Journal of Quantum Information Science, 2012, 2, 78-81
http://dx.doi.org/10.4236/jqis.2012.23013 Published Online September 2012 (http://www.SciRP.org/journal/jqis)

An Entanglement Criterion for States in Infinite Dimensional Bipartite Quantum Systems

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Received July 10, 2012; revised August 10, 2012; accepted September 10, 2012

ABSTRACT

In this paper, an entanglement criterion for states in infinite dimensional bipartite quantum systems is presented. We generalize some of separability criterion that was recently introduced by Wu and Anandan in (Phys. Lett. A, 2002, 297, 4-8) to infinite dimensional bipartite quantum systems. In addition, we give an example aimed to illustrate the application of the theorem.

Keywords: Entanglement Criterion; Infinite Dimensional Quantum Systems; Bochner Integral Representation

1. Introduction

Quantum entanglement plays a crucial role in the rapidly developing theory of quantum information and quantum computation [1]. In the cases of finite dimensional quantum systems, there are many methods to quantify the entanglement of bipartite and multipartite quantum systems [2-8]. However, most of them have not explicit formula, or it is hard to calculate. For the cases of infinite dimensional systems, the method of entanglement detection is a very difficult problem. But the case of infinite dimensional quantum systems can't be neglected since they do exist in quantum world [9]. Recently, Shengjun Wu and Jeeva Anandan [10] proposed a necessary criterion based on Pauli matrices re-presentation. Their result is as follows:

Let H_A, H_B be separable complex Hilbert spaces, $\dim H_A = 2$, $\dim H_B = n$ $(n < +\infty)$. By $S(H_A \otimes H_B)$ we denote the set of all states in $H_A \otimes H_B$. A given state ρ in $H_A \otimes H_B$ can be written as:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad (1)$$

where each ρ_{kl} $(k = 1,2; l = 1,2)$ is an n by n matrix, ρ can also be written as:

$$\rho = \begin{pmatrix} M_0 + M_z & M_x - iM_y \\ M_x + iM_y & M_0 - M_z \end{pmatrix} \quad (2)$$

or

$$\rho = I \otimes M_0 + \sigma_x \otimes M_x + \sigma_y \otimes M_y + \sigma_z \otimes M_z \quad (3)$$

where the four matrices

$$\begin{aligned} M_0 &= \frac{1}{2}(\rho_{11} + \rho_{22}), M_x = \frac{1}{2}(\rho_{12} + \rho_{21}) \\ M_y &= \frac{i}{2}(\rho_{12} - \rho_{21}), M_z = \frac{1}{2}(\rho_{11} - \rho_{22}) \end{aligned} \quad (4)$$

are n-dimensional Hermitian matrices. Let R be a 3-dimensional real matrix, and γ_R be a transformation on the density matrix ρ with the following form:

$$\gamma_R(\rho) = \begin{pmatrix} M_0 + M_z^R & M_x^R - iM_y^R \\ M_x^R + iM_y^R & M_0 - M_z^R \end{pmatrix} \quad (5)$$

where

$$\begin{pmatrix} M_x^R \\ M_y^R \\ M_z^R \end{pmatrix} = R \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \quad (6)$$

Shengjun Wu and Jeeva Anandan [10] give the following results:

1) If ρ is separable, then $\gamma_R(\rho)$ must be positively defined for any 3-dimensional real matrix R which satisfies $I - R^T R \geq 0$.

2) If ρ is separable, then

a) $M_0 - \boldsymbol{r} \cdot \boldsymbol{M}$ is positively defined for any vector $\boldsymbol{r} = (x, y, z)$ with $|\boldsymbol{r}| \leq 1$, $\boldsymbol{M} = (M_x, M_y, M_z)$;

b) $Tr\left(M_0^2 - M_x^2 - M_y^2 - M_z^2\right) \geq 0$;

where M_x, M_y, M_z are defined in Equation (4).

Then, a nature problem is arisen: whether or not there is counterpart result for the infinite-dimensional bipartite quantum systems? We find that the answer is "yes". The aim of the present paper is to establish this criterion for the infinite dimensional bipartite quantum systems.

The paper is organized as follows: In Section 2, we give the main results and the proof of the main results. In Section 3, we give an example to illustrate the application of the theorem.

