega)=1/2,$$

$$\left[F_{a_1a_3}\left(X\times\{x\}\right)\right](\omega)=\left[F_{a_2a_3}\left(X\times\{x\}\right)\right](\omega)=1/2,$$

$$\left[F_{a_1a_4}\left(X\times\{x\}\right)\right](\omega)=\left[F_{a_2a_4}\left(X\times\{x\}\right)\right](\omega)=1/2$$

$$\left(\forall x\in X,\forall\omega\in\Omega\times\Omega\right),$$

which is similar as in Remark 5; 1) or in (H).

4. Conclusions

In Bohr-Einstein debates (refs. [13] [14]), Einstein's standing-point (that is, "*the moon is there whether one looks at it or not*" (*i.e.*, physics holds without observers)) is on the side of the realistic world view in **Figure 1**. On the other hand, we think that Bohr's standing point (that is, "*to be is to be perceived*" (*i.e.*, there is no science without measurements)) is on the side of the linguistic world view in **Figure 1**.

In this paper, contrary to Bell's spirit (which inherits Einstein's spirit), we try to discuss Bell's inequality in Bohr's spirit (*i.e.*, in the framework of quantum language). And we show Theorem 6 (Bell's inequality in quantum language), which says the statement (F_2), that is,

(I_1) ($\equiv$(F_2)): If Bell's Inequality (5) (or (7)) is violated, then the combined observable does not exist, and thus, we cannot obtain the *measured value* (by the measurement of the combined observable).

Also, recall that Bell's original argument says, roughly speaking, that

(I_2) If the mathematical Bell's Inequality (8) is violated in Bell test experiment (the quantum case of Section 3.1), then *hidden variables* do not exist.

It should be note that the concept of "hidden variable" is independent of measurements, thus, the (I_2) is a philosophical statement in Einstein's spirit, on the other hand, the (I_1) is a statement in Bohr's spirit (*i.e.*, there is no science without measurements). It is sure that Bell's answer (I_2) is attractive philosophically, however, we believe in the scientific superiority of our answer (I_1). That is, we think that our (I_1) is a scientific representation of the philosophical (I_2). If so, we can, for the first time, understand Bell's inequality in science. That is, Theorem 6 is the true Bell's inequality. And we conclude that whether or not Bell's inequality holds does not depend on whether classical systems or quantum systems (in Section 3), but depend on whether the combined measurement exists or not (in Section 2).

We hope that our proposal will be examined from various points of view[1].

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Physical Limits of Computation

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Abstract

The paper deals with theoretical treatment of physical limits for computation. We are using some statements on base of min energy/bit, power delay product, Shannon entropy and Heisenberg uncertainty principle which result in about kTln(2) energy for a bit of information.

Keywords

Computation, Power Product, Entropy, SNL Limit

1. Introduction

Current computation technology is based on binary "thinking" and semiconductor hardware. The former has a very well developed and working theory, and thus it is expected to dominate informatics for some time. The scientific community is working on several solutions to replace the latter, which has consisted for the past several decades mainly of CMOS (complementary metal oxide semiconductor) architecture [1]. This research is necessary, because the currently available devices are slowly approaching their limits. But how long are we able to increase our computers' performance by replacing the technologies? This key question may be translated to the following: What is the minimal value of the important quantities in informatics?

2. Power-Delay Product

When the input voltage changes, logical circuits briefly dissipate power (usually a fixed amount). We call this the dynamic power. The power-delay product (PDP) is the dissipated energy per switching cycle. This quantity can be viewed as the sum of the individual switching events occurring during the cycle. The PDP is then connected to the switching time, whereby it is connected to the processors' heat generation and clock rate.

To calculate the PDP, let us consider the following thought experiment (**Figure 1**). The two switches are always in alternating positions. If I is the current, U_{out} is the output voltage and U_0 is the "upper" voltage, than the energy W required to charge the capacitor is [2]

$$W = \int_0^{\tau/2} I\left(U_0 - U_{out}\right)\mathrm{d}t\,. \tag{1}$$

It is easy to calculate I during this event as [2]

$$I = C\frac{\mathrm{d}U_{out}}{\mathrm{d}t}, \tag{2}$$

There equations related to experiment based on **Figure 1** are describing the charge-discharge processes on base of classical electronics.
where C is the capacitance. Combining these equations yields

$$W = CU_0^2/2\,. \tag{3}$$

This is only the first half of a switching cycle. For the second half it can be shown that the result is the same, so that the power-delay product in this case is [3] [4]

$$PDP = CU_0^2\,. \tag{4}$$

The time needed to perform the former cycle is [1]

$$\tau = CU_0/I\,. \tag{5}$$

3. Scaling Limits

We are closing in on the limit of Moore's 1st law (the 2nd will probably last longer). A good example why we cannot go on much longer with the exponential growth of the hardware's "device density" is the problem of conductances. The Moore' law—nowadays—looks like a bit problematique because the linewidth decreasing speed is NOT fully synchronized with density increasing speed.

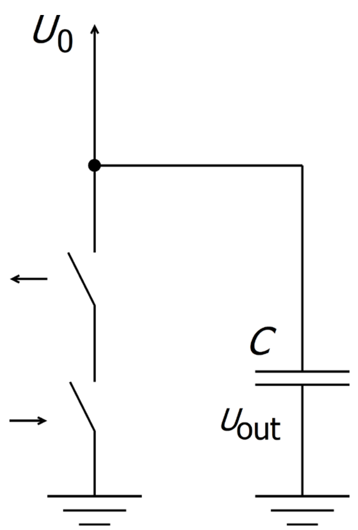

Figure 1. Thought experiment to calculate PDP.

3.1. Electrostatic Control

To have adequate control over the charge in a channel, it is necessary to maintain a distance of $l \ll L$, where L is the channel length and l is the thickness of the insulator. Unfortunately, l today is close to 1 nm, which means that the insulator is only several atoms thick. This places severe demands on the insulators [5].

3.2. Power Density

Removal of the heat generated by an integrated circuit has become perhaps the crucial constraint on the performance of modern electronics. The fundamental limit of the power density appears to be approximately 1000 W/cm^2. A power density of 790 W/cm^2 has already been achieved by using water cooling of a uniformly heated Si substrate with embedded micro channels (Note that the Sun's surface is around 6000 W/cm^2) [6].

4. Minimal Energy Dissipated Per Bit

To calculate the minimal energy required to generate a single bit of information, we need the entropy,

$$S = k \cdot \ln(\Omega), \tag{6}$$

where k is the Boltzmann constant and Ω is the degeneracy of the state. A more practical form of this quantity in binary fashioned systems is Shannon's entropy [7],

$$H = \log_2 \Omega, \tag{7}$$

which means $\Omega = 2^H$. Information from our viewpoint of humanity is the direct opposite of entropy: the larger the entropy, the more chaotic our system is. On the other hand, we are only able to gain more information from less chaotic (or more organized) systems.

To define the information quantitatively, we need further assumptions. Let us view the system (that we try to gain information from) as a set of events (or messages, or sometimes states). These messages have probabilities. Following Shannon's approach [7] [8], the definition of information (i) is

$$i_j = -\log_2 p_j, \tag{8}$$

where p is the probability of the j^{th} event that the message represents. We can see from this definition that a message is more valuable if it is more unlikely. Sometimes this quantity is also referred to as the uncertainty of the state. Usually the probabilities are equal, $p_j = 1/N$, where N is the number of possible events.

It can be shown that the formerly mentioned entropy can be also calculated with information i_j [8]:

$$H = \sum_j p_j \cdot i_j. \tag{9}$$

Computation is an information producing event, where it decreases the entropy of the computer. According to the 2nd law of thermodynamics, the whole universe's entropy may only increase. Labeling the environment as "e" and the computer as "c", this statement can be expressed as

$$\Delta S = \Delta S_e + \Delta S_c \geq 0\,, \tag{10}$$

which means that the traditional computation is an irreversible process. We obtain the heat by multiplying both sides by the temperature [6]:

$$\Delta Q = T \cdot \Delta S_e \geq -T \cdot \Delta S_c \tag{11}$$

Combining this with Equations ((6) and (7)), we obtain [9]

$$\Delta Q \geq -kT \cdot \Delta H_c \cdot \ln(2)\,. \tag{12}$$

This means that we need at least $kT\ln(2)$ of energy to generate a bit of information, which is the Shannon-Neumann-Landauer (SNL) limit. It is possible to interpret this result as the maximal efficiency of the information generating cycle. If we assume that the full energy invested is kT, then this efficiency is $\eta = \ln(2) = 0.693$. Applying Heisenberg's uncertainty principle to the SNL limit [5],

$$E_{\min}\tau \geq \hbar\,, \tag{13}$$

we get $\tau_{\min} \approx 0.04$ ps [3], although this theory is not proven so far.

5. Pursued Fields to Bypass the Limits

- Reversible computing (noise immunity is the main problem) [3] [10].
- New information tokens (spin of an electron, photons, etc.) [3] [11].
- Integration: switching from 2D circuits to 3D would increase the performance (If we are able to cool these 3D systems...) [3].
- Architecture.

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Study the Entanglement Dynamics of an Anisotropic Two-Qubit Heisenberg XYZ System in a Magnetic Field

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Abstract

We investigate the entanglement dynamics of an anisotropic two-qubit Heisenberg XYZ system with Dzyaloshinskii-Moriya (DM) interaction in the presence of both inhomogeneity of the external magnetic field b and intrinsic decoherence which has been studied. The behavior of quantum correlation and the degree of entanglement between the two subsystems is quantified by using measurement-induced disturbance (MID), negativity (N) and Quantum Discord (QD), respectively. It is shown that in the presence of an inhomogeneity external magnetic field occur the phenomena of long-lived entanglement. It is found that the initial state is the essential role in the time evolution of the entanglement.

Keywords

Different Dzyaloshinski-Moriya Interactions, Entanglement, Decoherence

1. Introduction

Nowadays, correlated systems represent one of the most important partners in the context of quantum communication [1] [2], quantum networks [3] [4] and quantum computers. Probably it is difficult to generate entangled systems with the same dimensions. This task may be difficult if the used devices are imperfect. Moreover, one success to generate maximum entangled states, but keeping them long-lived entangled is a very difficult task and may cost more [5] [6]. These states could be subject to noise channels [7] or dissipative environment [8]. Therefore, in the presence of a different type of noise, it is important to investigate the behavior of entanglement. There are several efforts which have been done to investigate the amount of survival entanglement of different systems pass through a different type of noise. Yu and Eberly investigated that the dynamics of entanglement between two qubits system interacting independently with classical or quantum noise displays the phenomena of entanglement decay and entanglement sudden death (ESD) [9] [10]. The dynamics of impurity and entanglement for a bipartite system that passes into Block channels were investigated in [11].

