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Surface Wave Echo in a Semi-Bounded Plasma

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Abstract

Plasma echo theory is revisited to apply it to a semi-bounded plasma. Spatial echoes associated with plasma surface wave propagating in a semi-bounded plasma are investigated by calculating the second order electric field produced by external charges and satisfying the boundary conditions at the interface. The boundary conditions are two-fold: the specular reflection condition and the electric boundary condition. The echo spots are determined in terms of the perpendicular coordinate to the interface and the parallel coordinate along which the wave propagates. This improves the earlier works in which only the perpendicular coordinate is determined. In contrast with the echo in an infinite medium, echoes in a bounded plasma can occur at various spots. The diversity of echo occurrence spots is due to the discontinuity of the electric field at the interface that satisfies the specular reflection boundary condition. Physically, the diversity appears to be owing to the reflections of the waves from the interface.

Keywords

Plasma Echo, Semi-Bounded Plasma, Boundary Condition

1. Introduction

Plasma echoes in an infinite plasma have long been known theoretically [1] [2] as well as experimentally [3]. Spatial echoes were theoretically investigated in a static situation where the non-propagating electric field is directed perpendicular to the interface of a semi-bounded plasma [4] [5]. If the perpendicular direction is designated as the x direction, the electric field E as well as the distribution function f is spatially one-dimensional: $E = E(x,t)$ and $f = f(x,v,t)$, where $x > 0$ ($x < 0$) is the plasma (vacuum) region. In this case, the corresponding Vlasov equation takes the form of a first order differential equation, and can be solved by satisfying the specular reflection boundary condition at the interface $x = 0$: $f(v,0) = f(-v,0)$ [6]. This

differential equation approach with the specular reflection boundary condition for a semi-bounded plasma has been shown to be entirely equivalent with the Fourier transform (with respect to x) under the recipe that the $E(x)$ is extended into the region $x < 0$ in an odd function manner, $E(x) = -E(-x)$ [5]. This odd function extension of $E(x)$ gives rise to a surface term in the Fourier transform of the Poisson equation, which plays a significant role in the determination of the echo spots. It appears that this surface term, which the earlier authors entirely neglected, gives rise to diversity of echo spots [5]. Physically, the surface term manifests the reflection of the electric field at the boundary.

The echo phenomena is the result of a quadratic interaction of the two primary waves launched by two external charges at different locations (spatial echoes) or different times (temporal echoes). In response to the external charges, the plasma distribution function $f(x,v,t)$ is modulated with the exponential phase $e^{ik(x-vt)}$, which is derived from the singularity at $\omega = kv$ of the linear response function. This term is called the free streaming term since $x = vt$ is the characteristic line of the Vlasov equation for a free particle. This rapidly modulating exponential phase makes the $f(x,v,t)$ more and more oscillatory as t or x increases, and consequently, $\int f \mathrm{d}v$ will become vanishingly small due to almost complete *phase mixing*. Therefore, in the first order, the phase mixing obliterates any appreciable effect on the macroscopic variable such as density perturbation. However, the second order distribution function which is a product of two first order distribution functions is not phase-mixed when or where the condition for a constructive interference is met, thereby the second order electric field does not vanish, resulting in an echo. It is evident from the expression for the product of two free-streaming exponentials $e^{ik_1(x_1-vt_1)}e^{ik_2(x_2-vt_2)}$ that a constructive interference can result in at a certain time (temporal echo) or a certain spot (spatial echo) such that $k_1 x_1 + k_2 x_2 = v(k_1 t_1 + k_2 t_2)$.

In this work, we investigate spatial echoes in a semi-bounded plasma, taking a full account of the boundary terms which originate from the oddly continuation of the electric field. This work is an extension of the earlier paper by Lee and Lee [5]; the distribution function and the electric field are now spatially two-dimensional, allowing for the z-dependance. Therefore, the echoes are associated with the surface wave which is propagating in the z-direction. The second order electric field endowed with the additional z-dependance can be Fourier-inverted by contour integration with unstraightforward analytic exercise, and delineating the echo condition requires extra complexity. The important boundary term is the discontinuity of the perpendicular electric field at the interface that is necessary to have the specular reflection boundary condition satisfied [5]. The diversity of echo occurrence spots has been experimentally reported [7] and can be explained by this boundary term. The identification of the echo spot associated with surface wave appears to be useful in experimental point of view [7].

2. Formulation of the Problem

We consider a plasma consisting of electrons and stationary ions, the latter forming the uniform background. The plasma is assumed to occupy the half-space $x \geq 0$. The region $x < 0$ is assumed to be a vacuum. The perturbed electron distribution function $f(\mathbf{r},\mathbf{v},t)$ and the electric field $E(\mathbf{r},t)$ will depend on x and z-coordinates with the y coordinate ignored since y direction has a translational invariance. We have the nonlinear Vlasov equation and the Poisson equation to describe the electrostatic perturbation:

$$\frac{\partial}{\partial t} f(\mathbf{v},\mathbf{r},t) + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m} E(\mathbf{r},t) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{1}$$

with $\mathbf{r} = \hat{x}x + \hat{z}z$, $\mathbf{v} = \hat{x}v_x + \hat{z}v_z$, $E = \hat{x}E_x + \hat{z}E_z$

$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 4\pi\left(-e\int \mathrm{d}^2 v f + \rho_0(x,t)\right) \tag{2}$$

where f is a two-dimensional distribution function, and ρ_0 represents the external charges:

$$\rho_0(x,z,t) = \left(\rho_1 e^{i\omega_1 t}\delta\left[k_0(x-L_1)\right]\delta\left[k_0(z-S_1)\right] + 1 \to 2\right) \tag{3}$$

k_0 is introduced to make the argument of the δ-function dimensionless, and $1 \to 2$ means the replica of the preceding term with the subscript 1 replaced by subscript 2. We solve the simultaneous Equations (1) and (2) for a given $\rho_0(x,t)$ as prescribed by Equation (3). In mathematical terms, we have an inhomogeneous system, driven by the source term in Equation (3). The responses f and E should be determined by ρ_0.

