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Interplay of Quantum Stochastic and Dynamical Maps to Discern Markovian and Non-Markovian Transitions

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ABSTRACT

It is known that the dynamical evolution of a system, from an initial tensor product state of system and environment, to any two later times, t_1, t_2 ($t_2 > t_1$), are both completely positive (CP) but in the intermediate times between t_1 and t_2 it need not be CP. This reveals the key to the Markov (if CP) and non Markov (if it is not CP) avataras of the intermediate dynamics. This is brought out here in terms of the quantum stochastic map A and the associated dynamical map B—without resorting to master equation approaches. We investigate these features with four examples which have entirely different physical origins: 1) A two qubit Werner state map with time dependent noise parameter; 2) Phenomenological model of a recent optical experiment (Nature Physics, 7, 931 (2011)) on the open system evolution of photon polarization; 3) Hamiltonian dynamics of a qubit coupled to a bath of N qubits; 4) Two qubit unitary dynamics of Jordan *et al*. (Phys. Rev. A 70, 052110 (2004) with initial product states of qubits. In all these models, it is shown that the positivity/negativity of the eigenvalues of intermediate time dynamical B map determines the Markov/non-Markov nature of the dynamics.

Keywords: Open System Dynamics; Non Markovianity; Not Completely Positive Maps

1. Introduction

Understanding the basic nature of dynamical evolution of a quantum system,which interacts with an inaccessible environment, attracts growing importance in recent years [1,2]. This offers the key to achieve control over quantum systems—towards their applications in the emerging field of quantum computation and communication [3]. While the overall system-environment state evolves unitarily, the dynamics governing the system is described by a completely positive (CP), trace preserving map [4-8].

Markov approximation holds when the future dynamics depends only on the present state—and not on the history of the system *i.e*. memory effects are negligible. The corresponding Markov dynamical map constitutes a trace preserving, CP, continuous one-parameter quantum semi-group [9,10]. Markov dynamics governing the evolution of the system density matrix is conventionally described by Lindblad-Gorini-Kossakowski-Sudarshan (LGKS) master equation [9,10] $\frac{d\rho}{dt} = L\rho$ where L is the time-independent Lindbladian operator generating the underlying quantum Markov semigroup. Generalized Markov processes are formulated in terms of time-dependent Lindblad generators and the associated trace preserving CP dynamical map is a two-parameter divisible map [11,12], which too corresponds to memory-less Markovian evolution.

Not completely positive (NCP) maps do make their presence felt in the open-system dynamics obtained from the joint unitary evolution—if the system and environment are in an initially quantum correlated state [13-16]. In such cases, the open-system evolution turns out to be non-Markovian [17]. However, the source of such non-Markovianity could not be attributed entirely to either initial system-environment correlations or their dynamical interaction or both. This issue gets refined if initial global state is in the tensor product form, in which case the sole cause of Markovianity/non-Markovianity could be attributed to dynamics alone. It is known that the time evolution of a subsystem from an initial tensor product form to two different later times, t_1, t_2 ($t_2 > t_1$), are both CP. However the dynamics in the intermediate time steps between t_1 and t_2 need not be CP. The quantum stochastic A and dynamical B maps—first introduced as a quantum extension of classical stochastic dynamics—by Sudarshan, Mathews, Rau and Jordan (SMRJ) [7,8] nearly five decades ago, offer an elegant approach to explore Markovian/non-Markovian nature of open system evolution. The interplay of A and B maps at intermediate times, to bring out the Markov or non-Markov avataras of open system evolution, is established in this paper.

To place these ideas succinctly, there are three basic aspects in open system quantum dynamics: 1) nature of dynamical interaction between the system and its environment, 2) role of initial correlations in system-environment state and 3) nature of dynamics at intermediate times. Last few years have witnessed intense efforts towards understanding these [11-32]. The third issue is the focus here to discern the Markov/non-Markov nature of dynamics in terms of intermediate time A and B maps.

The contents are organized as follows: In Section 2 some basic concepts [7,8] on A and B maps are given. The emergence of CP/NCP maps, at intermediate times, under open system dynamics is discussed in Section 3. Section 4 is devoted to a powerful link (brought out by Jamiolkowski isomorphism) between the B map and the dynamical state. Some illustrative examples of dynamical B map to investigate the CP/NCP nature of dynamics at intermediate times are discussed in Section 5. The examples are chosen from different origins: one based entirely from the general considerations of Jamiolkowski isomorphism; second one on the recent all-optical open system experiment to drive Markovian to non-Markovian transitions; the other two examples are based on open system Hamiltonian dynamics. In all these four examples, no master equation is employed in the deduction of Markov to non-Markov transitions—but the CP/NCP nature of the intermediate dynamical map (via the sign of the eigenvalue of the B map) has been invoked. Section 6 has some concluding remarks.