The intrinsic decoherence noise is one of the most important types of noise [12]. For example, M.-L. Hu and H.-L. Lian in [13] have investigated the quantum state transfer and the distribution of entanglement in the model of Milburns intrinsic decoherence. In [14], R. J. Amaro *et al.* have studied the interaction of a two-level atom and two fields, one of them is classical in the dispersive regime by using a model of intrinsic decoherence. By quantum, the possibility of reducing intrinsic decoherence is a superconducting circuit. Error detection is discussed by Zhong, *et al.* The effects of an inhomogeneous magnetic field on entanglement and teleportation in a two-qubit XXZ chain with intrinsic decoherence have been investigated [15] [16]. Therefore, we have motivated to investigate the behavior of quantum correlation for the system consists of two different dimensional subsystems in the presences of intrinsic decoherence. We need to investigate the effect of the dimensions of the system that passes through this type of noise on the degree of correlations between its subsystems. They use measurement-induced disturbance (MID), which does not include improvement methodology, to portray correlations as classical or quantum [17].

In this paper, we will investigate the quantum correlations base on MID in our model. As we probably are aware, the quantum entanglement of dense issue frameworks is a vital developing field as previously. Individuals have made a few examinations of quantum entanglement of thermal equilibrium states of spin chains subject to an external magnetic field at finite temperature [18] [19]. Likewise, the quantum correlations of two qubits with Dzyaloshinskii Moriya (DM) interaction, which can impact the phase transition, additionally have pulled in much consideration [20]. In this paper, in the examination with the warm quantum discord of two qubits [21] [22], we not just extend the examination on warm quantum correlation to blended turn (1/2, 1) XXZ display, yet in addition consider the impacts of DM interaction and outside magnetic field on a thermal quantum correlation measured by MID [23].

The rest of this paper is organized as follows. In Section 2, we introduce the Hamiltonian of the Heisenberg model with different DM interaction and present the exact solution of the model. In Section 3, is devoted to investigating the dynamics of entanglement and quantum correlation by means of negativity, measurement-induced disturbance [24] and Quantum Discord, respectively, Finally, we summarize our results in Section 4.

2. Model and Solution

The Hamiltonian H for a two-qubit anisotropic Heisenberg model with z-component interaction parameter D_z is

$$H = J(1+\gamma)\sigma_1^x\sigma_2^x + J(1-\gamma)\sigma_1^y\sigma_2^y + J_z\sigma_1^z\sigma_2^z + D_z(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x) + (B+b)\sigma_1^z + (B-b)\sigma_2^z \quad (1)$$

where J and J_z are the real coupling coefficients, γ is the anisotropic parameter. D_z is the z-component DM interaction parameter, and $\sigma^i\ (i = x, y, z)$ are Pauli matrices. B is the homogeneous part of the magnetic field and b describes the inhomogenity. The external magnetic fields and Dzyaloshinskii Moriya interaction are assumed to be along the z-direction. All the parameters are dimensionless. We get on the eigenvalues of the Hamiltonian H are given by

$$\begin{aligned} E_1 &= J_z + 2\sqrt{B^2 + J^2\gamma^2} \\ E_2 &= J_z - 2\sqrt{B^2 + J^2\gamma^2} \\ E_3 &= -J_z + 2\sqrt{b^2 + D_z^2 + J^2} \\ E_4 &= -J_z - 2\sqrt{b^2 + D_z^2 + J^2} \end{aligned} \quad (2)$$

We get on the eigenvectors of the Hamiltonian H are given by

$$|\psi_1\rangle = \frac{1}{\sqrt{1+\mu^2}}(\mu|11\rangle + |00\rangle) \quad (3)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{1+\nu^2}}(\nu|11\rangle + |00\rangle) \quad (4)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{1+X}}(N|10\rangle + |01\rangle) \quad (5)$$

$$|\psi_4\rangle = \frac{1}{\sqrt{1+Y}}(M|10\rangle + |01\rangle) \quad (6)$$

where,

$$\begin{gathered} \mu = \frac{B + \sqrt{B^2 + J^2\gamma^2}}{J\gamma}, \quad \theta = b + \sqrt{b^2 + D_z^2 + J^2} \\ \nu = \frac{B + \sqrt{B^2 + J^2\gamma^2}}{J\gamma}, \quad \phi = -b + \sqrt{b^2 + D_z^2 + J^2} \\ X = \frac{\theta^2}{Dz^2 + J^2}, \quad Y = \frac{\phi^2}{Dz^2 + J^2} \\ N = \frac{\theta(-ID_z + J)}{D_z^2 + J^2}, \quad M = \frac{\phi(ID_z - J)}{D_z^2 + J^2} \end{gathered} \quad (7)$$

The master equation describing the intrinsic decoherence under the Markovian approximations is given by

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] - \frac{\Gamma}{2}[H, [H, \rho(t)]] \quad (8)$$

where Γ is the intrinsic decoherence rate. The formal solution of the above master equation can be expressed as

$$\rho(t)=\sum_{k=0}^{\infty}\frac{(\Gamma t)^{k}}{k!}M^{k}\rho(0)M^{+k} \tag{9}$$

where $\rho(0)$ is the density operator of the initial system and M^{k} is defined by

$$M^{k}=H^{k}\mathrm{e}^{-iHt}\mathrm{e}^{-\frac{\Gamma t}{2}H^{2}} \tag{10}$$

According to Equation (9) it is easy to show that, under intrinsic decoherence, the dynamics of the density operator $\rho(t)$ for the above-mentioned system which is initially in the state $\rho(0)$ is given by

$$\begin{aligned}\rho(t)=\sum_{mn}&\exp\left[-\frac{\Gamma t}{2}(E_m-E_n)^2-i(E_m-E_n)t\right]\\&\times\langle\psi_m|\rho(0)|\psi_n\rangle|\psi_m\rangle\langle\psi_n|\end{aligned} \tag{11}$$

where E_m, E_n, $|\psi_m\rangle$, $|\psi_n\rangle$ are the eigenvalues and the corresponding eigenvectors of *H*. In the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ the time evolution of the density operator of the system will be obtained for two different initial states as:

The two qubits are initially in an entangled state $\rho(0)=|\phi\rangle\langle\phi|$, $|\phi\rangle=\cos(\alpha)|01\rangle+\sin(\alpha)|10\rangle$, we get

$$\rho(t)=\begin{pmatrix}0&0&0&0\\0&\rho_{22}&\rho_{23}&0\\0&\rho_{32}&\rho_{33}&0\\0&0&0&0\end{pmatrix} \tag{12}$$

$$\begin{aligned}\rho_{22}=\frac{1}{4y^2}\Big[&\big((b+y)(y-b\cos(2\alpha)+J\sin(2\alpha))\big)\\&-\left(\mathrm{e}^{-4iyt-8y^2t\Gamma}\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ+iD_zy)\sin(2\alpha)\right)\right)\\&-\left(\mathrm{e}^{4iyt-8y^2t\Gamma}\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ-iD_zy)\sin(2\alpha)\right)\right)\\&+\big((-b+y)(y+b\cos(2\alpha)-J\sin(2\alpha))\big)\Big]\end{aligned}$$

$$\begin{aligned}\rho_{23}=\frac{1}{4y^2}\Big[&\big((-iD_z+J)(y-b\cos(2\alpha)+J\sin(2\alpha))\big)\\&-\big((-iD_z+J)(y+b\cos(2\alpha)-J\sin(2\alpha))\big)\Big]+\frac{1}{4(iD_z+J)(y^2)}\\&\times\Big(\left(\mathrm{e}^{-4iyt-8y^2t\Gamma}(b+y)\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ+iD_zy)\sin(2\alpha)\right)\right)\\&+\left(\mathrm{e}^{4iyt-8y^2t\Gamma}(b-y)\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ-iD_zy)\sin(2\alpha)\right)\right)\Big)\end{aligned}$$

$$\begin{aligned}\rho_{32}=\frac{1}{4y^2}\Big[&\left((iD_z+J)\left(\sqrt{b^2+D_z^2+J^2}-b\cos(2\alpha)+J\sin(2\alpha)\right)\right)\\&-\big((iD_z+J)(y+b\cos(2\alpha)-J\sin(2\alpha))\big)\Big]+\frac{1}{4(iD_z+J)(y^2)}\\&\times\Big[\left(\mathrm{e}^{-4iyt-8y^2t\Gamma}(b-y)\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ+iD_zy)\sin(2\alpha)\right)\right)\\&+\left(\mathrm{e}^{4iyt-8y^2t\Gamma}(b+y)\left(\left(D_z^2+J^2\right)\cos(2\alpha)+(bJ-iD_zy)\sin(2\alpha)\right)\right)\Big]\end{aligned}$$

$$\rho_{33} = \frac{1}{4y^2(b+y)}\left[\left(D_z^2+J^2\right)\left(y-b\cos(2\alpha)+J\sin(2\alpha)\right)\right]$$
$$+\frac{1}{4y^2(-b+y)}\left[\left(D_z^2+J^2\right)\left(y+b\cos(2\alpha)-J\sin(2\alpha)\right)\right]$$
$$+\frac{1}{4y^2}\left[\left(e^{-4iyt-8y^2t\Gamma}\left(\left(D_z^2+J^2\right)\cos(2\alpha)+\left(bJ+iD_z y\right)\sin(2\alpha)\right)\right)\right.$$
$$\left.+\left(e^{4iyt-8y^2t\Gamma}\left(\left(D_z^2+J^2\right)\cos(2\alpha)+\left(bJ-iD_z y\right)\sin(2\alpha)\right)\right)\right]$$

3. Entanglement Evolutions

3.1. Negativity

There is some systems could be entangled but have a zero negativity and non-zero values of quantum correlations. This means that there are some quantum correlation cannot be predicted by using the negativity as a measure of entanglement. However, by using the measurement-induced disturbance as a measure of quantum correlation, one can quantify the unpredicted quantum correlation [24] [25]. Therefore, we are studying the effect of a type of noise called intrinsic decoherence on the negativity and the measurement-induced disturbance in this section, we investigate the behavior of entanglement in the presences of intrinsic decoherence, by means of negativity N. This measure states that if $\{\lambda_\mu\}$ represents the eigenvalues of $\rho_{ab}^{T_2}$, then the negativity is given by, $N=\sum_{i=1}^{4}|\lambda_i|-1$ where T_2 refers to the partial transposition for the second subsystem. By using this definition for our system, the negativity can be calculated explicitly as, $N=\max\left[0,-2\min[\lambda_i]\right]$. Thus in this letter, we use negativity as our measure of entanglement. The values of N range from zero to one: For a maximally-entangled when $N=1$, while for a unentangled state $N=0$.