The kinetic equation is supplemented by the kinematic boundary condition which we assume to be the specular reflection condition

$$f\left(v_x, v_z, x = 0, z\right) = f\left(-v_x, v_z, x = 0, z\right) \tag{4}$$

This specular reflection boundary condition is automatically satisfied by extending the electric field component $E_x(x)$ in odd function manner into the region $x < 0$, i.e., $E_x(-x) = -E_x(x)$. Assuming that the external perturbation is small, we solve Equations (1) and (2) by successive approximation. First, the linear solution of Equation (1) will be obtained for f with the boundary condition (4). Substituting this solution in Equation (2) yields an integral equation for the electric field which is solved by Fourier transform. Then the linear solution will be used to obtain the higher order solutions. We work only up to the second order. The higher order distribution function should also satisfy the boundary condition (4). The electric field should satisfy the electric boundary conditions: the normal component of the electric displacement $D_x(x)$ and the tangential electric field E_z are continuous across the interface. In this work, the Fourier transform is defined by

$$f(k, \omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, f(x, t) e^{-ikx + i\omega t}$$

$$f(x, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(k, \omega) e^{ikx - i\omega t}$$

Let us Fourier transform Equations (1)-(3) with respect to t and z to write

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) f(\mathbf{v}, \mathbf{k}, \omega) - \frac{e}{m} \int \frac{d\omega'}{2\pi} \int \frac{d^2 k'}{(2\pi)^2} \mathbf{E}(\mathbf{k} - \mathbf{k}', \omega - \omega') \cdot \frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}, \mathbf{k}', \omega') = 0 \tag{5}$$

$$i\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, \omega) + N(k_z, \omega) = 4\pi \left[-e \int f(\mathbf{k}, \mathbf{v}, \omega) d^2 v + \rho_0(\mathbf{k}, \omega) \right] \tag{6}$$

where

$$N(k_z, \omega) = E_x\left(0^-, k_z, \omega\right) - E_x\left(0^+, k_z, \omega\right)$$

is derived from the discontinuity of E_x at $x = 0$. This N-term is characteristic of a semi-bounded plasma and responsible for the diversity of surface wave echoes, as compared with an infinite plasma. The external charges are Fourier transformed to

$$\rho_0(\mathbf{k}, \omega) = \frac{2\pi}{k_0^2} \left[\rho_1 \delta(\omega + \omega_1) e^{-ik_x L_1} e^{-ik_z S_1} + 1 \rightarrow 2 \right] \tag{7}$$

Equations (5) and (6) constitute a set of nonlinear simultaneous equations. We solve the set of equations by successive approximations in terms of perturbation series:

$$f(\mathbf{k}, \mathbf{v}, \omega) = f_0(\mathbf{v}) + f^{(1)}(\mathbf{k}, \mathbf{v}, \omega) + f^{(2)}(\mathbf{k}, \mathbf{v}, \omega) + \cdots$$

$$\mathbf{E}(\mathbf{k}, \omega) = \mathbf{E}^{(1)}(\mathbf{k}, \omega) + \mathbf{E}^{(2)}(\mathbf{k}, \omega) + \cdots$$

Breaking down Equations (5) and (6) order by order, we have

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) f^{(1)}(\mathbf{k}, \mathbf{v}, \omega) = \frac{e}{m} \mathbf{E}^{(1)}(\mathbf{k}, \omega) \cdot \frac{df_0}{d\mathbf{v}} \tag{8}$$

$$i\mathbf{k} \cdot \mathbf{E}^{(1)}(\mathbf{k}, \omega) + N(k_z, \omega) = 4\pi \left[-e \int f^{(1)}(\mathbf{k}, \mathbf{v}, \omega) d^2 v + \rho_0(\mathbf{k}, \omega) \right] \tag{9}$$

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) f^{(2)}(\mathbf{k}, \mathbf{v}, \omega)$$
$$= \frac{e}{m} \mathbf{E}^{(2)}(\mathbf{k}, \omega) \cdot \frac{df_0}{d\mathbf{v}} + \frac{e}{m} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{dk_x'}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z'}{2\pi} \mathbf{E}^{(1)}(\mathbf{k} - \mathbf{k}', \omega - \omega') \cdot \frac{\partial}{\partial \mathbf{v}} f^{(1)}(\mathbf{k}', \omega', \mathbf{v}) \tag{10}$$

$$i\mathbf{k} \cdot \mathbf{E}^{(2)}(\mathbf{k}, \omega) = -4\pi e \int f^{(2)}(\mathbf{k}, \mathbf{v}, \omega) d^2 v \tag{11}$$

The quantity $N(k_z, \omega)$ in Equation (9) should be determined in terms of the vacuum field from the electric field boundary condition: electric displacement $D_x(x)$ is continuous across the interface $x = 0$,

$$D_x^{(1)}\left(0^+\right)=D_x^{(1)}\left(0^-\right) \tag{12}$$

where $D_x^{(1)}\left(0^-\right)$ equals to the vacuum electric field E_0.

3. Linear Solution

Equations (8) and (9), and $\nabla \times \boldsymbol{E} = 0$ give

$$\boldsymbol{E}^{(1)}\left(\boldsymbol{k},\omega\right)=\frac{i\boldsymbol{k}}{k^2\,\varepsilon\left(\boldsymbol{k},\omega\right)}\Big[N\left(k_z,\omega\right)-4\pi\rho_0\left(\boldsymbol{k},\omega\right)\Big] \tag{13}$$

$$\varepsilon\left(\boldsymbol{k},\omega\right)=1+\frac{\omega_p^2}{k^2}\int d^2 v\,\frac{\boldsymbol{k}\cdot\dfrac{df_0}{dv}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}} \tag{14}$$

is the dielectric function (ω_p is the plasma frequency). N is determined from the electric boundary condition as shown in the following. We need the normal component of electric displacement, $D_x(x)$ to enforce the boundary condition (12). By definition, $D_x\left(\boldsymbol{k},\omega\right)=E_x\left(\boldsymbol{k},\omega\right)+\dfrac{4\pi i}{\omega}J_x\left(\boldsymbol{k},\omega\right)$ where J is the current:

$J_x\left(\boldsymbol{k},\omega\right)=-e\int d^2 v\,v_x\,f\left(\boldsymbol{k},\omega,\boldsymbol{v}\right)$. We calculate

$$\frac{4\pi i}{\omega}J_x=\frac{4\pi i}{\omega}(-e)\int v_x f d^2 v=\frac{\omega_p^2}{\omega}\int d^2 v v_x\,\frac{E_j\dfrac{\partial f_0}{\partial v_j}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}$$

where we used Equation (8). The above quantity equals to $(\varepsilon-1)E_x$. Thus we have $D_x=\varepsilon E_x$. This statement can be most easily proved by assuming f_0 a Maxwellian. Use $\dfrac{\partial f_0}{\partial v_j}=-\dfrac{T}{m}v_j f_0$ and $\boldsymbol{E}\cdot\boldsymbol{v}=\dfrac{E_x}{k_x}\boldsymbol{k}\cdot\boldsymbol{v}$ to write for the last term

$$\frac{4\pi i}{\omega}J_x=-E_x\frac{\omega_p^2}{\omega}\frac{T}{m}\int\frac{v_x}{k_x}\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}f_0\,d^2 v$$

Put $\dfrac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}=-1+\dfrac{\omega}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}$. Then, (-1)-term vanishes upon integration, and we have

$$\frac{4\pi i}{\omega}J_x=-E_x\frac{\omega_p^2}{k_x}\frac{T}{m}\int\frac{v_x f_0}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}d^2 v=E_x\frac{\omega_p^2}{k_x}\int\frac{\dfrac{\partial f_0}{\partial v_x}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}d^2 v$$

$$=-E_x\,\omega_p^2\int\frac{f_0\,d^2 v}{\left(\omega-\boldsymbol{k}\cdot\boldsymbol{v}\right)^2}=E_x\frac{\omega_p^2}{k^2}\int d^2 v\,\frac{\boldsymbol{k}\cdot\dfrac{\partial f_0}{\partial v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\quad q.e.d.$$