2. Preliminary Ideas on Dynamical A and B Maps

The stochastic A and dynamical B maps [7,8] transform the initial system density matrix $\rho_S(t_0)$ to final density matrix $\rho_S(t)$ via,

$$\left[\rho_S(t)\right]_{b_1b_2} = \sum_{a_1a_2}\left[A(t,t_0)\right]_{b_1b_2;a_1a_2}\left[\rho_S(t_0)\right]_{a_1a_2} \tag{1}$$

$$\left[\rho_S(t)\right]_{b_1b_2} = \sum_{a_1a_2}\left[B(t,t_0)\right]_{b_1a_1;b_2a_2}\left[\rho_S(t_0)\right]_{a_1a_2} \tag{2}$$

$$a_1, a_2, b_1, b_2 = 1,2, \quad ,\mathrm{d}$$

where the realigned matrix B is defined by,

$$B_{b_1a_1;b_2a_2} = A_{b_1b_2;a_1a_2} \tag{3}$$

The requirement that the evolved density matrix $\rho_S(t)$ has unit trace and is Hermitian, positive semi-definite places the following conditions on A and B [7,8]:

Trace Preservation:

$$\sum_{b_1} A_{b_1b_1;a_1a_2} = \delta_{a_1a_2};\ \sum_{b_1} B_{b_1a_1;b_1a_2} = \delta_{a_1a_2}$$

Hermiticity:

$$A_{b_1b_2;a_1a_2} = A^*_{b_2b_1;a_2a_1};\ B_{b_1a_1;b_2a_2} = B^*_{b_2a_2;b_1a_1} \tag{4}$$

Positivity:

$$\sum_{a_1,a_2,b_1,b_2} x^*_{b_1}\, x_{b_2}\, A_{b_1b_2;a_1a_2}\, y_{a_1}\, y^*_{a_2} \ge 0$$

$$\sum_{a_1,a_2,b_1,b_2} x^*_{b_1}\, y_{a_1}\, B_{b_1a_1;b_2a_2}\, x_{b_2}\, y^*_{a_2} \ge 0$$

It may be readily identified that the dynamical B map is positive, Hermitian $d^2 \times d^2$ matrix with trace d—corresponding to CP evolution. We would also like to point out here that the composition of two stochastic A-maps, $A_1 * A_2$ transforming

$$\rho_S(t_0) \xrightarrow{A_1} \rho_S(t_1) \xrightarrow{A_2} \rho_S(t_2)$$

is merely a matrix multiplication, whereas it is not so in its B-form.

3. CP/NCP Nature of Intermediate Time A and B Maps

Let us consider unitary evolution of global system environment state $\rho_S(t_0)\otimes\rho_E(t_0)$ from an initial time t_0 to a final time t_2—passing through an intermediate instant t_1 (*i.e.* $t_0 < t_1 < t_2$). The A-map associated with t_0 to t_1 and that between t_0 to t_2 are identified as follows:

$$\begin{aligned}&\mathrm{Tr}_E\left[U(t_j,t_0)\rho_S(t_0)\otimes\rho_E(t_0)U^\dagger(t_j,t_0)\right]\\ &= A(t_j,t_0)\rho_S(t_0) = \rho_S(t_j),\ j = 1,2.\end{aligned} \tag{5}$$

The stochastic map $A(t_j,t_0)$ is completely positive (correspondingly the dynamical matrix $B(t_j,t_0)$ is positive). In order to identify the intermediate stochastic map $A(t_2,t_1)$, we make use of the composition law of unitary evolution $U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0)$:

$$\begin{aligned}&\mathrm{Tr}_E\Big[U(t_2,t_1)\left\{U(t_1,t_0)\rho_S(t_0)\otimes\rho_E(t_0)U^\dagger(t_1,t_0)\right\}\\ &\times U^\dagger(t_2,t_1)\Big] = A(t_2,t_0)\rho_S(t_0)\end{aligned} \tag{6}$$

However, this does not lead naturally to $A(t_2,t_0) = A(t_2,t_1)A(t_1,t_0)$ for the A-map. Invoking Markovian approximation (memoryless reservoir condition[1])