2. Some Notations and Main Results

In this section, we mainly generalize the finite dimensional results, which be proposed by Wu and Anandan [10], to infinite-dimensional bipartite quantum systems $H_A \otimes H_B$, where $\dim H_A = 2, \dim H_B = +\infty$.

Let's fix some notations. Let H_A, H_B be separable complex Hilbert spaces, by $S(H_A \otimes H_B)$, we denote the set of all states in $H_A \otimes H_B$. The set of all separable pure states in $H_A \otimes H_B$ is denoted by $S_{s-p}(H_A \otimes H_B)$. Throughout the paper we use the Dirac's symbols. The bra-ket notation $\langle \cdot | \cdot \rangle$ stands for the inner product in the given Hilbert spaces. Recall that a quantum state $\rho \in S(H_A \otimes H_B)$ $(\dim(H_A \otimes H_B) < +\infty)$, which is positive and has trace one, is said to be separable if ρ can be written as:

$$\rho = \sum_{i=1}^{n} p_i \rho_i \otimes \sigma_i \tag{7}$$

where ρ_i and σ_i are pure states in the subsystems H_A and H_B respectively, $p_i \ge 0, \sum_i p_i = 1$. Otherwise, ρ_i is called an entangled state. If $\dim(H_A \otimes H_B) = +\infty$, by Werner[8], a state acting on $H_A \otimes H_B$ is called separable if it can be approximated in the trace norm by the states of the form

$$\sigma_n = \sum_{i=1}^{n} p_i \rho_i \otimes \sigma_i \tag{8}$$

Furthermore, it is shown in [11] that any separable state ρ admits a representation of the Bochner interal

$$\rho = \int_{S_{S-p}} \phi\left(\rho^{H_A} \otimes \rho^{H_B}\right) \mathrm{d}\mu\left(\rho^{H_A} \otimes \rho^{H_B}\right) \tag{9}$$

where μ is a Borel probability measure on

$S_{s-p}(H_A \otimes H_B)$, and $\phi : S_{S-p} \to S_{S-p}$

is a measurable function. It is known that, from the definition of the Bochner integral, there exists a sequences of step function $\{\phi_n\}$, such that

$$\phi\left(\rho^{H_A} \otimes \rho^{H_B}\right) = \lim_{n \to +\infty} \phi_n\left(\rho^{H_A} \otimes \rho^{H_B}\right) \tag{10}$$

with respect to the trace norm. Where

$$\phi_n\left(\rho^{H_A} \otimes \rho^{H_B}\right) = \sum_{i=1}^{k_n} \chi_{E_i}\left(\rho^{H_A} \otimes \rho^{H_B}\right) \rho_i^{H_A} \otimes \rho_i^{H_B} \tag{11}$$

$\chi_{E_i}(\cdot)$ is the characteristic function of E_i, and $\{E_i\}_{i=1}^{k_n}$ is a partition of $S_{s-p}(H_A \otimes H_B)$. By E we denote the set of all partitions of $S_{s-p}(H_A \otimes H_B)$, Thus we have

$$\rho = \lim_{E_i \in E} \sum_i \mu(E_i) \rho_i^{H_A} \otimes \rho_i^{H_B} \tag{12}$$

with respect to the trace norm, as well as with respect to the Hilbert Schmidt norm. Where there exists an ensemble $\left\{p_i, \left|\psi_i^{H_A}\right\rangle\right\}$ (or $\left\{p_j, \left|\varphi_j^{H_B}\right\rangle\right\}$) of H_A (or H_B) such that

$$\rho_i^{H_A} = \left|\psi_i^{H_A}\right\rangle\left\langle\psi_i^{H_A}\right| \quad (\text{or } \rho_i^{H_B} = \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|) \tag{13}$$

Next, we give the main results as follows:

Theorem 2.1 Let H_A and H_B be separable complex Hilbert spaces, $\dim H_A = 2$, $\dim H_B = +\infty$. If $\rho \in S(H_A \otimes H_B)$ is a separable state, then

1) $\gamma_R(\rho)$ must be positively defined for any 3 by 3 real matrix R which satisfies $I - R^T R \ge 0$, where $\gamma_R(\rho)$ is defined in Equations (5) and (6).