Using the above result, we obtain

$$D_x^{(1)}\left(\boldsymbol{k},\omega\right)=\varepsilon\left(\boldsymbol{k},\omega\right)E_x^{(1)}\left(\boldsymbol{k},\omega\right)=\frac{ik_x}{k^2}\Big[N-4\pi\rho_0\left(\boldsymbol{k},\omega\right)\Big] \tag{15}$$

To invert Equation (15), we write

$$D_x^{(1)}\left(x,k_z,\omega\right)=\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\,e^{ik_x x}\frac{ik_x}{k^2}\left[N-\frac{8\pi^2}{k_0^2}\rho_1 e^{-ik_x L_1}\delta\left(\omega+\omega_1\right)e^{-ik_z S_1}+1\to 2\right] \tag{16}$$

In the above integral, we take the limit $x\to 0^+$. Evaluating the integral by residue theorem gives

$$\lim_{x\to 0^+}\int_{-\infty}^{\infty}dk_x\,\frac{k_x e^{ik_x x}}{k_x^2+k_z^2}=i\pi$$

Note that we set up the contour encircling the upper half plane since $x > 0$. When $Re\,k_z > 0$ ($Re\,k_z < 0$), the relevant pole located in the upper k_x-plane is $k_x = ik_z$ ($k_x = -ik_z$). In either case, the integral is found to be $i\pi$. Then, Equation (16) takes the form

$$D_x^{(1)}\left(0^+, k_z, \omega\right) = -\frac{N}{2} - \frac{4\pi}{k_0^2} \mathrm{e}^{-ik_z S_1} iI\left(L_1, k_z\right)\rho_1 \delta\left(\omega + \omega_1\right) + 1 \to 2 \tag{17}$$

$$\text{where} \quad I\left(L_1, k_z\right) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{\xi \mathrm{e}^{-i\xi L_1}}{\xi^2 + k_z^2} = -i\pi \mathrm{e}^{\pm k_z L_1} \tag{18}$$

where $+$ ($-$) sign corresponds to $Re\,k_z < 0$ ($Re\,k_z > 0$). The above equality can be easily proven by using the contour winding the lower half plane. Taking the limit $k_z \to 0$ gives the useful identity $\int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{\mathrm{e}^{-i\xi L_1}}{\xi} = -i\pi$, independently of $L_1(> 0)$. Clearly, this integral manifests the nature of a step function. By equating the quantity on the right hand side of Equation (17) to the x-component of the vacuum electric field ($\equiv E_0$), we obtain

$$N = -2E_0 - \frac{8\pi}{k_0^2}\rho_1 \delta\left(\omega + \omega_1\right)\mathrm{e}^{-ik_z S_1} iI\left(L_1, k_z\right) + 1 \to 2 \tag{19}$$

Using the above equation in Equation (13) gives

$$\boldsymbol{E}^{(1)}\left(\boldsymbol{k}, \omega\right) = \frac{i\boldsymbol{k}}{k^2\,\varepsilon\left(\boldsymbol{k}, \omega\right)}\left[-2E_0 - \frac{8\pi^2}{k_0^2}\rho_1 \delta\left(\omega + \omega_1\right)\,\mathrm{e}^{-ik_z S_1}\left(\frac{i}{\pi}I\left(L_1, k_z\right) + \mathrm{e}^{-ik_x L_1}\right) + 1 \to 2\right] \tag{20}$$

For an infinite plasma without boundary, we have $N = 0$ in Equation (13), and the plasma electric field is given by

$$\boldsymbol{E}^{(1)}\left(\boldsymbol{k}, \omega\right) = \frac{i\boldsymbol{k}}{k^2\,\varepsilon\left(\boldsymbol{k}, \omega\right)}\left[-\frac{8\pi^2}{k_0^2}\rho_1 \delta\left(\omega + \omega_1\right)\mathrm{e}^{-ik_z S_1}\mathrm{e}^{-ik_x L_1} + 1 \to 2\right]$$

Note that in Equation (20), the E_0-term and $I\left(L_1, k_z\right)$-term are the boundary terms which are non-existent in an infinite plasma.

In the static situation where the electric field is nonpropagating, we put $k_z = 0$ in Equation (20), and the electric field reduces to Equation (23) in Lee and Lee [5]:

$$E^{(1)}\left(\boldsymbol{k}, \omega\right) = \frac{-2i}{k\varepsilon\left(\boldsymbol{k}, \omega\right)}\left[E_0 + \frac{4\pi^2}{k_0^2}\rho_1 \delta\left(\omega + \omega_1\right)\left(1 + \mathrm{e}^{-ikL_1}\right) + 1 \to 2\right]$$

4. Second Order Solution and Echo Occurrence

Next, we deal with the second order equations, Equations (10) and (11). Using Equation (10) in Equation (11) yields, owing to the electrostatic nature of $\boldsymbol{E}^{(2)}$,

$$\boldsymbol{E}^{(2)}\left(\omega, \boldsymbol{k}\right) = \frac{4\pi e^2}{m}\frac{\boldsymbol{k}}{k^2\varepsilon\left(\boldsymbol{k}, \omega\right)}\int \mathrm{d}^2 v\, \frac{\boldsymbol{k}\cdot\boldsymbol{Q}}{\left(\omega - \boldsymbol{k}\cdot\boldsymbol{v}\right)^2} \tag{21}$$

where $\boldsymbol{Q}$ stands for

$$\boldsymbol{Q}\left(\omega, \boldsymbol{k}, \boldsymbol{v}\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_x'}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z'}{2\pi}\boldsymbol{E}^{(1)}\left(\omega - \omega', \boldsymbol{k} - \boldsymbol{k}'\right)f^{(1)}\left(\omega', \boldsymbol{k}', \boldsymbol{v}\right) \tag{22}$$