3.2. Quantum Correlation via Measurement-Induced Disturbance

By using $\Pi_k=\Pi_i^a\otimes\Pi_j^b$ and Π_i^a, Π_j^b are complete projective measurements consisting of one-dimensional orthogonal projections for parties a and b, we can apply local measurement $\{\Pi_k\}\left(\Pi_k\Pi_{k'}=\delta_{kk'}\Pi_k\right)$ and $\sum_k\Pi_k=1$, to any bipartite state ρ (of course, including thermal state). After the measurement, we get the state $\Pi(\rho)=\sum_{ij}\left(\Pi_i^a\otimes\Pi_j^b\right)\rho\left(\Pi_i^a\otimes\Pi_j^b\right)$ which is a classical state. If the measurement Π is induced by the spectral resolutions of the reduced states $\rho^a=\sum_i p_i^a\Pi_i^a$ and $\rho^b=\sum_i p_i^b\Pi_i^b$, the measurement leaves the marginal information invariant and is in a certain sense the least disturbing. In fact, $\Pi(\rho)$ is a classical state that is closest to the original state ρ since this kind of measurement can leave the reduced states invariant. One can use any reasonable distance between ρ and $\Pi(\rho)$ to measure the quantum correlation in ρ. In this article, we will use Luos method *i.e.*, quantum mutual information difference between ρ and $\Pi(\rho)$, to measure quantum correlation in ρ. The total correlation in a bipartite state ρ can be well quantified by the quantum mutual information $I(\rho)=S(\rho^a)+S(\rho^b)-S(\rho)$, and $I(\Pi(\rho))$, quantifies the classical correlations in ρ since $\Pi(\rho)$ is a classical state. Here $S(\rho)=-\sum_i \lambda_i \log_2 \lambda_i$ denotes the von Neumann entropy. So the quantum correlation can be quantified by the measurement induced disturbance $Q(\rho)= I(\rho)-I(\Pi(\rho))$. where

$$\Pi(\rho)=\begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} \tag{13}$$

3.3. Quantum Discord

Quantum discord is based on the difference between the quantum mutual information and the classical correlation. For a two-qubit quantum system, the total correlation is measured by their quantum mutual information $L(\rho_{ab})=S(\rho_a)+S(\rho_b)-S(\rho_{ab})$, where $\rho_{a(b)}$ and ρ_{ab} denote the reduced density matrix of $a(b)$ and the density of the bipartite system respectively, and $S(\rho)= -tr(\rho \log_2 \rho)$ is the von Neumann entropy. Quantum discord, which quantifies the quantumness of correlation between A and B, is then defined as the difference between the total correlation and classical correlation. For the X state described by the density matrix quantum discord (QD) is given as

$$QD=\min[D_1,D_2] \tag{14}$$

with

$$QD_j=\Gamma(\rho_{11}+\rho_{33})+\sum_{i=1}^{4}\lambda_i \log_2 \lambda_i + R_j$$

and

$$R_1=-\Gamma(\rho_{11}+\rho_{33})-\sum_{i=1}^{4}\rho_{ii}\log_2 \rho_{ii},$$

$$R_2=\Gamma(p) \tag{15}$$

$$p=\frac{1+\sqrt{\left[1-2(\rho_{33}+\rho_{44})\right]^2+4\left(|\rho_{14}|+|\rho_{23}|\right)^2}}{2}$$

where λ_i being the four eigenvalues of the density matrix ρ and $\Gamma(x)= -x\log_2 x-(1-x)\log_2(1-x)$. As the density matrix of our system in Equation (9) is X states, quantum discord can be evaluated by substituting from Equation (9) into Equation (11) after straightforward calculation, quantum discord reads

$$QD=\min[D_1,D_2] \tag{16}$$

with

$$D_1=\left(1/\log_2[4]\right)\Big(-(\rho_{22}+\rho_{33})\log_2[4]-2\rho_{22}\log_2[\rho_{22}]-2\rho_{33}\log_2[\rho_{33}]$$
$$+\left(\rho_{22}-\sqrt{4\rho_{23}\rho_{32}+(\rho_{22}-\rho_{33})^2}+\rho_{33}\right)$$

$$\times \log_2\left[\rho_{22} - \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right]$$
$$+\left(\rho_{22} + \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right)$$
$$\left.\times \log_2\left[\rho_{22} + \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right]\right)$$

$$D_2 = \frac{1}{\log_2[4]}\left(4p\tanh^{-1}[1-2p] + 4\rho_{33}\tanh^{-1}[1-2\rho_{33}]\right.$$
$$-(\rho_{22}+\rho_{33})\log_2[4] - 2\log_2[1-p] - 2\log_2[1-\rho_{33}]$$
$$+\left(\rho_{22} - \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right)$$
$$\times \log_2\left[\rho_{22} - \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right]$$
$$+\left(\rho_{22} + \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right)$$
$$\left.\times \log\left[\rho_{22} + \sqrt{4\rho_{23}\rho_{32} + (\rho_{22}-\rho_{33})^2} + \rho_{33}\right]\right)$$

In Figure 1 we plot the negativity N as a function of time t with different values DM interaction in the absence of a magnetic field. We see that negativity the movement starts from a point that varies according to different the initial states. We note that negativity rising to the highest value approaching one. We find that it oscillates and decay over time and with increased the DM interaction, it is seen that increased amplitude while reducing frequencies, the negativity N oscillates for the longest time before it reaches steady-going values. On the other side in the presence of an inhomogeneous external magnetic field, we find that the negativity N, the oscillation period decay and decay faster before reaching a stable values in Figure 2. In Figure 3 we plot the Measurement-induced disturbance (MID) as a function of time t with different values DM interaction in the absence of a magnetic field. We see that MID at the beginning of the track Moves from a point that varies according to Different the initial states. We note that negativity rising to the highest value at 0.62. We find that it oscillates and decay over time and with increased the DM interaction, it is seen that increased amplitude while reducing frequencies, MID oscillates for the longest time before it reaches steady-going values. On the other side in the presence of an inhomogeneous external magnetic field, we find that MID, the oscillation period decay and decay faster before reaching a stable values not equal to zero in Figure 4. In Figure 5 we plot quantum discord (QD) as a function of time t with different values DM interaction in the absence of a magnetic field. We see that QD At the beginning of the track moves from a point that varies according to different the initial states. We note that negativity rising to the highest value approaching one. We find that it oscillates and decay over time and with increased the DM interaction, it is seen that amplitude is greater with reducing frequencies; QD oscillates for the longest time before it reaches steady-going values. On the other side in the presence of inhomogeneous external magnetic field, we find that MID, the oscillation period decay

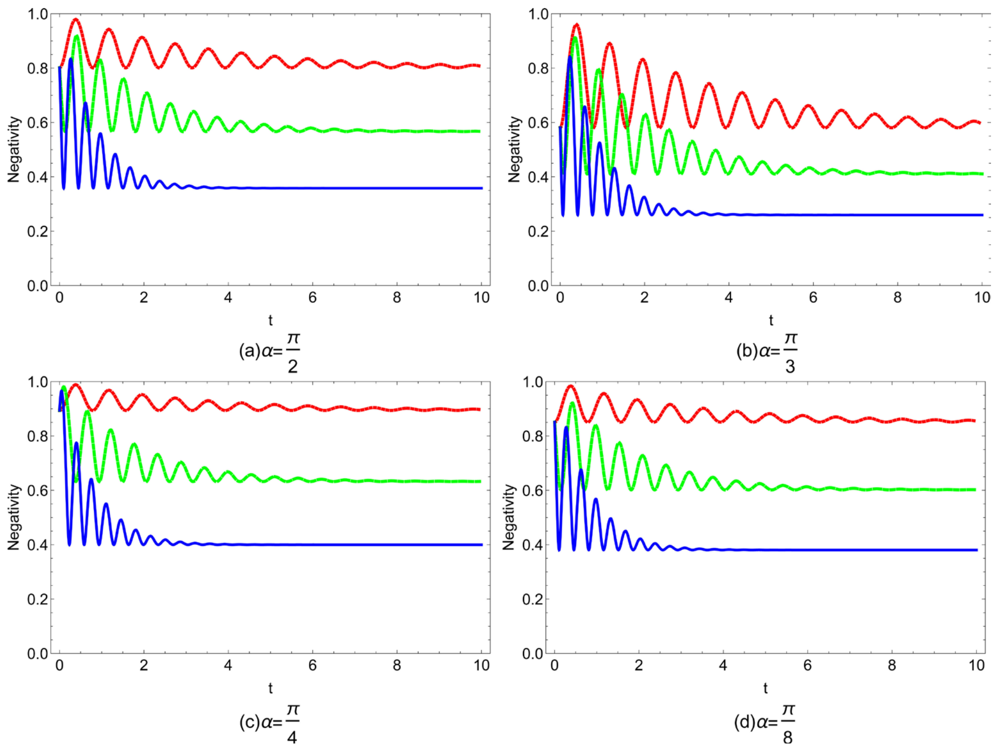

Figure 1. The negativity N as a function of time t. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 0$).

