

A Basis for Causal Scattering Waves, Relativistic Diffraction in Time Functions

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Abstract

Relativistic diffraction in time wave functions can be used as a basis for causal scattering waves. We derive such exact wave function for a beam of Dirac and Klein-Gordon particles. The transient Dirac spinors are expressed in terms of integral defined functions which are the relativistic equivalent of the Fresnel integrals. When plotted versus time the exact relativistic densities show transient oscillations which resemble a diffraction pattern. The Dirac and Klein-Gordon time oscillations look different, hence relativistic diffraction in time depends strongly on the particle spin.

Keywords

Diffraction in Time, Relativistic Diffraction in Time, Causal Scattering Basis, Transient Quantum Processes

1. Introduction

Similarities between optics and quantum mechanics have long been recognized. One example of this symmetry was obtained by Moshinsky [1] who addressed the following non-relativistic, quantum, 1D shutter problem. Consider a monoenergetic beam of free particles moving parallel to the x -axis. For negative times, the beam is interrupted at $x = 0$ by a perfectly absorbing shutter perpendicular to the beam. Suddenly, at time $t = 0$, the shutter is opened, allowing for $t > 0$ the free time-evolution of the beam of particles. What is the transient density observed at a distance x from the shutter? The shutter problem implies solving, as an initial value problem, the time-dependent Schrödinger equation with an initial condition given by

$$\psi(x, 0) = e^{ikx} \theta(-x), \quad (1)$$

where $\theta(x)$ denotes the step function defined as: $\theta(x) = (1 \text{ if } x > 0)$ or $(0 \text{ if } x < 0)$. For $x \geq 0$, Moshinsky proves that the free propagation of the beam has the exact solution given by:

$$M(x, t; k) = \frac{1}{\sqrt{2i}} e^{i(kx - \omega t)} \left\{ \frac{1+i}{2} + \int_0^{\xi(x,t)} e^{i\pi u^2/2} du \right\} \theta(t), \tag{2}$$

where the integral is the complex Fresnel function: $\int_0^{\xi} \exp(i\pi u^2/2) du \equiv C(\xi) + iS(\xi)$ and ξ is given by $\xi(x, t) \equiv \sqrt{m/\pi\hbar t} (\hbar kt/m - x)$. For $x \geq 0$ the probability density ρ is then

$$\rho(x, t) = \frac{1}{2} \left[C(\xi) + \frac{1}{2} \right]^2 + \frac{1}{2} \left[S(\xi) + \frac{1}{2} \right]^2 \theta(t). \tag{3}$$

The right-hand side in Equation (3) is similar to the mathematical expression for the light intensity in the optical Fresnel diffraction by a straight edge [2]. For a fixed position $x = 1$, the plot of the probability density $\rho(1, t)$ as a function of time is shown in Figure 1.

These temporal oscillations are a pure quantum phenomenon, and similar oscillations arise at the moment of closing and opening gates in nanoscopic circuits [3]. With adequate potentials added to the model, it has been used to study transient dynamics of tunneling matter waves [4]-[7], and the transient responses to abrupt changes of the interaction potential in semiconductor structures and quantum dots [8] [9]. For a review on the subject see [10] [11]. There is, in summary, a strong motivation for a thorough understanding of transient time oscillation in beams of matter.

One of the main problems in physics is to find, for the S matrix of an interaction, restrictions which proceed from general principles such as causality [12]. Notice that there is a close relation between diffraction in times wave functions and those wave functions which are needed for a causal description. From Equation (2) we see that for $x \geq 0$ the wave function $M(x, t; k)$ is causal and the shutter solution can then be used as a basis function for causal scattering. Indeed, for an arbitrary function, $f(x) \equiv \int_{-\infty}^{\infty} F(k) e^{ikx} dk$, and assuming an initial condition given by:

$$\Psi(x, 0) = f(x) \theta(-x), \tag{4}$$

then the free time evolution of the initial condition becomes

$$\Psi(x, t) = \theta(t) \int_{-\infty}^{\infty} M(x, t; k) F(k) dk. \tag{5}$$

It is evident that if we want a relativistic solution for $\Psi(x, t)$, we need, instead of $M(x, t; k)$, the corresponding relativistic solution to the shutter problem.

As far as we know nobody has ever reported the exact relativistic solution to the shutter problem. Moshinsky worked this problem and gave an approximated answer. In a couple of articles [13] [14], he discussed the shutter problem using the Klein-Gordon and the Dirac equations. Using approximated solutions Moshinsky arrives to the conclusion that only for the Schrödinger equation the wave function ψ does resemble the expression that appear in the optical theory of diffraction. In his conclusions [13], Moshinsky emphatically denies the existence

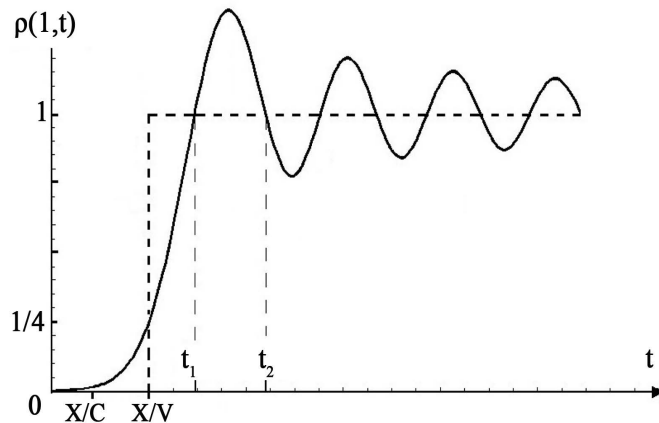


Figure 1. Probability density for non-relativistic diffraction in time.

of diffraction in time in the relativistic case. In the case of photons this is obviously true, the d'Alembert's solution does not allow such time oscillations. However, for particles with mass different from zero, in full disagreement with Moshinsky's conclusions, we report here that relativistic diffraction in time oscillations is indeed present.