Substituting the first order solutions [Equations (8) and (20)], into the above equations, we can write $\boldsymbol{E}^{(2)}$ in the form,

$$\boldsymbol{E}^{(2)}\left(\omega, \boldsymbol{k}\right) = \frac{e^3}{2\pi^2 m^2}\frac{\boldsymbol{k}}{k^2\varepsilon\left(\boldsymbol{k}, \omega\right)}\int\frac{\mathrm{d}^2 v}{\left(\omega - \boldsymbol{k}\cdot\boldsymbol{v}\right)^2}\int\mathrm{d}k_z'\int\mathrm{d}k_x'\int\mathrm{d}\omega' \times \frac{\boldsymbol{k}\cdot\left(\boldsymbol{k} - \boldsymbol{k}'\right)}{\varepsilon\left(\omega - \omega', \boldsymbol{k} - \boldsymbol{k}'\right)\left(\boldsymbol{k} - \boldsymbol{k}'\right)^2}\frac{\boldsymbol{k}'\cdot\dfrac{\mathrm{d}f_0}{\mathrm{d}\boldsymbol{v}}}{k'^2\varepsilon\left(\omega', \boldsymbol{k}'\right)\left(\omega' - \boldsymbol{k}'\cdot\boldsymbol{v}\right)}AB, \tag{23}$$

$$A = -2E_0 - \frac{8\pi^2}{k_0^2}\rho_1\delta(\omega' + \omega_1)e^{-ik_z'S_1}\left(\frac{i}{\pi}I(L_1, k_z') + e^{-ik_x'L_1}\right) + 1 \to 2 \tag{24}$$

$$B = -2E_0 - \frac{8\pi^2}{k_0^2}\rho_1\delta(\omega - \omega' + \omega_1)e^{-i(k_z - k_z')S_1}\left(\frac{i}{\pi}I(L_1, k_z - k_z') + e^{-i(k_x - k_x')L_1}\right) + 1 \to 2 \tag{25}$$

where I stands for the exponential function as given by Equation (18). Since we don't know yet which sign should be chosen, we keep on using the symbol I. Equation (23) is to be used for investigation of echo occurrence. The various cross terms in the product (AB) are the candidates of echo resonances to see if the condition for vanishing phase can be met.

We choose to investigate a cross term which is 1-term in A multiplied by 2-term in B. With this term, the t-inversion of Equation (23) can be easily carried out by simply putting $\omega' \to -\omega_1$ and $\omega \to -\omega_3$:

$$E^{(2)}(t, k) = \beta e^{i\omega_3 t}e^{-ik_z S_2}\frac{k}{k^2\varepsilon(-\omega_3, k)}\int\frac{d^2v}{(\omega_3 + k\cdot v)^2}\int dk_z'\int dk_x'$$

$$\times\frac{k\cdot(k - k')}{(k - k')^2\varepsilon(-\omega_2, k - k')}\frac{k'\cdot\frac{df_0}{dv}}{k'^2\varepsilon(-\omega_1, k')(\omega_1 + k'\cdot v)}e^{ik_z'(S_2 - S_1)} \tag{26}$$

$$\times\left(\frac{i}{\pi}I(L_1, k_z') + e^{-ik_x'L_1}\right)\left(\frac{i}{\pi}I(L_2, k_z - k_z') + e^{-i(k_x - k_x')L_2}\right)$$

where

$$\omega_3 = \omega_1 + \omega_2, \ \beta = \frac{-16\pi e^3\rho_1\rho_2}{k_0^4 m^2} \tag{27}$$

In the above equation, we can assume that the poles associated with the dielectric functions contribute negligibly in the $\int dk$ - or $\int dk'$-integral. [The dominant contribution comes from the free-streaming poles.] Also we assume f_0 to be a Maxwellian. Then we have

$$\frac{k'\cdot\frac{df_0}{dv}}{\omega_1 + k'\cdot v} = -\frac{m}{T}f_0\left(1 - \frac{\omega_1}{\omega_1 + k'\cdot v}\right)$$

where 1 can be assumed to contribute nothing to the inversion integral in the following, due to phase mixing. Thus, Equation (26) can be further simplified as

$$E^{(2)}(t, k) = \frac{\beta m\omega_1}{T}e^{i\omega_3 t}e^{-ik_z S_2}\frac{k}{k^2\varepsilon(-\omega_3, k)}\int\frac{d^2vf_0}{(\omega_3 + k\cdot v)^2}\int dk_z'\int dk_x'$$

$$\times\frac{k\cdot(k - k')}{(k - k')^2\varepsilon(-\omega_2, k - k')k'^2\varepsilon(-\omega_1, k')(\omega_1 + k'\cdot v)} \tag{28}$$

$$\times e^{ik_z'(S_2 - S_1)}\left(\frac{i}{\pi}I(L_1, k_z') + e^{-ik_x'L_1}\right)\left(\frac{i}{\pi}I(L_2, k_z - k_z') + e^{-i(k_x - k_x')L_2}\right)$$

Let us write explicitly the inversion integral of Equation (28) with respect to k:

$$E^{(2)}(t, x, z) = \frac{\beta m\omega_1}{4\pi^2 T}e^{i\omega_3 t}\int\frac{d^2vf_0}{v_x^2}\int dk_z e^{ik_z(z - S_2)}\int dk_x\frac{k}{k^2\varepsilon(-\omega_3, k)}\frac{e^{ik_x x}}{\left(k_x + \frac{k_z v_z + \omega_3}{v_x}\right)^2}$$

$$\times\int dk_z' e^{ik_z'(S_2 - S_1)}\int dk_x'\frac{k\cdot(k - k')}{(k - k')^2\varepsilon(-\omega_2, k - k')k'^2\varepsilon(-\omega_1, k')(\omega_1 + k'\cdot v)} \tag{29}$$

$$\times\left(\frac{i}{\pi}I(L_1, k_z') + e^{-ik_x'L_1}\right)\left(\frac{i}{\pi}I(L_2, k_z - k_z') + e^{-i(k_x - k_x')L_2}\right)$$

This equation will be examined in view of the possibility of the vanishing phase.

(1) First, we shall consider the interference of two exponential terms in Equation (29):

$$e^{-ik'_x L_1} e^{-i(k_x - k'_x)L_2} = e^{ik'_x(L_2 - L_1)} e^{-ik_x L_2}$$

The important singularities are: the double pole at $k_x = -(k_z v_z + \omega_3)/v_x$ and the simple poles associated respectively with $k^2 = 0$ and $k'^2 = 0$ and $\omega_1 + k_x v_x + k_z v_z = 0$. We shall consider only these four poles. Singularities at $\varepsilon = 0$ are not important. Therefore we can put

$$\frac{\boldsymbol{k} \cdot (\boldsymbol{k} - \boldsymbol{k}')}{(\boldsymbol{k} - \boldsymbol{k}')^2} = \frac{1}{2}$$

and all the ε's can be taken out of the integral. The residue at the double pole is obtained by taking $\partial/\partial k_x$

$$\left[\text{Integrand} \times \left(k_x + \frac{k_z v_z + \omega_3}{v_x} \right)^2 \right] \text{ and substituting for } k_x = -(k_z v_z + \omega_3)/v_x. \text{ Here it is sufficient to differentiate}$$

only the exponential functions because they yield asymptotically dominant result. [Or integrate by parts with respect to k_x.] Thus let us calculate

$$\Phi = \frac{i(x - L_2)}{v_x} \int dk_z e^{ik_z(z - S_2)} \int dk_x \frac{\boldsymbol{k}}{k^2} \frac{e^{ik_x(x - L_2)}}{k_x + (k_z v_z + \omega_3)/v_x}$$
$$\times \int dk'_z \frac{e^{ik'_z(S_2 - S_1)}}{k'^2} \int dk'_x \frac{e^{ik'_x(L_2 - L_1)}}{k'_x + (k'_z v_z + \omega_1)/v_x} \tag{30}$$

Then $\boldsymbol{E}^{(2)}$ is obtained by $e^{i\omega_3 t} \int d^2 v \, \Phi f_0 / v_x^2$, suppressing the unessential factor.