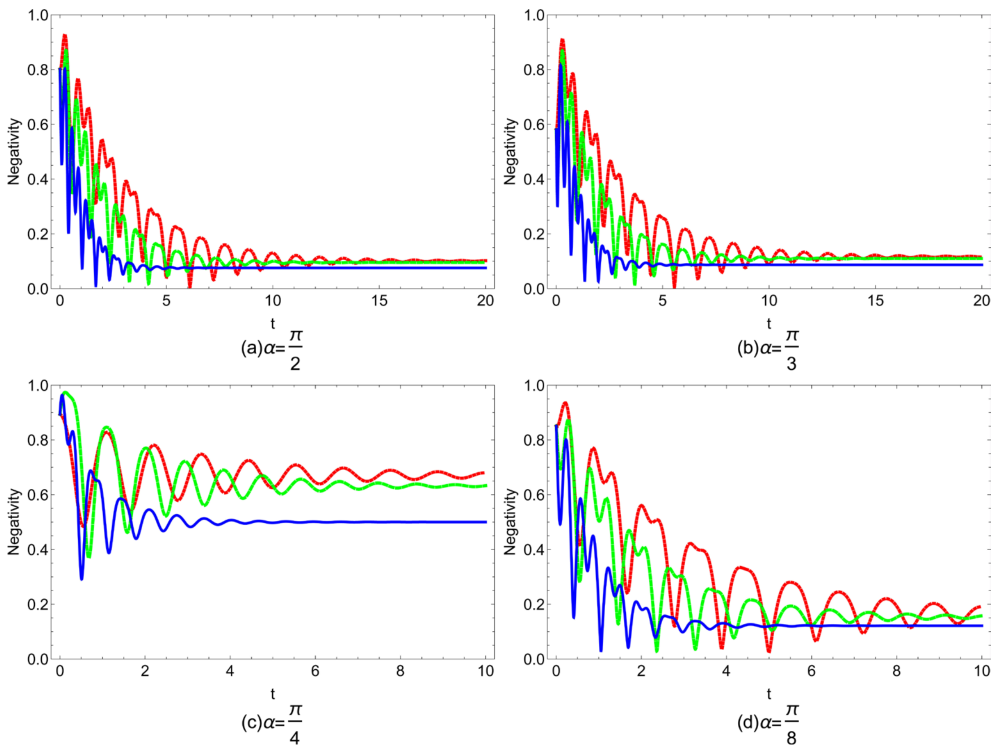

Figure 2. The negativity N as a function of time t. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 1$).

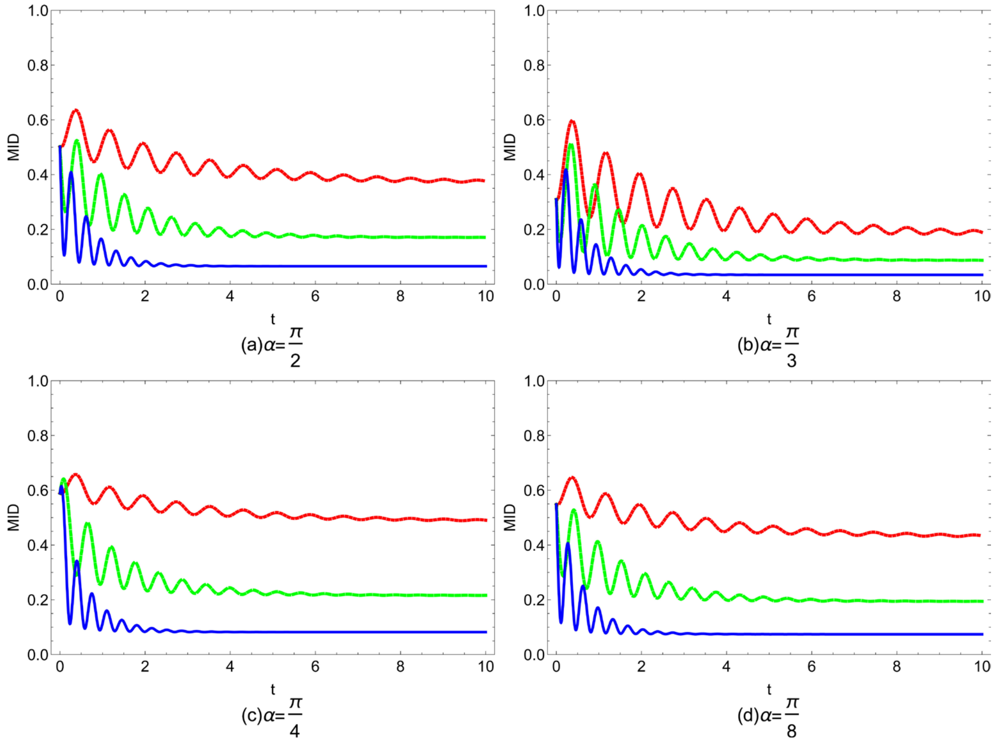

Figure 3. The behavior of measurement-induced disturbance (MID) as a function of time *t*. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 0$).

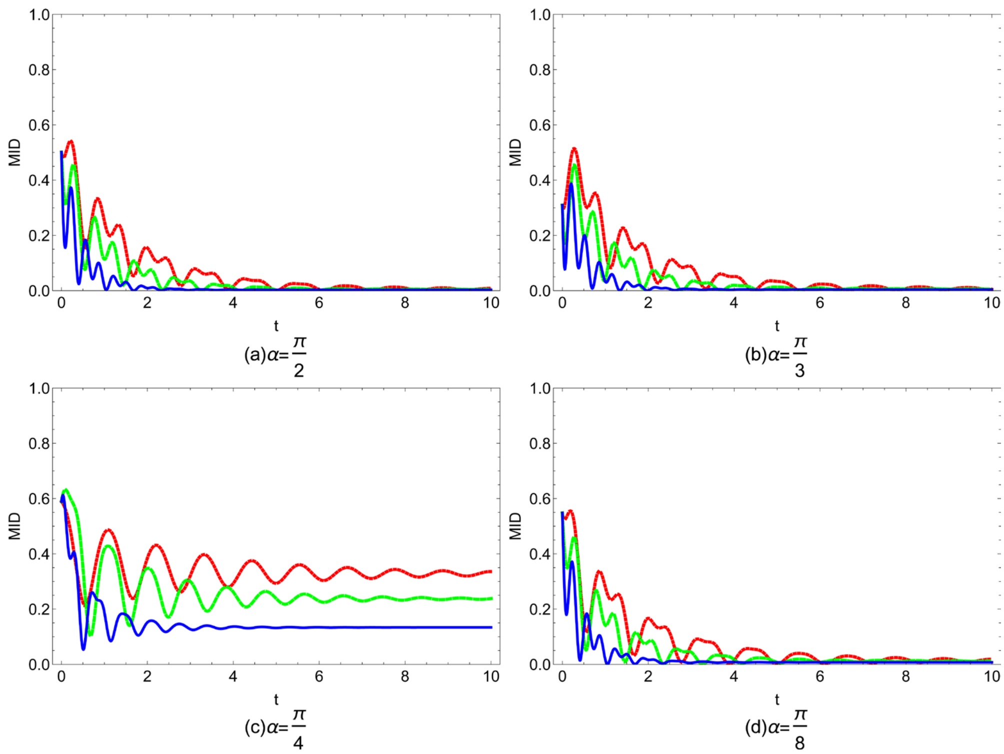

Figure 4. The behavior of measurement-induced disturbance (MID) as a function of time *t*. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 1$).

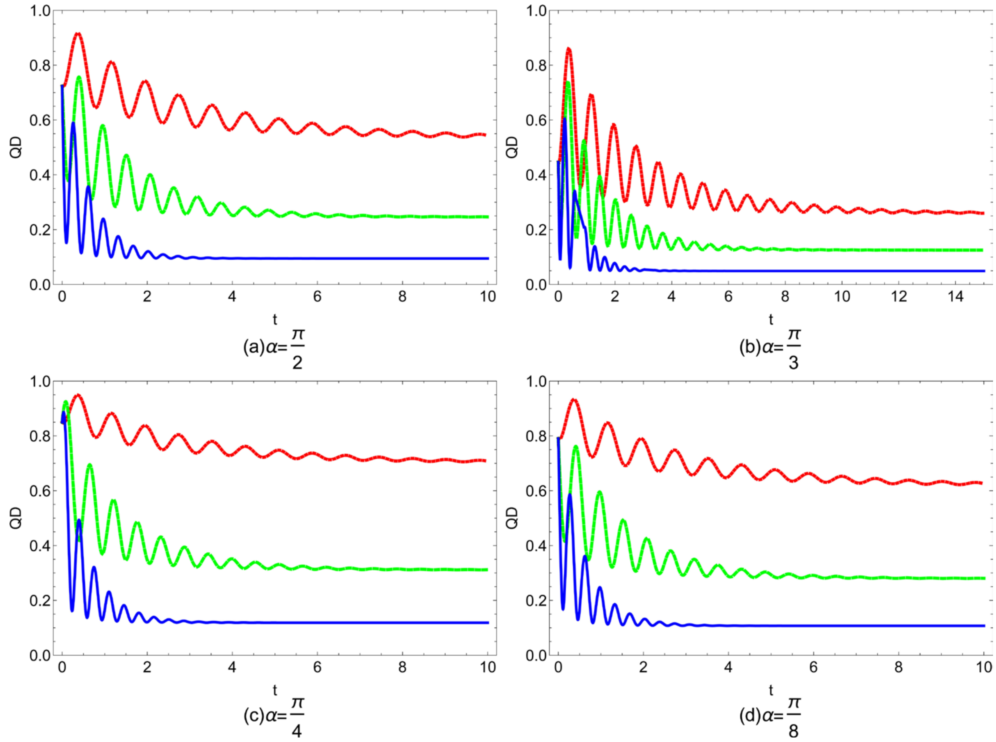

Figure 5. The behavior of Quantum Discord (QD) as a function of time t. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 0$).