2) $M_0 - \boldsymbol{r} \cdot \boldsymbol{M}$ is positively defined for any vector $\boldsymbol{r} = (x, y, z)$ with $|\boldsymbol{r}| \le 1$, $\boldsymbol{M} = (M_x, M_y, M_z)$ and

$$Tr\left(M_0^2 - M_x^2 - M_y^2 - M_z^2\right) \ge 0, \quad M_0, M_x, M_y, M_z$$

are defined in Equation (4).

Proof. 1) Since ρ is separable, according by Equations (9)-(12), we have

$$\rho = \lim_{E_i \in E} \sum_i \mu(E_i) \left|\psi_i^{H_A}\right\rangle\left\langle\psi_i^{H_A}\right| \otimes \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \tag{14}$$

where $\left|\psi_i^{H_A}\right\rangle\left\langle\psi_i^{H_A}\right|$ are pure states of H_A, $\left|\varphi_i^{H_B}\right\rangle\left\langle\varphi_i^{H_B}\right|$ are pure states of H_B, respectively. Furthermore, according by Bloch representation [1], we have

$$\begin{aligned}
\rho &= \lim_{E_i \in E} \sum_i \mu(E_i) \frac{1}{2}(I + \boldsymbol{r}_i \cdot \boldsymbol{\sigma}) \otimes \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\
&= \frac{1}{2} \lim_{E_i \in E} \Bigg[I \otimes \sum_i \mu(E_i) \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\
&\quad + \sigma_x \otimes \sum_i \mu(E_i) x_i \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\
&\quad + \sigma_y \otimes \sum_i \mu(E_i) y_i \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\
&\quad + \sigma_z \otimes \sum_i \mu(E_i) z_i \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \Bigg]
\end{aligned} \tag{15}$$

with respect to the trace norm, where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrixes and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\boldsymbol{r}_i = (x_i, y_i, z_i)$ are real vectors on the Bloch sphere and satisfies $x_i^2 + y_i^2 + z_i^2 = 1$. Comparing Equations (3) and (15), we have

$$M_0 = \frac{1}{2} \lim_{E_i \in E} \left[\sum_{i=1} \mu(E_i) \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \right]$$

$$M_x = \frac{1}{2} \lim_{E_i \in E} \left[\sum_{i=1} \mu(E_i) x_i \left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \right]$$

$$M_y = \frac{1}{2}\lim_{E_i \in E}\left[\sum_{i=1}\mu(E_i)y_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right]$$

$$M_z = \frac{1}{2}\lim_{E_i \in E}\left[\sum_{i=1}\mu(E_i)z_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right]$$

$$\begin{aligned}\gamma_R(\rho) = \frac{1}{2}\lim_{E_i \in E}[& I\otimes\sum_i\mu(E_i)\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\ & +\sigma_x\otimes\sum_i\mu(E_i)x_i'\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\ & +\sigma_y\otimes\sum_i\mu(E_i)y_i'\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right| \\ & +\sigma_z\otimes\sum_i\mu(E_i)z_i'\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\end{aligned} \tag{16}$$

where

$$\begin{pmatrix} x_i' \\ y_i' \\ z_i' \end{pmatrix} = R\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \tag{17}$$

Since for any 3-dimensional real matrix R which satisfies $I - R^T R \geq 0$, means that $(x_i')^2 + (y_i')^2 + (z_i')^2 \leq 1$, on the other hand, according by Equation (16), we have

$$\gamma_R(\rho) = \frac{1}{2}\lim_{E_i \in E}\left[A\otimes\sum_i\mu(E_i)\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right],$$

where

$$A = \begin{pmatrix} 1+z_i' & x_i'-iy_i' \\ x_i'+iy_i' & 1-z_i' \end{pmatrix} \tag{18}$$

It is obvious that we have $A \geq 0$, in fact since $(x_i')^2 + (y_i')^2 + (z_i')^2 \leq 1$, so $\gamma_R(\rho)$ is still a density matrix, *i.e.*, $\gamma_R(\rho) \geq 0$.