Integral $\int dk'_x$ can be easily done by picking up the pole at

$$k'_x = -\frac{k'_z v_z + \omega_1}{v_x} \tag{31}$$

For definiteness we assume $L_2 > L_1$. Then the contour in k'_x-plane should encircle the upper half k'_x-plane, and in order for the pole to lie in the upper k'_x-plane, the imaginary part of $(k'_z v_z / v_x)$ should be negative. Now k'^2 is only a function of k'_z per Equation (31), and we can write the second part of Equation (30) as

$$J' \equiv \int dk'_z \frac{e^{ik'_z(S_2 - S_1)}}{k'^2} \int dk'_x \frac{e^{ik'_x(L_2 - L_1)}}{k'_x + (k'_z v_z + \omega_1)/v_x}$$
$$= -\frac{\pi v_x}{\omega_1} e^{-i(L_2 - L_1)\omega_1/v_x} \int dk'_z e^{i\theta'} \left(\frac{1}{k'_z - k'_{z+}} - \frac{1}{k'_z - k'_{z-}} \right) H\left(-\frac{v_z}{v_x} \operatorname{Im} k'_z \right) \tag{32}$$

where

$$k'_{z\pm} = -\frac{\omega_1}{v^2}(v_z \pm i v_x) \tag{33}$$

$$\theta' = S_2 - S_1 + (L_1 - L_2)\frac{v_z}{v_x} \tag{34}$$

and $H(x)$ is a step function; $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$.

The contour in $\int dk'_z$-integral depends on the sign of θ': when $\theta' > 0$ ($\theta' < 0$), the contour must wind the upper (lower) k'_z-plane. The location of the poles depends upon the sign of v_x. Sorting out the relevant cases, we carry out the integral for J':

$$\theta' > 0 : J' = \frac{-2i\pi^2 v_x}{\omega_1} e^{i(L_1 - L_2)\omega_1/v_x} \left[e^{i\theta' k'_{z+}} H(v_z) H(-v_x) - e^{i\theta' k'_{z-}} H(-v_z) H(v_x) \right] \tag{35}$$

$$\theta' < 0 : J' = \frac{2i\pi^2 v_x}{\omega_1} e^{i(L_1 - L_2)\omega_1/v_x} \left[e^{i\theta' k'_{z+}} H(v_z) H(v_x) - e^{i\theta' k'_{z-}} H(-v_z) H(-v_x) \right] \tag{36}$$

Next, taking on the first part of the integral in Equation (30) $\left(\int dk_z \left(\cdots\right)\right)$, we have two cases:

1) $x < L_2$

In this case, the k_x contour must encircle the lower half plane and the k_x-integral does not vanish under the provision $Im\left(k_z v_z / v_x\right) > 0$. Then, the integral can be written as

$$J \equiv \int dk_z e^{ik_z(z - S_2)} \int dk_x \frac{\mathbf{k}}{k^2} \frac{e^{ik_x(x - L_2)}}{k_x + \left(k_z v_z + \omega_3\right)/v_x}$$

$$= \frac{\pi v_x}{\omega_1} e^{-i(x - L_2)\omega_3/v_x} \int dk_z e^{i\theta} \left(\frac{1}{k_z - k_{z+}} - \frac{1}{k_z - k_{z-}}\right) H\left(\frac{v_z}{v_x} Im\, k_z\right) \left[-\hat{\mathbf{x}} \frac{k_z v_z + \omega_3}{v_x} + z\hat{k}_z\right] \tag{37}$$

where

$$k_{z\pm} = -\frac{\omega_3}{v^2}\left(v_z \pm i v_x\right) \tag{38}$$

$$\theta = z - S_2 + \left(L_2 - x\right)\frac{v_z}{v_x} \tag{39}$$

Analogously to the foregoing calculation in J', the above integral depends on the sign of θ:

$$\theta > 0 : J = \frac{2i\pi^2 v_x}{\omega_3} e^{i(L_2 - x)\omega_3/v_x} \left[\mathbf{k}_{z+} e^{i\theta k_{z+}} H\left(-v_x\right) H\left(-v_z\right) - \mathbf{k}_{z-} e^{i\theta k_{z-}} H\left(v_x\right) H\left(v_z\right)\right] \tag{40}$$

$$\theta < 0 : J = \frac{-2i\pi^2 v_x}{\omega_3} e^{i(L_2 - x)\omega_3/v_x} \left[\mathbf{k}_{z+} e^{i\theta k_{z+}} H\left(v_x\right) H\left(-v_z\right) - \mathbf{k}_{z-} e^{i\theta k_{z-}} H\left(-v_x\right) H\left(v_z\right)\right] \tag{41}$$

2) $x > L_2$

Repeating a similar analysis, we obtain

$$\theta > 0 : J = \frac{-2i\pi^2 v_x}{\omega_3} e^{i(L_2 - x)\omega_3/v_x} \left[\mathbf{k}_{z+} e^{i\theta k_{z+}} H\left(-v_x\right) H\left(v_z\right) - \mathbf{k}_{z-} e^{i\theta k_{z-}} H\left(v_x\right) H\left(-v_z\right)\right] \tag{42}$$

$$\theta < 0 : J = \frac{2i\pi^2 v_x}{\omega_3} e^{i(L_2 - x)\omega_3/v_x} \left[\mathbf{k}_{z+} e^{i\theta k_{z+}} H\left(v_x\right) H\left(v_z\right) - \mathbf{k}_{z-} e^{i\theta k_{z-}} H\left(-v_x\right) H\left(-v_z\right)\right] \tag{43}$$

where

$$\mathbf{k}_{z\pm} = k_{z\pm}\left(\mp i\hat{\mathbf{x}} + \hat{z}\right) \tag{44}$$

Now, we have to multiply J and J'. In doing it, note that $H\left(x\right)H\left(x\right) = H\left(x\right)$ and $H\left(x\right)H\left(-x\right) = 0$. Nonzero results surviving the velocity integral are obtained in the following four cases:

a) $x < L_2$, $\theta < 0$, $\theta' > 0$; b) $x < L_2$, $\theta > 0$, $\theta' < 0$; c) $x > L_2$, $\theta > 0$, $\theta' > 0$; d) $x > L_2$, $\theta < 0$, $\theta' < 0$