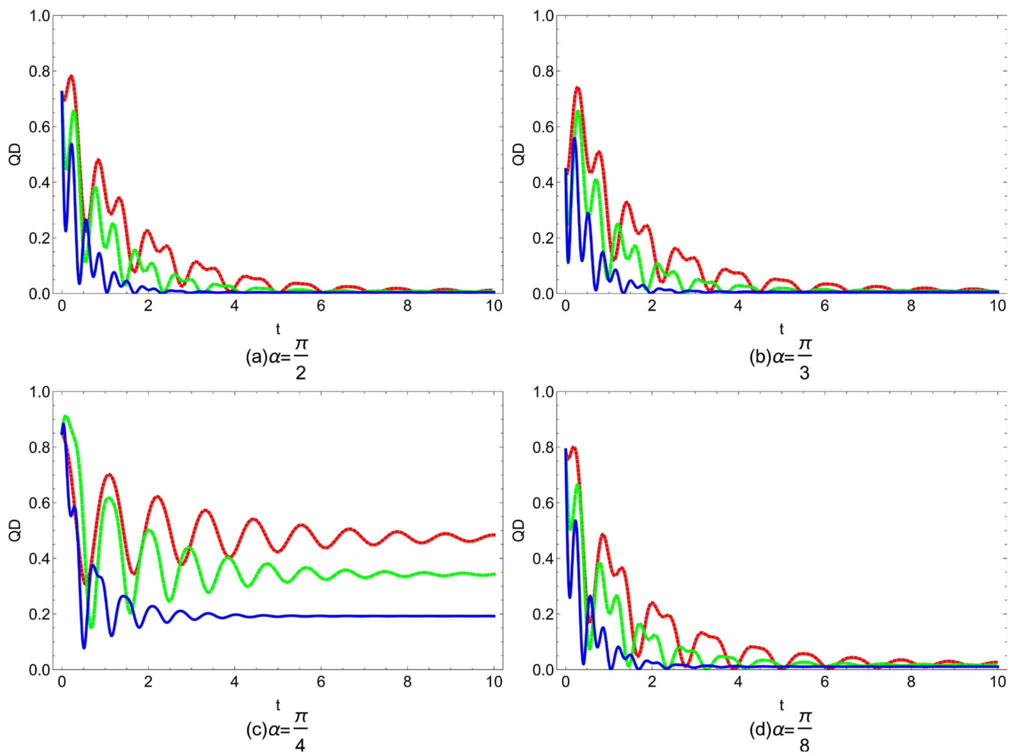

Figure 6. The behavior of Quantum Discord (QD) as a function of time t. The dotted, dashed and solid curves are evaluated for $D = 0,1,2$, respectively in the initial states $\alpha = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{8}$, respectively ($\gamma = 2$, $J = 1$, $Jz = 1$, $\Gamma = 0.02$, $b = 0$).

and decay faster before arrive at a steady-going non-zero value for the long-time case for different values of the DM interaction in **Figure 6**, which means that the inhomogeneous external magnetic field is a positive component to the entanglement when the partial anisotropic parameter of the system is at a fixed non-zero value. They obtained the similar result that a proper external magnetic field can protect the entanglement from the destructive effect of the intrinsic decoherence.

4. Conclusion

In the presence of both external magnetic field and intrinsic decoherence, we have treated the entanglement dynamics of an anisotropic two-qubit Heisenberg XYZ system with Dzyaloshinskii-Moriya interaction which has been studied. We found that the initial state of the system plays an important role in the time evolution of the entanglement. The magnetic field has an effective role in maintaining the intertwining for a long time and non-analysis. The negativity, MID, and QD for different DM interaction will arrive at a steady-going non-zero value for the long-time case, which means that the external magnetic field b is a positive component to the entanglement when the partial anisotropic parameter of the system is at a fixed non-zero value.

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Entanglement of Moving and Non-Moving Two-Level Atoms

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Abstract

In this paper we study the dynamics of the atomic inversion, von Neumann entropy and entropy squeezing for moving and non-moving two-level atoms interacting with a Perelomov coherent state. The final state of the system using specific initial conditions is obtained. The effects of Perelomov and detuning parameters are examined in the absence and presence of the atomic motion. Important phenomena such as the collapse and revival are shown to be very sensitive to the variation of the Perelomov parameter in the presence of detuning parameter. The results show that the Perelomov parameter is very useful in generating a high amount of entanglement due to variation of the detuning parameter.

Keywords

Atomic Inversion, Von Neumann Entropy, Atomic Motion, Entropy Squeezing

1. Introduction

Quantum entanglement is one of the most outstanding features of quantum mechanics. Quantum entangled states, as a fundamental physical resource of quantum information processing [1], are widely used, in quantum computation and quantum communication [2], quantum cryptography [3] [4], quantum teleportation [5], etc. Also, there is a lot of attention that has been focused on information entropies as a measure of entanglement in quantum information such as von Neumann entropy [6], Linear entropy, Shannon information entropy [7] and atomic Wehrl entropy [8]. On the other hand, we find that the most important problems in quantum optics are the studies of different systems interaction such as field-atom, atom-atom and the field-field. These problems have been extensively considered in a huge number of papers; see for example [9]-[26]. This is due to the richness of many different phenomena that have been observed in the laboratory.

In fact, these different kinds of interactions have been classified from the point of view of Lie algebra depending on the nature of the interaction. For example, the Hamiltonian which represents the interaction between two fields is described in the form of the parametric frequency converter which is of $SU(2)$ Lie algebra type. While the Hamiltonian which represents the non-degenerate parametric amplifier is of $SU(1,1)$ Lie algebra type. In this context a system describes the interaction between $SU(2)$ and $SU(1,1)$ Lie algebra, in which a Hamiltonian of the following form was treated

$$H = \omega \hat{K}_z + \frac{\omega_0}{2}\hat{\sigma}_z + gf(z)\left[\hat{\sigma}_+\hat{K}_- + \hat{K}_+\hat{\sigma}_-\right], \tag{1}$$

where ω and ω_0 are the field and atomic frequencies, respectively, $\hat{\sigma}_z$ and $\hat{\sigma}_\pm$ are the atomic pseudospin operators that obey the commutation relation

$$\left[\hat{\sigma}_+, \hat{\sigma}_-\right] = \hat{\sigma}_z, \quad \left[\hat{\sigma}_z, \hat{\sigma}_\pm\right] = \pm 2\hat{\sigma}_\pm \tag{2}$$

while g is the atom field coupling constant, $f(z)$ denotes a shape function of cavity field mode.

We restrict our study to the atomic motion along the z-axis so that the z-dependence of the field-mode function should be taken into account. The atomic motion can be incorporated in the usual way, *i.e.*

$$f(z) \to f(vt) = p_1 + \sin\left(\frac{p_2 \pi v t}{L}\right), \tag{3}$$

where v denotes the atomic motion velocity, p_1 and p_2 are the atomic motion parameters well, if we put $p_1 = 1$ and $p_2 = 0$, then the shape function takes the form

$$\Omega(t) = \int_0^t f(vt')\,\mathrm{d}t' = t \tag{4}$$

which means, there is no atomic motion inside the cavity, but if $p_1 = 0$ and $p_2 = p$, where p represents the number of half-wave lengths of the field mode inside a cavity of the length L, the shape function for a particular choice of the atomic motion velocity $v = \frac{gL}{\pi}$ will be

$$\Omega(t) = \int_0^t f(vt')\,\mathrm{d}t' = \frac{1}{pg}\left[1 - \cos(pgt)\right] \tag{5}$$

Over the last two decades much attention has been focused on the properties of the Jaynes-Cummings model JCM for a moving atom. The theoretical efforts have been stimulated by experimental progress in cavity QED. Besides the experimental drive, there also exists a theoretical motivation to include atomic motion effect to JCM because its dynamics becomes more interesting. A number of authors have treated the JCM in the presence of atomic motion by the use of analytic approximations [27] [28] [29] [30] and numerical calculations [31].

The solution in the presence of atomic motion is not only of theoretical interest, but also important from a practical point of view for cold atoms. The important demonstration of the quantum collapse and revival phenomena was observed in a one-atom maser by Rempe *et al.* [32]. Some research groups were unable to build experimental setups capable of enhancing the coupling of an atom with a single field mode, simultaneously suppressing other modes.

In this article, we consider the extension of the problem by considering the two-level interaction with $SU(1,1)$ quantum system. We focus on the effect of the Perelomov parameter, field-mode structure parameter and detuning parameter on the evolution of the atomic inversion, von Neumann entropy in the case of absence and presence of the atomic motion effect.

We organize the material of this paper as follows: in Section 2 we introduce our Hamiltonian model which represents the interaction between $SU(1,1)$ and $SU(2)$, Then we derive the effective two-level atom Hamiltonian model, and we use the evolution operator method to find an exact expression of the wave function at the time $t>0$. We devote Section 3 to discuss the atomic inversion in order to see the change that would occur in its behavior during interaction. While in Section 4 we discuss the degree of entanglement for the atomic system via von Neumann entropy and entropy squeezing. Finally, we draw a summary in Section 5.

2. The System Hamiltonian

The Hamiltonian which describe the interaction between a single two-level atom and $SU(1,1)$ quantum system take the following form

$$H=\omega\hat{K}_z+\frac{\omega_0}{2}\hat{\sigma}_z+gf(z)\left[\hat{\sigma}_+\hat{K}_-+\hat{K}_+\hat{\sigma}_-\right], \tag{6}$$

while $\hat{K}_\pm$ and $\hat{K}_z$ satisfy the following commutation relation

$$\left[\hat{K}_z,\hat{K}_\pm\right]=\pm\hat{K}_\pm,\quad \left[\hat{K}_-,\hat{K}_+\right]=2\hat{K}_z \tag{7}$$

The Heisenberg equation of motion for any operator $\hat{O}$ is given by

$$i\frac{\mathrm{d}\hat{O}}{\mathrm{d}t}=\left[\hat{O},H\right],\quad (\hbar=1) \tag{8}$$

thus, the equations of motion for $\hat{\sigma}_z$ and $\hat{K}_z$ are given by

$$\frac{\mathrm{d}\hat{\sigma}_z}{\mathrm{d}t}=-i\left[\hat{\sigma}_z,H\right]=2igf(vt)\left(\hat{K}_+\hat{\sigma}_--\hat{\sigma}_+\hat{K}_-\right). \tag{9}$$

$$\frac{\mathrm{d}\hat{K}_z}{\mathrm{d}t}=-i\left[\hat{K}_z,H\right]=igf(vt)\left(-K_+\hat{\sigma}_-+\hat{\sigma}_+\hat{K}_-\right). \tag{10}$$

$$\frac{\mathrm{d}\hat{K}_z}{\mathrm{d}t}+\frac{1}{2}\frac{\mathrm{d}\hat{\sigma}_z}{\mathrm{d}t}=0 \tag{11}$$

from the above equation, we can see that $N=\hat{K}_z+\frac{1}{2}\hat{\sigma}_z$ is constant of motion, therefore, the Hamiltonian takes the following form

$$H=\omega N+H_I, \tag{12}$$

where

$$H_I = \frac{\Delta}{2}\hat{\sigma}_z + gf(vt)\left(\hat{\sigma}_+\hat{K}_- + \hat{K}_+\hat{\sigma}_-\right), \tag{13}$$

where $\Delta = \omega_0 - \omega$ is the detuning parameter. We note that, $[N, H_I] = 0$, therefore $[N, H] = [H, H_I] = 0$, *i.e.* N and H_I are the constants of motion. Where the time evolution operator is defined as

$$U(t) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{14}$$

where

$$\begin{aligned} u_{11} &= \cos(\hat{\mu}_1 t) - i\frac{\Delta}{2\mu_1}\sin(\hat{\mu}_1 t), \quad u_{12} = -i\frac{g}{\hat{\mu}_1}\sin(\hat{\mu}_1 t)\hat{K}_- \\ u_{21} &= -i\frac{g}{\hat{\mu}_2}\sin(\hat{\mu}_2 t)\hat{K}_+, \quad u_{22} = \cos(\hat{\mu}_2 t) + i\frac{\Delta}{2\hat{\mu}_2}\sin(\mu_2 t) \end{aligned} \tag{15}$$

and

$$\hat{\mu}_j^2 = \frac{\Delta^2}{4} + v_j, \quad j = 1, 2, \quad v_1 = g^2\hat{K}_-\hat{K}_+, \quad v_2 = g^2\hat{K}_+\hat{K}_- \tag{16}$$