[1]The dynamical evolution of the system density matrix $\rho_S(t_0)\to\rho_S(t)$ is not a local unitary operation, when memoryless reservoir approximation' holds—but it is governed by an irreversible, stochastic $A(t, t_0)$ map.

$$\left\{U\left(t_1,t_0\right)\rho_S\left(t_0\right)\otimes\rho_E\left(t_0\right)U^{\dagger}\left(t_1,t_0\right)\right\}\approx\rho_S\left(t_1\right)\otimes\rho_E\left(t_1\right)$$

the LHS of Equation (6) may be expressed as,

$$\begin{aligned}&\mathrm{Tr}_E\left[U\left(t_2,t_1\right)\rho_S\left(t_1\right)\otimes\rho_E\left(t_1\right)U^{\dagger}\left(t_2,t_1\right)\right]\\&=A\left(t_2,t_1\right)\rho_S\left(t_1\right)\end{aligned}\tag{7}$$

Further, substituting $j=1$ in Equation (5) and expressing $\rho_S\left(t_0\right)=A^{-1}\left(t_1,t_0\right)\rho_S\left(t_1\right)$ in Equation (6) the intermediate A map $A\left(t_2,t_1\right)$ is identified:

$$A\left(t_2,t_1\right)=A\left(t_2,t_0\right)A^{-1}\left(t_1,t_0\right).\tag{8}$$

In other words, when the environment is passive (Markovian dynamics), the intermediate A-map has the divisible composition as in Equation (8). In such cases $A\left(t_2,t_1\right)$ is ensured to be CP—otherwise it is NCP, and hence non-Markovian. Correspondingly, the intermediate B-map $B\left(t_2,t_1\right)$ is positive if the dynamics is Markovian; negative eigenvalues of $B\left(t_2,t_1\right)$ imply non-Markovianity.

4. The *B* Map and the Jamiolkowski Isomorphism

The Jamiolkowski isomorphism [6] provides an insight that the B-map is directly related to a $\mathrm{d}^2\times\mathrm{d}^2$ system-ancilla bipartite density matrix. More specifically, the action of the map $A^{\mathrm{Id}}\otimes A$ on the maximally entangled system-ancilla state

$$\left|\psi_{\mathrm{ME}}\right\rangle=\frac{1}{\sqrt{\mathrm{d}}}\sum_{i=0}^{\mathrm{d}-1}\left|i,i\right\rangle$$

results in the density matrix ρ_{ab} which may be identified to be related to the dynamical B-map *i.e.*,

$$\rho_{ab}=\left[A^{\mathrm{Id}}\otimes A\right]\left|\psi_{ME}\right\rangle\left\langle\psi_{ME}\right|\to\frac{1}{d}B\tag{9}$$

gives an explicit matrix representation for the B-map. (Here A^{Id} is the identity A-map, which leaves the ancilla undisturbed). In detail, we have

$$\begin{aligned}&\left(\rho_{ab}\right)_{a_1b_1;a_2b_2}\\&=\sum_{a_1',a_2',b_1',b_2'}\left[A^{\mathrm{Id}}\otimes A\right]_{a_1b_1a_2b_2;a_1'b_1'a_2'b_2'}\left[\left|\psi_{ME}\right\rangle\left\langle\psi_{ME}\right|\right]_{a_1'b_1';a_2'b_2'}\\&=\frac{1}{\mathrm{d}}\sum_{a_1',a_2',b_1',b_2'}\delta_{a_1,a_1'}\delta_{a_2,a_2'}A_{b_1b_2;b_1'b_2'}\delta_{a_1',b_1'}\delta_{a_2',b_2'}\\&=\frac{1}{\mathrm{d}}A_{b_1b_2;a_1a_2}=\frac{1}{\mathrm{d}}B_{b_1a_1;b_2a_2}\end{aligned}\tag{10}$$

or,

$$\left(\rho_{ba}\right)_{b_1a_1;b_2a_2}=\frac{1}{\mathrm{d}}B_{b_1a_1;b_2a_2}$$

(Here we have used

$$\left[A^{\mathrm{Id}}\otimes A\right]_{a_1b_1a_2b_2;a_1'b_1'a_2'b_2'}=\delta_{a_1,a_1'}\delta_{a_2,a_2'}A_{b_1b_2;b_1'b_2'}$$

and

$$\begin{aligned}\left[\left|\psi_{ME}\right\rangle\left\langle\psi_{ME}\right|\right]_{a_1',b_1';a_2',b_2'}&=\frac{1}{\mathrm{d}}\sum_{i,j=1}^{\mathrm{d}}\left\langle a_{1'}b_{1'}\middle|i,i\right\rangle\left\langle j,j\middle|a_2',b_2'\right\rangle\\&=\frac{1}{\mathrm{d}}\delta_{a_1',b_1'}\delta_{a_2',b_2'}\end{aligned}$$

in the second line of Equation (10)).