Let us first consider case a). Using Equations (35) and (41), we obtain

$$\Phi = \frac{\left(2\pi^2 v_x\right)^2}{\omega_1 \omega_3} \frac{i\left(x - L_2\right)}{v_x} \left[H\left(v_x\right) H\left(-v_z\right) k_{z+} \exp\left(i\theta_0 + i\theta k_{z+} + i\theta' k'_{z-}\right)\right.$$

$$\left. + H\left(-v_x\right) H\left(v_z\right) k_{z-} \exp\left(i\theta_0 + i\theta k_{z-} + i\theta' k'_{z+}\right)\right] \tag{45}$$

where

$$\theta_0 = \left(L_2 - x\right)\frac{\omega_3}{v_x} + \left(L_1 - L_2\right)\frac{\omega_1}{v_x} \tag{46}$$

Using Equations (34), (39), and (46), we can obtain the exponential phases:

$$i\theta_0 + i\theta k_{z-} + i\theta' k'_{z+} = \varphi'_r + i\varphi_i \tag{47}$$

$$i\theta_0 + i\theta k_{z+} + i\theta' k'_{z-} = -\varphi'_r + i\varphi_i \tag{48}$$

with

$$\varphi'_r = \frac{1}{v^2}\left[v_x\omega_3(S_2 - z) + v_x\omega_1(S_2 - S_1) + v_z\omega_3(x - L_2) + v_z\omega_1(L_1 - L_2)\right] \tag{49}$$

$$\varphi_i = \frac{1}{v^2}\left[v_z\omega_3(S_2 - z) + v_z\omega_1(S_1 - S_2) + v_x\omega_3(L_2 - x) + v_x\omega_1(L_1 - L_2)\right] \tag{50}$$

Thus Equation (45) can be written in the form

$$\Phi = \frac{\left(2\pi^2 v_x\right)^2}{\omega_1\omega_3}\frac{i(x - L_2)}{v_x}e^{i\varphi_i}\left[H(v_x)H(-v_z)\boldsymbol{k}_{z+}\exp(-\varphi'_r) + H(-v_x)H(v_z)\boldsymbol{k}_{z-}\exp(\varphi_{r'})\right] \tag{51}$$

Therefore the velocity integrals in Equation (29) survive the phase mixing when $\varphi_i = 0$, that is,

$$\omega_3(z - S_2) + \omega_1(S_2 - S_1) = 0, \quad \omega_3(L_2 - x) + \omega_1(L_1 - L_2) = 0 \tag{52}$$

$$\text{or} \quad x = \frac{\omega_1 L_1 + \omega_2 L_2}{\omega_1 + \omega_2}, \quad z = \frac{\omega_1 S_1 + \omega_2 S_2}{\omega_1 + \omega_2} \tag{53}$$

where an echo is given rise to. The electric field $\boldsymbol{E}^{(2)}$ can be obtained by velocity integral in the form (see Equation (29))

$$\boldsymbol{E}^{(2)}(t, x, z) = \left[\int_0^\infty dv_x \int_{-\infty}^0 dv_z \boldsymbol{k}_{z+} e^{-\varphi'_r} + \int_{-\infty}^0 dv_x \int_0^\infty dv_z \boldsymbol{k}_{z-} e^{\varphi'_r}\right](\cdots) \tag{54}$$

where $(\cdots)$ denotes the obvious integrand.

Next, let us calculate case (b). Using Equations (36) and (40) gives

$$\begin{aligned}\Phi = &\frac{\left(2\pi^2 v_x\right)^2}{\omega_1\omega_3}\frac{i(x - L_2)}{v_x}\Big[H(-v_x)H(-v_z)\boldsymbol{k}_{z+}\exp\left(i\theta_0 + i\theta k_{z+} + i\theta' k'_{z-}\right)\\ &+ H(v_x)H(v_z)\boldsymbol{k}_{z-}\exp\left(i\theta_0 + i\theta k_{z-} + i\theta' k'_{z+}\right)\Big]\end{aligned} \tag{55}$$

This equation is identical with Equation (45) if $H(v_x)$ and $H(-v_x)$ are interchanged in the latter. Thus, this case can give rise to an echo at the same spot as predicted by Equation (53). The corresponding electric field is obtained by a similar velocity integral to Equation (54) but over different range of v_x.

The cases (a) and (b) predict the same echo spot because they yield the same imaginary phase φ_i. One more task: the various inequality conditions set forth to specify the contour in the contour integrations need to be checked against the echo coordinate found in Equation (54). Let us consider the inequalities $\theta < 0$ and $\theta' > 0$ postulated in the case (a). Using Equation (52), the inequality $\theta < 0$ can be written in the form

$$S_1 - S_2 + \frac{v_z}{v_x}(L_2 - L_1) < 0$$

which is the condition $\theta' > 0$. Therefore the conditions $\theta < 0$ and $\theta' > 0$ imply each other. Also we can ascertain that the echo x-coordinate is in accord with the condition $x < L_2$. So in cases (a) and (b), the premise and the result are consistent. For the cases of (c) and (d), we state without repeating a similar algebra that the imaginary part of the phase is still obtained by Equation (50) [the real part of the phase is different]. Although the echo spot is predicted by the same equation as Equation (53), these cases of (c) and (d) are not acceptable because the conditions $\theta > 0$ and $\theta' > 0$ or $\theta < 0$ and $\theta' < 0$ are contradictory to each other. We have the conclusion: an echo occurs where $x < L_2$ and the echo coordinates are predicted by Equation (53).

(2) Next, we consider the product of two boundary terms, $I(L_1, k'_z)I(L_2, k_z - k'_z)$ in Equation (29):

$$\begin{aligned}\boldsymbol{E}^{(2)}(t, x, z) = &Ce^{i\omega_3 t}\int\frac{d^2v f_0}{v_x^2}\int dk_z \exp\left[ik_z(z - S_2)\right]\int dk_x \frac{\boldsymbol{k}}{k^2}\frac{\exp[ik_x]}{\left(k_x + \left(k_z v_z + \omega_3\right)/v_x\right)^2}\\ &\times\int dk'_z e^{ik'_z(S_2 - S_1)}I(L_1, k'_z)I(L_2, k_z - k'_z)\int\frac{dk'_x}{k'^2\left(\omega_1 + \boldsymbol{k}'\cdot\boldsymbol{v}\right)}\end{aligned} \tag{56}$$

where C is a nonessential constant factor. For definiteness, we assume $S_2 - S_1 > 0$. $\int dk_x'$-integral and $\int dk_x$-integral can be done easily by picking up the relevant poles, and we can write

$$\boldsymbol{E}^{(2)}\left(t,x,z\right) = ix\, Ce^{i\omega_3 t} \int \frac{d^2 v f_0}{v_x^3} \exp\left[-ix\frac{\omega_3}{v_x}\right] \int dk_z \frac{\boldsymbol{k}}{k^2} \exp\left[ik_z\left(z - S_2 - x\frac{v_x}{v_z}\right)\right]$$
$$\times \int \frac{dk_z'}{k'^2} e^{ik_z'(S_2 - S_1)} I\left(L_1, k_z'\right) I\left(L_2, k_z - k_z'\right) \tag{57}$$