Let us assume the initial state of the atom in excited state $|1\rangle$ and the cavity field mode is in the Perelomov coherent state [33] [34].

$$|\mu;k\rangle = \sum_{m=0}^{\infty} Q_m |m,k\rangle, \quad Q_m = \left(1-|\mu|^2\right)^k \left(\frac{\Gamma(2k+m)}{m!\Gamma(2k)}\right)^{\frac{1}{2}} \mu^m, \tag{17}$$

where, μ the Perelomov parameter and

$$\begin{aligned} \hat{K}_z|m,k\rangle &= (m+k)|m,k\rangle, \\ \hat{K}_+|m,k\rangle &= \sqrt{(m+1)(m+2k)}|m+1,k\rangle, \\ \hat{K}_-|m,k\rangle &= \sqrt{m(m+2k-1)}|m-1,k\rangle, \end{aligned} \tag{18}$$

Therefore, we can write the wave function at the time $t > 0$ in the form

$$|\psi(t)\rangle = |C(t)\rangle|1\rangle + |S(t)\rangle|2\rangle, \tag{19}$$

where,

$$|C(t)\rangle = \sum_{m=0}^{\infty} X_1(m,t)|m,k\rangle, \quad |S(t)\rangle = \sum_{m=0}^{\infty} X_2(m,t)|m+1,k\rangle, \tag{20}$$

here,

$$\begin{aligned} X_1(m,t) &= Q_m\left[\cos(\mu_1 t) - i\frac{\Delta}{2\mu_1}\sin(\mu_1 t)\right], \\ X_2(m,t) &= -iQ_m\frac{g\sqrt{(m+1)(m+2k)}}{\mu_1}\sin(\mu_1 t), \end{aligned} \tag{21}$$

where,

$$\mu_1 = \sqrt{\frac{\Delta^2}{4} + g^2(m+1)(m+2k)}, \quad \mu_2 = \sqrt{\frac{\Delta^2}{4} + g^2 m(m+2k-1)}. \tag{22}$$

The atomic reduced density operator for the system is given by

$$\rho_A(t) = Tr_f \left|\psi(t)\right\rangle\left\langle\psi(t)\right| = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} \rho_{11} &= \sum_{m=0}^{\infty} \left|X_1(m,t)\right|^2, \quad \rho_{22} = \sum_{m=0}^{\infty} \left|X_2(m,t)\right|^2 \\ \rho_{12} &= \sum_{m=0}^{\infty} X_1(m+1,t) X_2^*(m,t), \quad \rho_{21} = \rho_{12}^* \end{aligned} \tag{24}$$

3. Atomic Inversion

Atomic inversion can be considered as the simplest important quantity to be calculated. It is related to the difference between the probabilities of finding the atom in the upper state $|1\rangle$ and in the ground state $|2\rangle$. Using Equation (20), we can calculate the time evolution of the atomic inversion as

$$W(t) = \sum_{m=0}^{\infty} \left[\left|X_1(m,t)\right|^2 - \left|X_2(m,t)\right|^2 \right] \tag{25}$$

In Figure 1, we have plotted the atomic inversion against the time gt, for fixed value of the Bargmann index $k = 5$. This, in fact, would help us to examine the effect of the other involved parameters. For instance, we consider the case in which the Perelomov parameter $\mu = 0.5$ and the detuning $\Delta = 0$. In this case the function fluctuates around the value *zero* and exhibits periods of revivals, see Figure 1(a).

When we increase the value of μ to 0.7 a dramatic change can be seen in the function behavior. In this case the function fluctuates around the value *zero* and exhibits periods of collapses and revivals, however the revival periods start to spread out as time increases. In the meantime we observe a decrease in the

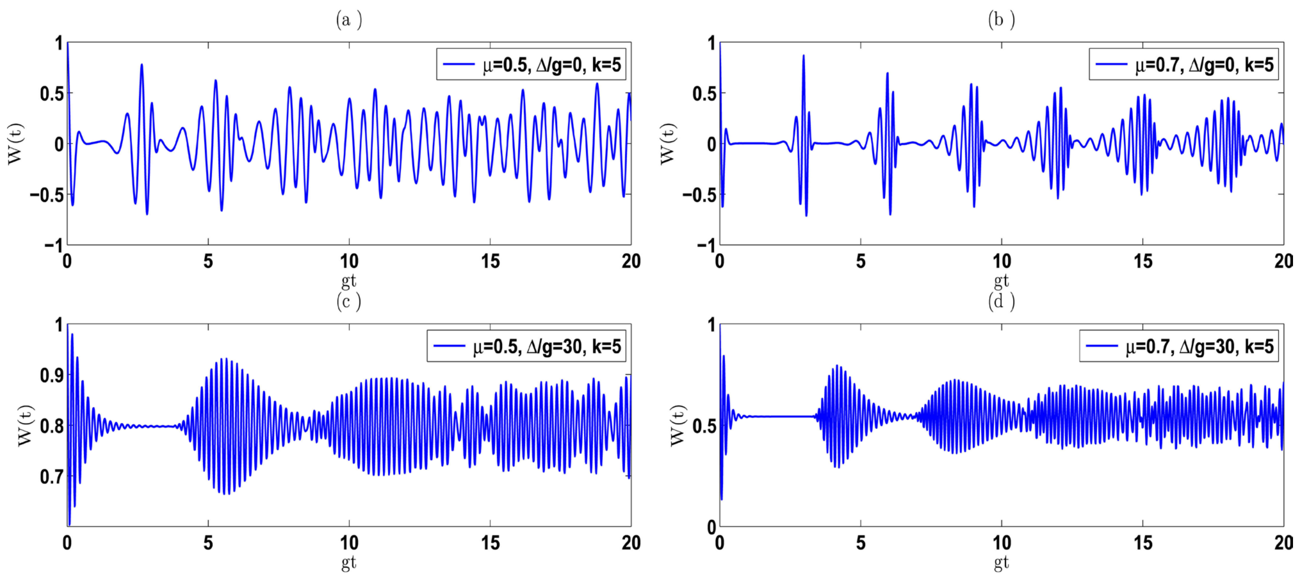

Figure 1. Effects of the detuning Δ and the Perelomov μ parameters on the evolution of the atomic inversion. The atom is initially in the excited state and the field mode in the Perelomov coherent state with the Bargmann index $k = 5$ and the atomic motion is neglected.

amplitude of each period of the revival (slow decay) and we observe somewhat a compress in the fluctuations during the revival periods see Figure 1(b). On the other hand, Figure 1(c) displays the case when the detuning is taken into account, where $\Delta/g = 30$ and $\mu = 0.5$ a dramatic change can be seen in the function behavior. In this case the function is shifted upwards and fluctuates around 0.8 and the atom most of the time in its upper state. When we increase the value of μ to 0.7 a more increment in the value of the function amplitude is observed. We also realize that the function is shifted downwards and fluctuates around 0.55 see Figure 1(d).

In order to discuss effects of atomic motion and field mode structure on the atomic inversion, we have plotted in Figure 2 the atomic inversion against the time gt, for fixed value of the Bargmann index $k = 5$ and absence of detuning parameter. In Figure 2(a) and Figure 2(b), the atomic motion is considered with $p = 1$ and different values of μ. One can see that as the values of μ decrease, the revival patterns also increase, while in Figure 2(a) and Figure 2(c), with $\mu = 0.5$ and different values of p. One can see that as the values of p increase, the revival patterns also increase. Also, we see that the $g\Omega(t)$ is a periodical function on the scale time gt with period $\frac{2\pi}{p}$.