In other words, Jamiolkowski isomorphism maps *every completely positive dynamical map B* acting on d dimensional space to a positive definite $\mathrm{d}^2\times\mathrm{d}^2$ bipartite density matrix ρ_{ab} (See Equation (10))—whose partial trace (over the first subsystem—as seen from the trace preservation property on dynamical map B (as in Equation (4)) is a maximally disordered state. One such set of bipartite $\mathrm{d}\times\mathrm{d}$ density matrices belong to the class that are invariant under $U\otimes U$ [33]—which constitute the well-known Werner density matrices. One may now identify several toy models of dynamical B maps—including the two qubit Werner state example motivated by the above remark—to investigate the nature of intermediate time dynamics.

In view of the connection established between dynamical map B with the resultant bipartite density matrix we identify the following: when we consider the evolution of a system—which is initially uncorrelated with its environment—from t_0 to two different later times t_1, t_2, ($t_2>t_1$) the corresponding dynamical maps $B\left(t_1,t_0\right)$ and $B\left(t_2,t_0\right)$ are both CP—and would correspond to *physical* bipartite density matrices under Jamiolkowski isomorphism. On the other hand, at an intermediate time t_1 the system and environment may get correlated (*i.e.* when Markov approximation $\rho_{SE}\left(t_1\right)\approx\rho_S\left(t_1\right)\otimes\rho_E\left(t_2\right)$ does not hold). Consequently, further evolution from t_1 to t_2 is not ensured to be CP [13-16] and hence the corresponding intermediate time dynamical map $B\left(t_2,t_1\right)$ does not correspond to a legitimate bipartite density matrix under Jamiolkowski isomorphism. Non-positive eigenvalues of intermediate time dynamical map $B\left(t_2,t_1\right)$ capture intermediate system-environment correlations, revealing in turn, non-Markovianity of the underlying open system dynamics.

5. Examples

In this section we present specific examples chosen to illustrate the features of intermediate dynamical maps: 1) A toy model map inspired by Jamiolkowski isomorphism (which associates any bipartite density matrix consisting of a maximally disordered subsystem with a dynamical B map). This is not based on any Hamiltonian underpinning.

2) Recent optical experiment by Liu *et al.*, [34] on open system evolution of photon polarization to bring out non-Markovianity features is reinterpreted in terms of NCP nature of the intermediate dynamical map. 3) Intermediate dynamical map in the Hamiltonian evolution of a two-level system coupled to N two-level systems [32]. 4) Open system dynamics arising from a two qubit unitary evolution [13].

5.1. A Toy Model Dynamical Map

The two qubit Werner density matrix is a natural choice for a prototype of dynamical B-map—arising from general considerations of the Jamiolkowski isomorphism:

$$B(t,0)=\frac{[1-p(t)]}{2}I_2\otimes I_2+\frac{p(t)}{2}\left|\Psi^{(-)}\right\rangle\left\langle\Psi^{(-)}\right| \quad (11)$$

with a time dependent noise parameter $0\le p(t)\le 1$, and $\left|\Psi^{(-)}\right\rangle=\frac{1}{\sqrt{2}}(|0,0\rangle-|1,1\rangle)$ is the Bell state. For a dynamical map, time dependence in $p(t)$ occurs due to the underlying Hamiltonian evolution. This state is especially important in that it exhibits both separable and entangled states, as its characteristic parameter $p(t)$ is varied. Its use here as a valid B-map is novel in identifying transitions between Markovianity and non-Markovianity in the dynamics as captured from their intermediate time behaviour.