where

$$k'^2 = k_z'^2 + \frac{\left(\omega_1 + k_z' v_z\right)^2}{v_x} \tag{58}$$

$$k = -x\frac{k_z v_z + \omega_3}{v_x} + zk_z \tag{59}$$

The contour of $\int dk_z'$-integral should encircle the upper k_z'-plane. Since the relevant singularity should be located in the upper k_z'-plane, the residue is calculated from $k_z' = k_{z+}'$ for $v_x < 0$ and $k_z' = k_{z-}'$ for $v_x > 0$. $k_{z\pm}'$ are defined in Equation (33). Then the last integral ($\int dk_z'$) in Equation (57) can be carried out in the form

$$\int dk_z'\left(\cdots\right) = \frac{-\pi v_x}{\omega_1}\Big[\exp\left[ik_{z+}'\left(S_2 - S_1\right)\right] I\left(L_1, k_{z+}'\right) I\left(L_2, k_z - k_{z+}'\right) H\left(-v_x\right)$$
$$- \exp\left[ik_{z-}'\left(S_2 - S_1\right)\right] I\left(L_1, k_{z-}'\right) I\left(L_2, k_z - k_{z-}'\right) H\left(v_x\right)\Big] \equiv J\left(k_z\right) \tag{60}$$

where

$$I\left(L_1, k_{z\pm}'\right) = -i\pi\left(\exp\left[L_1 k_{z\pm}'\right] H\left(v_z\right) + \exp\left[-L_1 k_{z\pm}'\right] H\left(-v_z\right)\right) \quad \text{(see Equation (18))} \tag{61}$$

To carry out $\int dk_z\left(\cdots\right)$ in Equation (59), let us assume that $z - S_2 - x\dfrac{v_z}{v_x} \equiv \vartheta < 0$.

Now we are ready to evaluate $\int dk_z$-integral in Equation (59):

$$\int dk_z\left(\cdots\right) = \int dk_z \frac{\boldsymbol{k} e^{ik_z \vartheta}}{k^2} J\left(k_z\right) = \frac{iv_x}{2\omega_3}\int dk_z\, \boldsymbol{k} J\left(k_z\right)\left[\frac{e^{ik_z \vartheta}}{k_z - k_{z+}} - \frac{e^{ik_z \vartheta}}{k_z - k_{z-}}\right]$$
$$= \frac{\pi v_x}{\omega_3}\left[\boldsymbol{k}\left(k_{z+}\right) e^{ik_{z+}\vartheta} J\left(k_{z+}\right) H\left(v_x\right) - \boldsymbol{k}\left(k_{z-}\right) e^{ik_{z-}\vartheta} J\left(k_{z-}\right) H\left(-v_x\right)\right] \tag{62}$$

Using Equation (60) in Equation (62) yields

$$\int dk_z\left(\cdots\right) = \frac{\pi^2 v_x^2}{\omega_1 \omega_3}\Big[\boldsymbol{k}\left(k_{z+}\right)\exp\left[ik_{z+}\vartheta\right]\exp\left[ik_{z-}'\left(S_2 - S_1\right)\right] I\left(L_1, k_{z-}'\right) I\left(L_2, k_{z+} - k_{z-}'\right) H\left(v_x\right)$$
$$+ \boldsymbol{k}\left(k_{z-}\right)\exp\left[ik_{z-}\vartheta\right]\exp\left[ik_{z+}'\left(S_2 - S_1\right)\right] I\left(L_1, k_{z+}'\right) I\left(L_2, k_{z-} - k_{z+}'\right) H\left(-v_x\right)\Big] \tag{63}$$

where we have $Re\left(k_{z+} - k_{z-}'\right) = Re\left(k_{z-} - k_{z+}'\right) = -\dfrac{\omega_2}{v^2} v_z$. Therefore, we obtain

$$I\left(L_2, k_{z+} - k_{z-}'\right) = -i\pi\left(\exp\left[L_2\left(k_{z+} - k_{z-}'\right)\right] H\left(v_z\right) + \exp\left[-L_2\left(k_{z+} - k_{z-}'\right)\right] H\left(-v_z\right)\right) \tag{64}$$

$$I\left(L_2, k_{z-} - k_{z+}'\right) = -i\pi\left(\exp\left[L_2\left(k_{z-} - k_{z+}'\right)\right] H\left(v_z\right) + \exp\left[-L_2\left(k_{z-} - k_{z+}'\right)\right] H\left(-v_z\right)\right) \tag{65}$$

The above two equations and Equation (61) yield

$$I\left(L_1, k_{z-}'\right) I\left(L_2, k_{z+} - k_{z-}'\right) = -\pi^2\left[\exp\left[L_2 k_{z+} + \left(L_1 - L_2\right) k_{z-}'\right] H\left(v_z\right) + \exp\left[-L_2 k_{z+} - \left(L_1 - L_2\right) k_{z-}'\right] H\left(-v_z\right)\right] \tag{66}$$

$$I\left(L_{1},k_{z+}'\right)I\left(L_{2},k_{z-}-k_{z+}'\right)=-\pi^{2}\left[\exp\left[L_{2}k_{z-}+\left(L_{1}-L_{2}\right)k_{z+}'\right]H\left(v_{z}\right)+\exp\left[-L_{2}k_{z-}-\left(L_{1}-L_{2}\right)k_{z+}'\right]H\left(-v_{z}\right)\right] \quad (67)$$

Now we are ready to carry out the velocity integral in Equation (59) by substituting Equation (63) into it. Because of the step functions $H\left(v_{x}\right)$ and $H\left(v_{z}\right)$, the velocity integral consists of four parts corresponding to $H\left(\pm v_{x}\right)H\left(\pm v_{z}\right)$. Since we are interested in the echo spots, we pay attention only to the exponential phases:

$$H\left(v_{x}\right)H\left(-v_{z}\right)\mathrm{e}^{\varphi_{1}},H\left(v_{x}\right)H\left(v_{z}\right)\mathrm{e}^{\varphi_{2}},H(-v_{x})H\left(v_{z}\right)\mathrm{e}^{\varphi_{3}},H\left(-v_{x}\right)H\left(-v_{z}\right)\mathrm{e}^{\varphi_{4}} \quad (68)$$

Straightly we can identify:

$$\varphi_{1}=-ix\frac{\omega_{3}}{v_{x}}+ik_{z+}\vartheta+ik_{z-}'\left(S_{2}-S_{1}\right)-L_{2}k_{z+}+\left(L_{2}-L_{1}\right)k_{z-}' \quad (69)$$

$$\varphi_{2}=-ix\frac{\omega_{3}}{v_{x}}+ik_{z+}\vartheta+ik_{z-}'\left(S_{2}-S_{1}\right)+L_{2}k_{z+}+\left(L_{1}-L_{2}\right)k_{z-}' \quad (70)$$

$$\varphi_{3}=-ix\frac{\omega_{3}}{v_{x}}+ik_{z-}\vartheta+ik_{z+}'\left(S_{2}-S_{1}\right)+L_{2}k_{z-}+\left(L_{1}-L_{2}\right)k_{z+}' \quad (71)$$

$$\varphi_{4}=-ix\frac{\omega_{3}}{v_{x}}+ik_{z-}\vartheta+ik_{z+}'\left(S_{2}-S_{1}\right)-L_{2}k_{z-}+\left(L_{2}-L_{1}\right)k_{z+}' \quad (72)$$