4. Von Neumann Entropy (S_a)

In this section we study the evaluation of the von Neumann entropy, defined as

$$S = -Tr\{\rho \ln \rho\} \tag{26}$$

where ρ is the density operator for a given quantum system. For a pure state, $S = 0$, but when $S \neq 0$ the system is in a mixed state. The entropies of the atomic and the field systems can be defined through the corresponding reduced

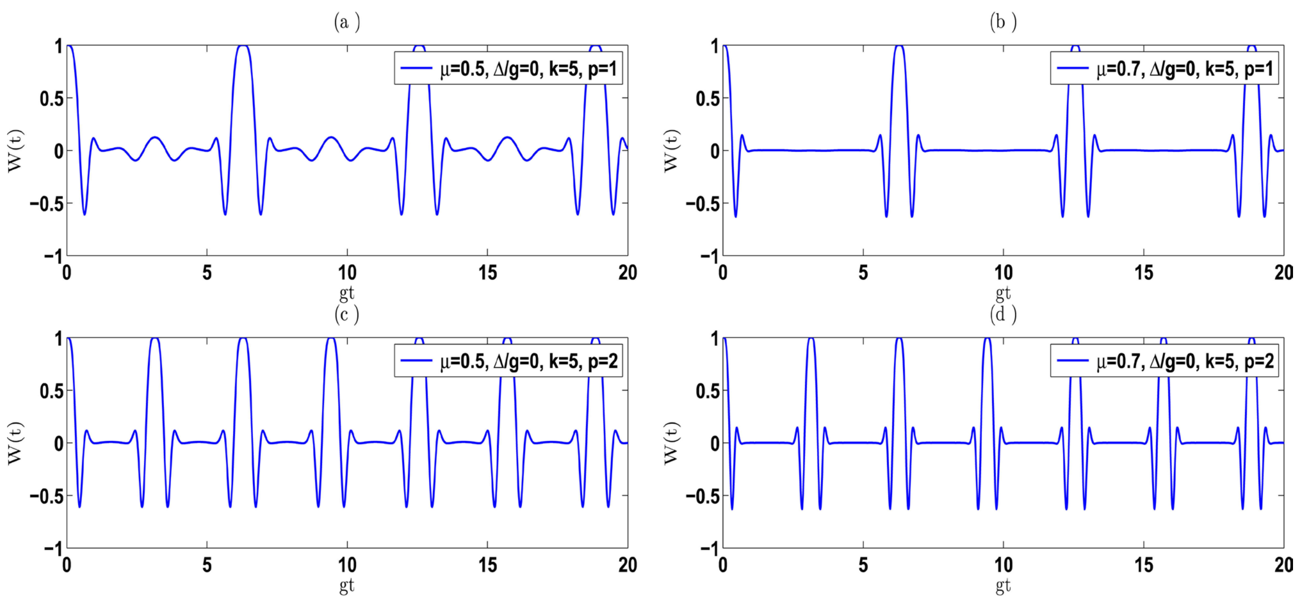

Figure 2. The same as Figure 1 but the atomic motion is taken into account and the absence of detuning parameter.

operators [35]. The time evolution of the field entropy (von Neumann entropy) carries information about the atom-field entanglement. For the system in which both the atom and the field starts from the decoupled pure states, the atomic and the field entropies are equal and may be expressed in terms of the eigenvalues $\lambda^{\pm}$ of the reduced field density operator through the relation.

$$\text{entropy} = -\left(\lambda^{(+)} \ln \lambda^{(+)} + \lambda^{(-)} \ln \lambda^{(-)}\right) \tag{27}$$

To discuss the entanglement of the atomic system with the field we plot **Figure 3** for the fixed value of $k = 5$ and for different values of the other involved parameters. **Figure 3(a)** displays the evolution of the von Neumann entropy in the absence of the detuning parameter and the Perelomov parameter $\mu = 0.7$. We can see that the entanglement reaches the maximum (0.66) at the beginning and then gets back to its minimum (0.035) gradually and after that it returns back and oscillates irregular during the "revival" period. In the "collapse" period, the system tends to return the initial pure state. However, it is observed that as the detuning parameter $\Delta/g = 30$ is introduced the maximum value of the von Neumann entropy is decreased and the minimum value increased in contrast to the case of absence of the detuning parameter see **Figure 3(a)** & **Figure 3(c)**. On the other hand to show how the information entropy are affected by the increasing of the Perelomov parameter we set the Perelomov parameter $\mu = 0.97$ in **Figure 3(b)** and **Figure 3(d)** we can see that increase in the Perelomov parameter leads to an increase in the maximum value (0.69) and the minimum (0.572) entanglement and then the function S_a oscillates regularly around the maximum entanglement but it does not return the initial pure state anymore.

Figure 4 illustrate the dynamical properties of the von Neumann entropy

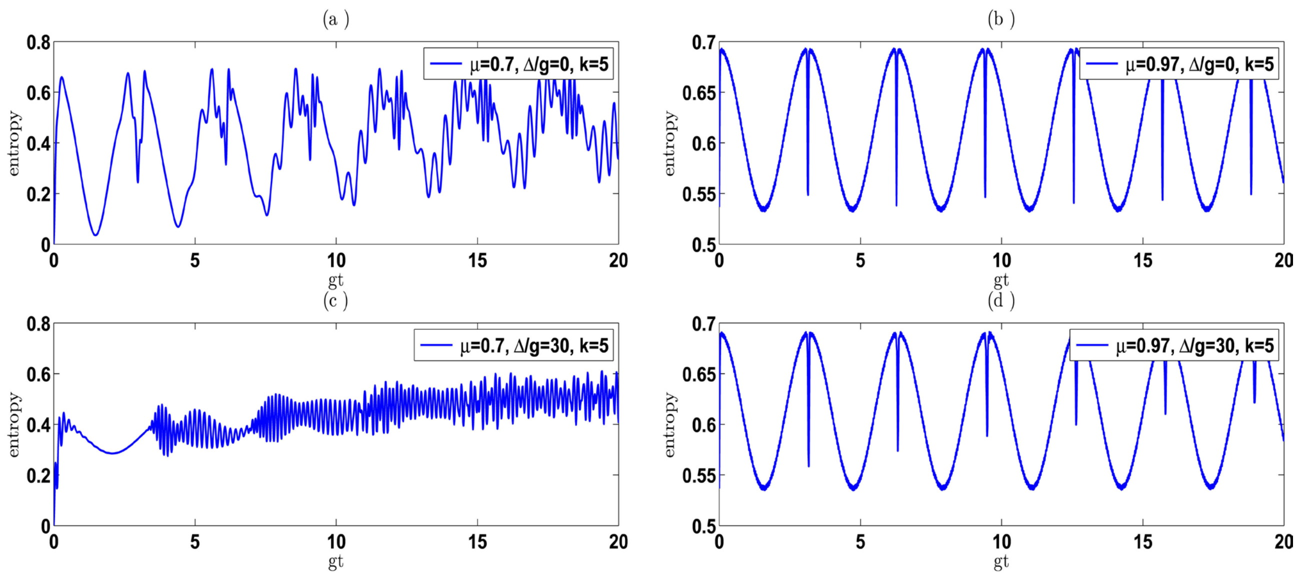

Figure 3. Effects of the detuning Δ and the Perelomov μ parameters on the evolution of the von Neumann entropy. The atom is initially in the excited state and the field mode in the Perelomov coherent state with the Bargmann index $k = 5$ and the atomic motion is neglected.

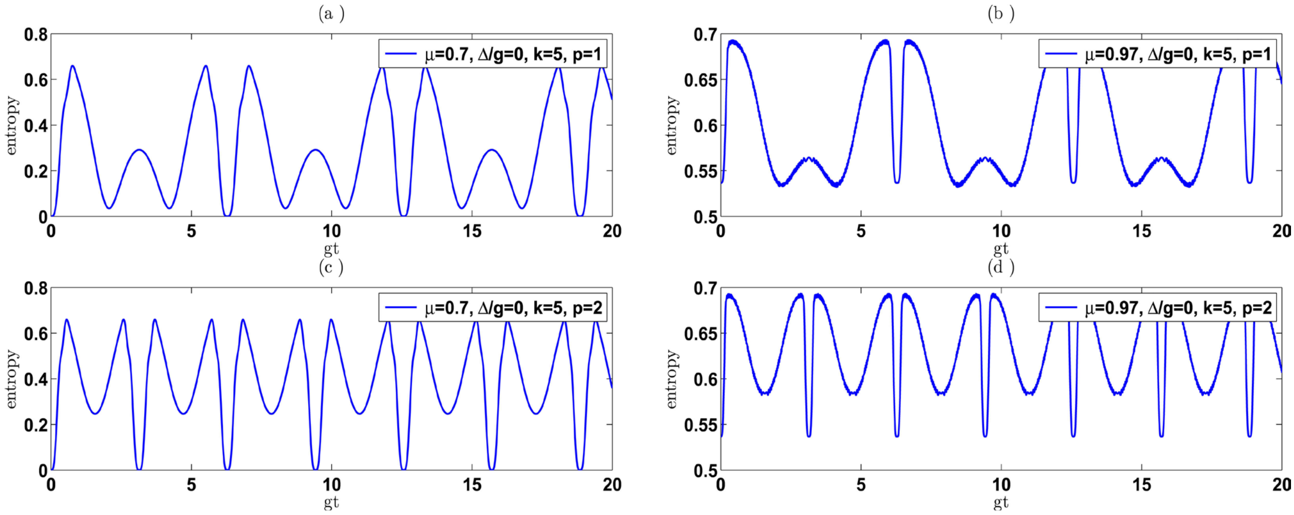

Figure 4. The same as **Figure 3** but the atomic motion is taken into account and the absence of detuning parameter.

when atomic motion is taken into account. From these figures, we can conclude that: 1) the atomic motion leads to the periodic evolution of the field entropy; 2) an increase of the parameter p leads to the shortening of the evolution periodicity of the von Neumann entropy.

Entropy Squeezing Properties

Now we study the squeezing of the atomic entropy, where we can express the entropy squeezing of the two-level atom by using the quantum information entropy theory. The information entropy of the Pauli operators $\langle\hat{\sigma}_\alpha\rangle(\alpha = x, y, z)$

$$\langle\hat{\sigma}_x\rangle = 2\,\mathrm{Re}(\rho_{12}), \quad \langle\hat{\sigma}_y\rangle = 2\,\mathrm{Im}(\rho_{12}), \quad \langle\hat{\sigma}_z\rangle = \rho_{11} - \rho_{22} \tag{28}$$

for a two-level atom system is

$$H(\hat{\sigma}_\alpha) = -\sum_{i=1}^{2} P_i(\hat{\sigma}_\alpha)\ln P_i(\hat{\sigma}_\alpha), \tag{29}$$

where $P_i(\hat{\sigma}_\alpha) = \langle\psi_{\alpha i}|\rho|\psi_{\alpha i}\rangle$ $(i = 1, 2)$, which are the probability distributions for two possible measurements of an operator $\hat{\sigma}_\alpha$. $H(\hat{\sigma}_x), H(y)$, and $H(z)$ satisfy

$$H(\hat{\sigma}_x) + H(\hat{\sigma}_y) \geq 2\ln 2 - H(\hat{\sigma}_z), \tag{30}$$

which may also be rewritten as

$$\delta H(\hat{\sigma}_x)\,\delta H(\hat{\sigma}_y) \geq \frac{4}{\delta H(\hat{\sigma}_z)}, \tag{31}$$

where $\delta H(\hat{\sigma}_\alpha) = \exp(H(\hat{\sigma}_\alpha))$. The squeezing of the atom is determined by using the EUR Equation (31) named entropy squeezing. The fluctuation in the component $(\hat{\sigma}_\alpha, \alpha = x, y)$ of the atomic dipole is said to be squeezed in entropy if the information entropy $H(\hat{\sigma}_\alpha)$ satisfies the following condition:

$$E(\hat{\sigma}_\alpha) = \delta H(\hat{\sigma}_\alpha) - \frac{2}{\sqrt{|\delta H(\hat{\sigma}_z)|}} < 0, \quad (\alpha = x, y). \tag{32}$$