On evaluating the corresponding A-map $A(t,0)$ (expressed in the standard $\{|0,0\rangle,|0,1\rangle,||1,0\rangle,|1,1\rangle\}$ basis) *i.e.*,

$$A(t,0)=\begin{pmatrix}\frac{1+p(t)}{2} & 0 & 0 & 0\\ 0 & \frac{1-p(t)}{2} & -\frac{p(t)}{4} & 0\\ 0 & -\frac{p(t)}{4} & \frac{1-p(t)}{2} & 0\\ 0 & 0 & 0 & \frac{1+p(t)}{2}\end{pmatrix}$$

one can obtain the intermediate dynamical map $A(t_2,t_1)=A(t_2,0)A^{-1}(t_1,0)$. The intermediate time B-map $B(t_2,t_1)$ is given by

$$B(t_2,t_1)=\frac{1}{2}\left(1-\frac{p(t_2)}{p(t_1)}\right)I_2\otimes I_2+\frac{2p(t_2)}{p(t_1)}\left|\Psi^{(-)}\right\rangle\left\langle\Psi^{(-)}\right|$$

Its eigenvalues are

$$\lambda_1=\lambda_2=\lambda_3=\frac{1}{2}\left(1-\frac{p(t_2)}{p(t_1)}\right)$$

and

$$\lambda_4=\frac{1}{2}\left(1+\frac{3p(t_2)}{p(t_1)}\right)$$

A choice $p(t)=\cos^{2M}(at)$ for any $M\ge 1$ leads to NCPness of the intermediate map—as the eigenvalues $\lambda_{1,2,3}\equiv\lambda$ of $B(t_2,t_1)$ may assume negative values —and hence non-Markovian dynamics ensues. We have plotted the negative eigenvalue λ of $B(t_2,t_1)$ as a function of $\mu=t_2/t_1$ and for typical values of $M=1,3,5$ in **Figure 1**. This reveals transitions from Markovianity to non-Markovianity and back in this model.

Another choice $p(t)=e^{-\alpha t}$ corresponds to a CP intermediate map—resulting entirely in a Markovian process. In this case, we also find that $A(t_2,t_1)=A(t_2-t_1)$ and this forms a Markov semigroup. However, if $p(t)=e^{-\alpha t^{\beta}}$, $(\beta\ne 1)$, the intermediate map is still CP (and hence Markovian), though $A(t_2,t_1)\ne A(t_2-t_1)$ and therefore, it does not constitute a one-parameter semigroup.

Furthermore, we wish to illustrate through this toy model that concurrence of

$$\rho_{ab}(t)=\frac{1}{\mathrm{d}}B(t,0)$$

(given by $C=(3p(t)-1)/2$) can never increase as a result of Markovian evolution. This is because ensuing dynamics is a local CP map on the system. Any temporary regain of system-ancilla entanglement during the course of evolution is clearly attributed to the back-flow from environment to the system—which is a signature of non-Markovian process. This feature is displayed in **Figure 2** by plotting the concurrence of $\rho_{ab}(t)$ for different choices of $p(t)$.

5.2. Optical Experiment

Recently, Liu *et al.* [34] reported an optical experiment on the open quantum system constituted by the polarization degree of freedom of photons (system) coupled to the frequency degree of freedom (environment). They reported transition between Markovian and non-Markovian regimes. It may be pointed out that in this optical experiment non-Markovianity is characterized in terms of increase of the distinguishability of quantum states, which signifies reverse flow of information from environment back to the system [28]—and not in terms of deviation from divisibility [11,12]. In this paper we would analyze the non-Markovian nature of dynamics in terms of the negative eigenvalues of intermediate time dynamical map $B(t_2,t_1)$.

The dynamical evolution of the horizontal and vertical polarization states $|H\rangle,|V\rangle$ of the photon is captured by the following transformation:

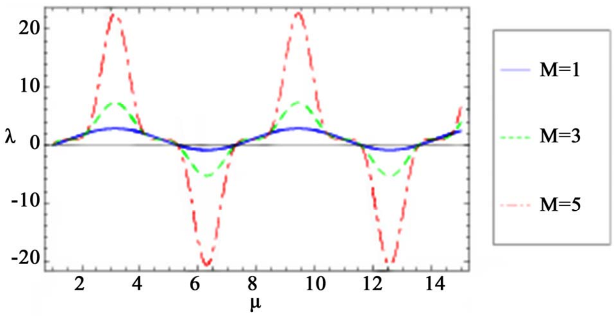

Figure 1. A plot of the eigenvalue λ of $B(t_2,t_1)$ versus $\mu = t_2/t_1$ for different values of *M*. The dynamics is non-Markovian when λ assumes negative values and otherwise it is Markovian.