From above, the imaginary phases are obtained as

$$Im\,\varphi_{1}=Im\,\varphi_{3}=\frac{v_{z}}{v^{2}}\left[\omega_{1}\left(S_{1}-S_{2}\right)-\omega_{3}\left(z-S_{2}\right)\right]+\frac{v_{x}}{v^{2}}\left[L_{2}\left(\omega_{1}+\omega_{3}\right)-L_{1}\omega_{1}-x\,\omega_{3}\right] \quad (73)$$

$Im\,\varphi_{2}=Im\,\varphi_{4}$ is obtained from $Im\,\varphi_{1}$ by replacing $L_{1}\rightarrow-L_{1}$ and $L_{2}\rightarrow-L_{2}$. Putting $Im\,\varphi_{1,2}=0$, we obtain the echo spots as

$$z_{echo}=\frac{\omega_{1}S_{1}+\omega_{2}S_{2}}{\omega_{3}} \quad (74)$$

$$x_{echo}=\frac{\pm L_{2}\left(2\omega_{1}+\omega_{2}\right)\mp L_{1}\omega_{1}}{\omega_{3}} \quad (75)$$

In Equation (75), x_{echo} corresponding to upper signs and x_{echo} corresponding to lower signs are mutually exclusive because if one of them is inside the plasma the other is necessarily is outside the plasma. We add that the condition $\vartheta<0$ amounts to $\frac{\omega_{1}}{\omega_{3}}\left(S_{1}-S_{2}\right)<x\frac{v_{z}}{v_{x}}$, which poses no problem in as much as we have ample liberty in choosing the sign of v_{z}/v_{x}.

5. Discussion

In Section 3, the plasma electric field was determined in terms of the vacuum electric field. Judicious application of the boundary conditions at the interface enables one to determine the plasma electric field entirely in terms of the external charges without introducing the vacuum electric field E_{0}. Inverting Equation (13), we can write

$$D_{x}\left(0,\omega\right)=-\frac{N}{2}-i\frac{4\pi}{k_{0}^{2}}\rho_{1}\delta\left(\omega+\omega_{1}\right)\mathrm{e}^{-ik_{z}S_{1}}I\left(k_{z},L_{1}\right) \quad (76)$$

$$E_{z}\left(0,\omega\right)=iN\int\frac{k_{z}}{k^{2}\varepsilon}\frac{\mathrm{d}k_{x}}{2\pi}-i\frac{4\pi}{k_{0}^{2}}\rho_{1}\delta\left(\omega+\omega_{1}\right)\mathrm{e}^{-ik_{z}S_{1}}I_{\varepsilon}\left(k_{z},L_{1}\right) \quad (77)$$

$$\text{where}\quad I_{\varepsilon}\left(k_{z},L_{1}\right)=\int\frac{k_{z}\mathrm{d}k_{x}}{k^{2}\varepsilon}\mathrm{e}^{-ik_{x}L_{1}} \quad (78)$$

Next, we turn to the vacuum solution.

$$E_{x}\left(x,k_{z}\right)=-B'k_{z}\mathrm{e}^{k_{z}x}\equiv E_{0}\mathrm{e}^{k_{z}x} \quad (79)$$

where E_0 is the vacuum electric field, the quantity designated by the same symbol E_0 in Equation (19).

$$E_z(x, k_z) = -iB'k_z \mathrm{e}^{k_z x} \tag{80}$$

Continuity of E_z across the interface $x = 0$ gives that Equation (22) equals to $-iB'k_z$. Also continuity of D_x across $x = 0$ yields that Equation (20) equals to $-B'k_z$. Eliminating B' between these two equations gives N in the form

$$N = \frac{\dfrac{8\pi}{k_0^2}\rho_1\delta(\omega + \omega_1)\mathrm{e}^{-ik_z S_1}\left(I_\varepsilon(k_z, L_1) - iI(k_z, L_1)\right) + 1 \to 2}{1 + \int \dfrac{\mathrm{d}k_x}{\pi}\dfrac{k_z}{k^2\varepsilon}} \tag{81}$$

Substituting the above equation into Equation (13) yields

$$\boldsymbol{E}(\boldsymbol{k}, \omega) = \frac{8\pi^2}{k_0^2}\frac{i\boldsymbol{k}}{k^2\varepsilon}\rho_1\delta(\omega + \omega_1)\mathrm{e}^{-ik_z S_1}\frac{-iI(k_z, L_1)/\pi - \mathrm{e}^{-ik_x L_1}}{1 + \int \dfrac{\mathrm{d}k_x}{\pi}\dfrac{k_z}{k^2\varepsilon}} \tag{82}$$

Equation (82) should be compared with Equation (20). Eliminating the vacuum field introduces the denominator $1 + \int \dfrac{\mathrm{d}k_x}{\pi}\dfrac{k_z}{k^2\varepsilon}$ in Equation (82). In fact, the relation

$$1 + \int \frac{\mathrm{d}k_x}{\pi}\frac{k_z}{k^2\varepsilon} = 0 \tag{83}$$

is the electrostatic dispersion relation of the surface wave in a semi-bounded plasma [10].

In the investigation of echo occurrence, E_0 in Equation (20) can be discarded because echoes are given rise to by interference of influences of the external charges. This amounts to saying that the denominator $\left(1 + \int \dfrac{\mathrm{d}k_x}{\pi}\dfrac{k_z}{k^2\varepsilon}\right)$ doesn't play any role in the determination of echo locations.

Equations (53) and (74) and (75) are the main results of this work in locating the echo spots associated with the surface wave in a semi-bounded plasma launched by the oscillating external charges at $(x, z) = (L_1, S_1)$ and (L_2, S_2). In the static situation, the z-coordinate is irrelevant. The echo spot given by Equation (53) corresponds to x_{echo} in Equation (45) in Lee and Lee [5]. The echo spot given by Equations (74) and (75) is surface wave-proper. Our search for the echo spots are not exhaustive; we put aside many other product terms in (AB) in Equations (24) and (25). It appears that we have diversity of echoes in a bounded plasma, which was also experimentally reported [7]. The diversity seems to be due to reflections of the wave at the interface.

In reality, bounded plasmas are usual rather than exceptional. Important literatures to get acquainted with this field are References [8] and [9], among others. Surface wave dispersion relation in a plasma slab is derived in Ref. [10]. An exact nonlinear solution of a surface wave excited by external charges is obtained in Ref. [11].

Acknowledgements

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