By using $\rho_A(t)$, we can obtain the information entropies of the atomic operators $\hat{\sigma}_x, \hat{\sigma}_y$ and $\hat{\sigma}_z$ as follows:

$$\begin{aligned} H(\hat{\sigma}_x) &= -(\xi_1 \ln \xi_1 + \xi_2 \ln \xi_2), \\ H(\hat{\sigma}_y) &= -(\xi_3 \ln \xi_3 + \xi_4 \ln \xi_4), \\ H(\hat{\sigma}_z) &= -(\rho_{22} \ln \rho_{22} + \rho_{11} \ln \rho_{11}), \end{aligned} \tag{33}$$

where

$$\begin{aligned} \xi_1 &= \frac{1}{2} + \mathrm{Re}(\rho_{12}), \quad \xi_2 = \frac{1}{2} - \mathrm{Re}(\rho_{12}), \\ \xi_3 &= \frac{1}{2} + \mathrm{Im}(\rho_{12}), \quad \xi_4 = \frac{1}{2} - \mathrm{Im}(\rho_{12}), \end{aligned} \tag{34}$$

We discuss the effects of the detuning parameter, the Perelomov parameter, the atomic motion and the field-mode structure on the properties of the entropy squeezing. The time evolution of the squeezing factors, $E(\hat{\sigma}_x), E(\hat{\sigma}_y)$ are shown in Figure 5 for the atom initially in the excited state and the field in a Perelomov coherent state with Bargmann index $k = 5$ in the absence of the atomic motion. We see from Figure 5(b) that both $E(\hat{\sigma}_x)$ and $E(\hat{\sigma}_y)$ predict no squeezing in the variables $\hat{\sigma}_x$ and $\hat{\sigma}_y$ when the Perelomov parameter $\mu = 0.97$. A comparison of frame (a) in Figure 5 shows that, entropy squeezing in every period of the evolution of $E(\hat{\sigma}_y)$ while still no squeezing in the behavior of the entropy squeezing factor $E(\hat{\sigma}_x)$ when the Perelomov parameter $\mu = 0.7$. From these we can conclude the following an decrease in parameter μ results in not only the spread of revival periods but also squeezing of the evolution of $E(\hat{\sigma}_y)$. On the other hand Figure 5(c) shows the effect of the detuning parameter on the time evolution of the squeezing factors $E(\hat{\sigma}_x)$ and $E(\hat{\sigma}_y)$ with $\mu = 0.7$. We see that the squeezing in

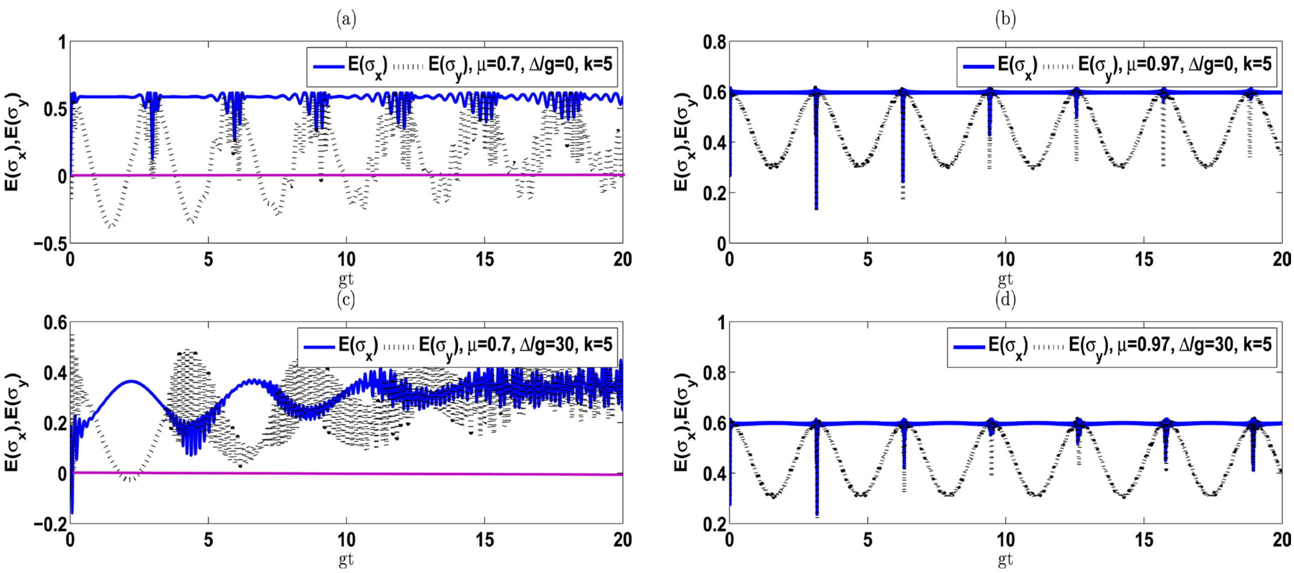

Figure 5. Effects of the detuning Δ and the Perelomov μ parameters on the evolution of the entropy squeezing factors $E(\hat{\sigma}_x)$ and $E(\hat{\sigma}_y)$. The atom is initially in the excited state and the field mode in the Perelomov coherent state with the Bargmann index $k = 5$ and the atomic motion is neglected.

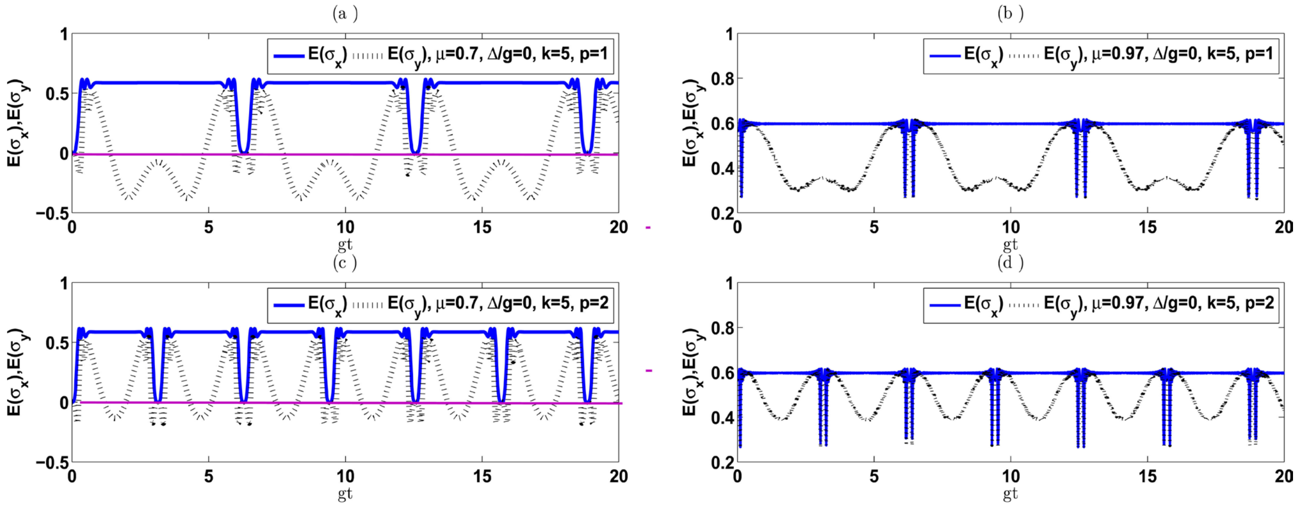

Figure 6. The same as **Figure 5** but the atomic motion is taken into account and the absence of detuning parameter.

$E(\hat{\sigma}_x)$ and $E(\hat{\sigma}_y)$ disappears quickly and will not reappear anymore. While in **Figure 5(d)** we observe the squeezing Completely disappeared when $\mu = 0.97$. From these we can conclude the following an increases in parameter μ leads to the disappearance of the squeezing in the absence and presence of the detuning parameter. Finally, to discuss the impact of the atomic motion and the field-mode structure, on the evolution of the entropy squeezing factors $E(\hat{\sigma}_x)$ and $E(\hat{\sigma}_y)$ we have plotted **Figure 6** against the time t for fixed value of the Bargmann index $k = 5$ and $\Delta = 0$. This, in fact, would help us to examine the effect of the atomic motion and the field-mode structure. In **Figure 6** we show the influence of the atomic motion and the field-mode structure parameter on the time evolution of $E(\hat{\sigma}_x)$. It does not show the squeezing in every period of the evolution of $E(\hat{\sigma}_x)$. **Figure 6(a)** and **Figure 6(c)** illustrate the time evolution of $E(\hat{\sigma}_y)$ when the atom is moving at a speed $v = gl/\pi$ and the field-mode structure parameters are $p = 1, p = 2$ respectively. In **Figure 6(a)** ($p = 1$), the range of entropy squeezing time increases more distinctly than that in **Figure 5(a)**, but the value of maximal squeezing remains the same. From **Figure 6(c)** ($p = 2$) one can observe that the duration of entropy squeezing decreases and the degree of squeezing weakens as p increases. From these we can conclude that atomic motion causes the curve of the evolution of $E(\hat{\sigma}_y)$ to change. The period of the entropy squeezing and the duration of the information entropy squeezing are determined by the field-mode structure.

5. Summary

We have introduced the problem of the interaction between a two-level atom and a quantum system. We have used the Perelomov generalized coherent state as the initial state for the $SU(1,1)$ quantum system, while we considered the atom to be initial in the excited state. The wave function is calculated via the evolution operator and the result is used to obtain the atomic density operator. The effects of the different parameters such as detuning parameter, atomic motion and Perelomov parameter on the atomic inversion, von Neumann entropy and Entropy squeezing have been studied. It is shown that some new features such as: 1) the Perelomov parameter is very useful in generating a high amount of entanglement in the absence and presence of the detuning parameter; 2) the atomic inversion is sensitive to the variation of the Perelomov parameter in the presence of detuning parameter; 3) the atomic motion and field-mode structure parameter play an important role on the time evolution of the atomic inversion, entropy squeezing and von Neumann entropy.

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