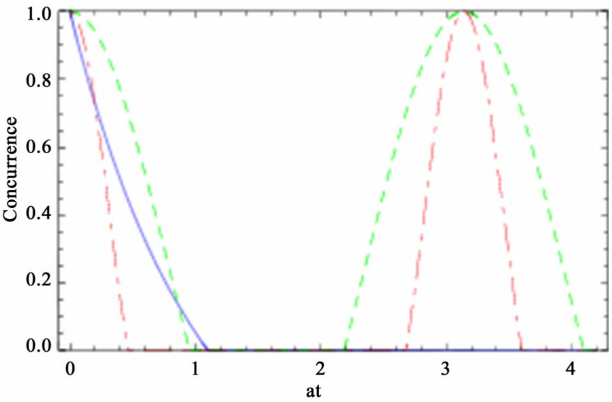

Figure 2. Concurrence $C = (3p(t)-1)/2$ of the system-ancilla state $\rho_{ab}(t) = \frac{[1-p(t)]}{4} I_2 \otimes I_2 + p(t)|\psi_{ME}\rangle\langle\psi_{ME}|$, vs. scaled time *at*, for the following choices i) Markov process: $p(t) = e^{-at}$ (solid line) and ii) non-Markov process: $p(t) = \cos^{2M}(at)$, $M = 1$ (dashed line) and $M = 5$ (dot-dashed line). Note that there is a death and re-birth of entanglement (dash, dot-dashed lines) due to back-flow from environment.

$$
\begin{array}{cc}
|H\rangle\langle H\rangle & |H\rangle\langle H| \\
|V\rangle\langle V\rangle & |V\rangle\langle V| \\
|H\rangle\langle V\rangle & \kappa^*(t)|H\rangle\langle V| \\
|V\rangle\langle H\rangle & \kappa(t)|V\rangle\langle H|
\end{array}
$$

Here $\kappa(t)$ denotes the decoherence function, magnitude of which is modelled as,

$$
\begin{aligned}
|\kappa(t)| &= \frac{e^{-\frac{1}{2}\sigma^2(\Delta nt)^2}}{1+A_\theta}\sqrt{1+A_\theta^2+2A_\theta\cos(\Delta\omega\cdot\Delta nt)} \\
&= e^{-\frac{1}{2}\sigma^2(\Delta nt)^2}\sqrt{1-4A_1(1-A_1)\sin^2\left(\frac{\Delta\omega\cdot\Delta nt}{2}\right)}
\end{aligned}
\tag{14}
$$

where $\Delta n = n_V - n_H$ is the difference between the refractive indices of vertically and horizontally polarized light; $\Delta\omega = \omega_2 - \omega_1$ distance between two frequency peaks (for details see [34]), $A_1 = \frac{1}{1+A_\theta}, 0 \le A_1 \le 1$.

The corresponding A and B maps (in the $\{HH, HV, VH, VV\}$ basis) are readily identified to be,

$$
A(t,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa^*(t) & 0 & 0 \\ 0 & 0 & \kappa(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
B(t,0) = \begin{pmatrix} 1 & 0 & 0 & \kappa^*(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \kappa(t) & 0 & 0 & 1 \end{pmatrix}
$$

We construct the intermediate time dynamical map $B(t_2,t_1)$ from the corresponding $A(t_2,t_1) = A(t_2,0)A^{-1}(t_1,0)$ to obtain,

$$
B(t_2,t_1) = \begin{pmatrix} 1 & 0 & 0 & \frac{\kappa^*(t_2)}{\kappa^*(t_1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\kappa(t_2)}{\kappa(t_1)} & 0 & 0 & 1 \end{pmatrix}. \tag{16}
$$

Eigenvalues of $B(t_2,t_1)$ are given by,

$$
\lambda_{1,4} = 1 \pm \left|\frac{\kappa(t_2)}{\kappa(t_1)}\right|, \quad \lambda_{2,3} = 0. \tag{17}
$$

The eigenvalue λ_4 can assume negative values indicating Markovian/non-Markovian regimes. A plot of the negative eigenvalue as a function of A_1, for different ratios t_2/t_1 (for the choice of parameters $\Delta\omega = 1.6\times10^{13}$ Hz, $\sigma = 1.8\times10^{12}$ Hz, which are employed in Ref. [34]) is given in **Figure 3** where one can clearly see the Markovian ($\lambda_4 \ge 0$) and non-Markovian ($\lambda_4 < 0$) regimes.