2) $M_0 - \boldsymbol{r}\cdot\boldsymbol{M}$

$$\begin{aligned} &= M_0 - xM_x - yM_y - zM_z \\ &= \frac{1}{2}\lim_{E_i \in E}\left[\sum_i\mu(E_i)\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right] \\ &\quad - x\frac{1}{2}\lim_{E_i \in E}\left[\sum_{i=1}\mu(E_i)x_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right] \\ &\quad - y\frac{1}{2}\lim_{E_i \in E}\left[\sum_{i=1}\mu(E_i)y_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right] \\ &\quad - z\frac{1}{2}\lim_{E_i \in E}\left[\sum_i\mu(E_i)z_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right] \\ &= \frac{1}{2}\lim_{E_i \in E}\left[\sum_i\mu(E_i)z_i\left|\phi_i^{H_B}\right\rangle\left\langle\phi_i^{H_B}\right|\right]\left[1 - xx_i - yy_i - zz_i\right]\end{aligned}$$

Since $|\boldsymbol{a}\cdot\boldsymbol{b}| \leq |\boldsymbol{a}|\cdot|\boldsymbol{b}|$, so we have $1 - xx_i - yy_i - zz_i \geq 0$ and $M_0 - \boldsymbol{r}\cdot\boldsymbol{M} \geq 0$.

On the other hand, we have

$$\begin{aligned}&\mathrm{Tr}\left(M_0^2 - M_x^2 - M_y^2 - M_z^2\right) \\ &= \frac{1}{4}\lim_{E_i \in E}\left[\sum_i\sum_j\mu(E_i)\mu(E_j)\right. \\ &\qquad \left.\times\left(1 - x_i x_j - y_i y_j - z_i z_j\right)\left|\left\langle\phi_i^{H_B}\middle|\phi_i^{H_B}\right\rangle\right|^2\right] \geq 0\end{aligned}$$

This completes the proof.

3. Example

Next, we give an example to illustrate the application of the Theorem 2.1. We consider a bipartite infinite dimensional state with the following forms:

$$\rho = x\left|00'\right\rangle\left\langle 00'\right| + \frac{1-x}{2}\left(\left|01'\right\rangle + \left\langle 10'\right|\right)\left(\left\langle 01'\right| + \left\langle 10'\right|\right) \tag{19}$$

with $\rho \in S(H_A \otimes H_B)$

$$\dim H_A = 2, \dim H_B = +\infty, \quad 0 \leq x \leq 1,$$

where $\{|0\rangle, |1\rangle\}$ is the orthogonal real basis of H_A,

$\{|0'\rangle, |1'\rangle, \cdots\}$ is the orthogonal real basis of H_B.

For simplicity, we assume $\dim H_B = n(2 \leq n \leq +\infty)$. In this case, we can obtain that

$$M_0 = \begin{pmatrix} \frac{1+x}{4} & 0 & \cdots & 0 \\ 0 & \frac{1-x}{4} & \cdots & 0 \\ 0 & 0 & & \\ & & & \end{pmatrix}$$

$$M_x = \begin{pmatrix} 0 & \frac{1-x}{4} & \cdots & 0 \\ \frac{1-x}{4} & 0 & \cdots & 0 \\ 0 & 0 & & \\ & & & \end{pmatrix}$$

$$M_y = \frac{i}{2}\begin{pmatrix} 0 & \frac{x-1}{2} & \cdots & 0 \\ \frac{1-x}{2} & 0 & \cdots & 0 \\ 0 & 0 & & \\ & & & \end{pmatrix}$$

$$M_z = \begin{pmatrix} \frac{3x-1}{4} & 0 & \cdots & 0 \\ 0 & \frac{1-x}{4} & \cdots & 0 \\ 0 & 0 & & \\ & & & \end{pmatrix}$$

According by Theorem 2.1, by a straightforward calculation, we obtain the following result, if

$$0 < x < \frac{1}{5},$$

then

$$Tr\left(M_0^2 - M_x^2 - M_y^2 - M_z^2\right) < 0 \quad (20)$$

so ρ is entangled.

Remark: It is obvious that this criterion of Theorem 2.1 is weaker than PPT criterion [4], in fact if $x \neq 1$, ρ is also entangled, but this criterion give us a method to detect the entanglement for states in infinite bipartite quantum systems.

4. Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (11171249) and the Natural Science Foundation of Shanxi Province (2011021002-2). The authors also wish to give their thanks to the referees for their comments to improve the presentation of this paper.

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