5.3. Hamiltonian Evolution of a Two Level System Coupled to a Bath of *N* Spins

We now present a Hamiltonian model, which give rise to explicit structure of time dependence in the open system evolution. Interaction Hamiltonian considered here is [28,32]

$$
H = \frac{A}{\sqrt{N}}\sigma_z\sum_{k=1}^{N}\sigma_{kz}. \tag{18}
$$

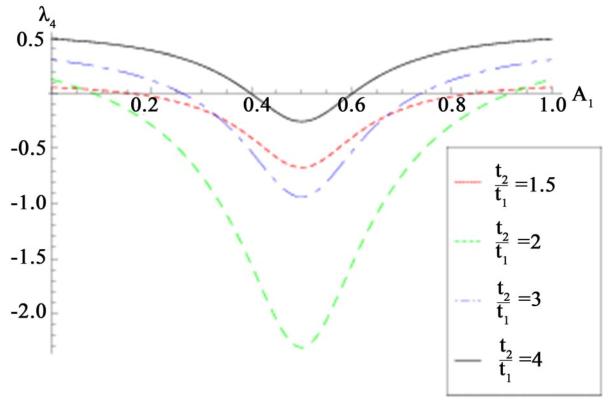

Figure 3. A plot of the eigenvalue λ_4 of $B(t_2,t_1)$ versus A_1 for different values of $\mu = t_2/t_1$ and for the choice of parameters $\Delta\omega = 1.6 \times 10^{13}$ Hz, $\sigma = 1.8 \times 10^{12}$ Hz.

This is a simplified model of a hyperfine interaction of a spin—1/2 system with N spin—1/2 nuclear environment in a quantum dot. Taking the initial system-environment state to be $\rho_S(0) \otimes \frac{I_{2^N}}{2^N}$, the dynamical A-map is obtained by evaluating

$$\mathrm{Tr}_E\left[U(t,0)\rho_S(0)\otimes\frac{I_{2^N}}{2^N}U^\dagger(t,0)\right]$$

(where $U(t,0) = \mathrm{Exp}[-iHt]$):

$$A(t,0) = \frac{1}{2}(1-x(t))\sigma_z\otimes\sigma_z + \frac{1}{2}(1+x(t))I_2\otimes I_2, \quad x(t) = \cos^N\left(\frac{2At}{\sqrt{N}}\right). \tag{19}$$

From this, the intermediate map $A(t_2,t_1)$ (see Equation (8)) and in turn the corresponding $B(t_2,t_1)$ may be readily evaluated. We obtain,

$$B(t_2,t_1) = \frac{1}{2}(I_2\otimes I_2 + \sigma_z\otimes\sigma_z) + \frac{x(t_2)}{2x(t_1)}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y). \tag{20}$$

The eigenvalues of $B(t_2,t_1)$ are

$$0,0,1\pm\frac{x(t_2)}{x(t_1)}.$$

Clearly, the intermediate time dynamics exhibits NCP nature as one of the eigenvalues *i.e.* $\lambda = 1 - \frac{x(t_2)}{x(t_1)}$ of $B(t_2,t_1)$ can assume negative values.

We illustrate regimes of Markovianity/non-Markovianity revealed via positive/negative values of λ (plot ted as a function of $\mu = t_2/t_1$) in **Figure 4**.

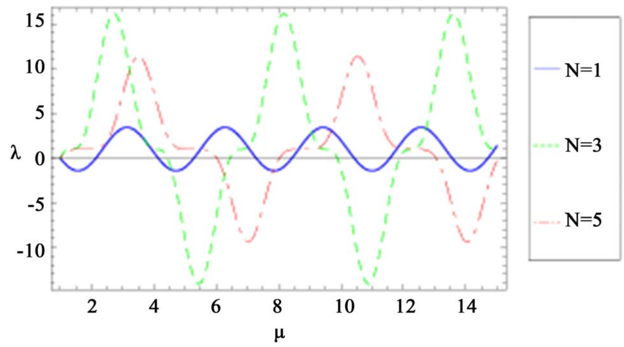

Figure 4. The variation of the eigenvalue λ of $B(t_2,t_1)$ (as a function of $\mu = t_2/t_1$) from positive to negative values and back with the passage of time for different values of *N*.

5.4. Two Qubit Unitary Evolution

We now consider the open system dynamics arising from the unitary evolution [13]

$$U(t,0) = e^{-it[\omega\sigma_z\otimes\sigma_x]} = \cos(\omega t/2)I_2\otimes I_2 - i\sin(\omega t/2)\sigma_z\otimes\sigma_x \tag{21}$$

on the system-environment initial state

$$\rho_{SE}(0) = \rho_S(0)\otimes\rho_E(0) = \frac{1}{2}(I_2+\sigma_x)\otimes\frac{1}{2}(I_2+\sigma_z).$$

The $A(t,0)$ map is given by,

$$A(t,0) = \frac{1}{2}(1+\cos(\omega t))I_2\otimes I_2 + \frac{1}{2}(1-\cos(\omega t))\sigma_z\otimes\sigma_z. \tag{22}$$

Following Equation (8), we obtain

$$B(t_2,t_1) = \frac{1}{2}(I_2\otimes I_2 + \sigma_z\otimes\sigma_z) + \frac{\cos(\omega t_2)}{2\cos(\omega t_1)}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y). \tag{23}$$

The eigenvalues of the B-map are given by $0, 0, 1\pm\left|\frac{\cos\omega t_2}{\cos\omega t_1}\right|$. The eigenvalue $\lambda = 1\pm\left|\frac{\cos\omega t_2}{\cos\omega t_1}\right|$ can assume negative values—bringing out the non-Markovian features prevalent in the dynamical process. **Figure 5** illustrates the transitions from Markovianity to non-Markovianity. This model, with initially correlated states, has been explored before in Refs. [13,17] and the dynamical map turned out to be NCP throughout not merely in the intermediate time interval).

6. Summary

In conclusion, a few remarks on a variety of definitions

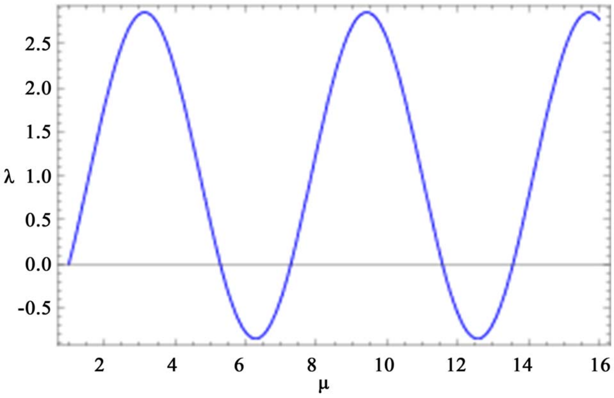

Figure 5. The plot of the eigenvalue $\lambda = 1 - \left|\frac{\cos\omega t_2}{\cos\omega t_1}\right|$ as a function of $\mu = t_2/t_1$. The periodic transitions of λ from positive to negative values indicates the transition of the process from Markovian to non-Markovian.

of non-Markovianity in the recent literature may be recalled here. Mainly the focus has been towards capturing the violation of semi-group property [17,27] or more recently—its two-parameter generalization viz the divisibility of the dynamical map [11,12]. Yet another measure, where non-Markovianity [28] is attributed to increase of distinguishability of any pairs of states (as a result of the partial back-flow of information from the environment into the system) and is quantified in terms of trace distance of the states. It has been shown that the two different measures of non-Markovianity—one based on the divisibility of the dynamical map [12] and the other based upon the distinguishability of quantum states [28] —need not agree with each other [29,30]. A modified version of the criterion of Ref. [12] was proposed recently [32]. In this paper we have established the interplay of stochastic A and dynamical B maps at intermediate times, revealing Markovian/non-Markovian regimes. We have explored four different examples revealing the features of intermediate time maps originating from variety of physical mechanisms: 1) A toy model map inspired by general considerations based on Jamiolkowski isomorphism—which explores a two qubit Werner state with time-dependent noise parameter as a dynamical map; 2) A reinterpretation of the phenomenological model explaining the recent optical experiment by Liu *et al*., [34] in terms of NCP nature of the intermediate B map; 3) Hamiltonian evolution describing the hyperfine interacttion of a spin—1/2 system with N spin—1/2 nuclear environment in a quantum dot [32] displaying Markovian/non-Markovian behavior and 4) Unitary evolution of Jordan *et al*., [13]—wherein initial system-environment two qubit is chosen in a product state. Here too, intermediate time dynamical map exhibits Markov/non-Markov regimes. It is interesting to note that the dynamics had been identified to be NCP throughout not merely in the intermediate time interval—when initially correlated states were employed [13,17]. Placing these two results together, brings forth that the source of non-Markovianity in this model is attributable entirely to the unitary dynamics—rather than initial correlations of system-environment qubits. We have thus exposed the underlying features of intermediate time A and B maps to bring out clearly if the dynamics relies on past history of the states or not.

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