The purpose of the present paper is to derive the exact solutions for the Dirac and Klein-Gordon shutter problems. The exact transient Dirac spinors are expressed in terms of integral-defined-functions which are the relativistic equivalent of the Fresnel integrals. In partial agreement with Moshinsky's conclusions we find that indeed the relativistic densities do not resemble the mathematical expression for intensity of light that appears in the theory of diffraction in Optics. In spite of this, when our exact relativistic densities are plotted versus time, the plots show transient oscillations which resemble a diffraction pattern. For this reason in this article we claim that impressive diffractions in time oscillations do exist in the relativistic realm. Furthermore, the Dirac and Klein-Gordon densities look quite different, which implies that relativistic diffraction in time distinguishes between spin 0 and 1/2.

2. The Dirac Shutter Problem

Consider, for relativistic particles of spin 1/2, the shutter problem. We want to find out the spinor wave function $\psi(z, t) = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ which is the solution of the one-dimensional Dirac equation:

$$\left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial(ct)} + \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial z} + \mu \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right\} \psi = 0, \quad (6)$$

where σ_z is the 2×2 Pauli matrix and $\mu^{-1} \equiv \hbar/mc \equiv \lambda_c$ the Compton length. The initial condition corresponds, for $t \leq 0$, to a plane wave to the left of the shutter and zero to the right. Three quantum numbers are needed to classify the Dirac free particle solutions, namely, the momentum $\mathbf{p} \equiv \hbar \mathbf{k}$, the positive or negative energies $\epsilon = \pm \hbar \omega$, where $\omega = c(k^2 + \mu^2)^{1/2}$, and helicity $\Lambda_S = \mathbf{S} \cdot \mathbf{p}/p$. We select the initial condition assuming a positive energy $\epsilon = +\hbar \omega$ and a plane wave propagating along the z direction $\mathbf{k} = (0, 0, k)$. As for the initial helicity,

$$\Lambda_S = S_z = \frac{\hbar}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}, \quad (7)$$

we choose the initial state with a well defined direction of spin, for instance parallel to the direction of motion, $S_z = +1/2$. Then in the shutter problem we have an incident plane wave given by

$$\psi(z, t) = N \left(1, 0, \frac{k}{\mu + \omega/c}, 0 \right)^T e^{i(kz - \omega t)} \theta(-z) \quad (t \leq 0). \quad (8)$$

For free particles, the helicity Λ_S is a constant of motion. The initial direction of spin, $S_z = 1/2$, will be conserved at all positive times. As a consequence the two components of the wave function (ψ_2 and ψ_4) which are zero at the initial time will remain zero at all positive times:

$$\psi_2(z, t) = \psi_4(z, t) = 0. \quad (t \geq 0). \quad (9)$$

In terms of the remaining two components (ψ_1, ψ_3) the Dirac shutter problem is the solution of the equation,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial(ct)} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial z} + \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix} = 0, \quad (10)$$

with the initial condition:

$$\begin{pmatrix} \psi_1(z, 0) \\ \psi_3(z, 0) \end{pmatrix} = \begin{pmatrix} N \\ \gamma \end{pmatrix} e^{ikz} \theta(-z), \quad (11)$$

where $\gamma \equiv Nk/(\mu + \omega/c)$. The normalization factor N is chosen as $N \equiv \sqrt{1 + \omega/c\mu}$; in this way the probability density ρ and the density current \mathbf{j}/c transform initially, as a four-vector $(\rho, \mathbf{j}/c) = (2/\mu)(\omega/c, \mathbf{k})$.

We use the Compton length $\lambda_C \equiv \hbar/mc$ to define dimensionless variables: $\chi \equiv z/\lambda_C$, $\tau \equiv ct/\lambda_C$, $\kappa \equiv k\lambda_C$, and $\Omega \equiv \omega\lambda_C/c = \sqrt{1+\kappa^2}$. Using these variables we derive in [Appendix A](#) the exact solution of the Dirac shutter problem. To simplify the notation, for n integer, we denote by G_n the integral-defined complex functions

$$G_n(\chi, \tau) \equiv \int_{\chi}^{\tau} du e^{i\Omega(\tau-u)} \frac{J_n(\sqrt{u^2 - \chi^2})}{(u^2 - \chi^2)^{n/2}} \equiv C_n + iS_n \quad (\tau \geq \chi) \quad (12)$$

where $J_n(z)$ is the Bessel function of the first kind of order n . Notice that $z=0$ is a removable singularity for $J_n(z)/z^n$, hence the functions G_n are analytic. In fact the integrand can be explicitly written analytic by eliminating the denominator. Indeed, using repeatedly the recurrence relation for the Bessel functions, $2nJ_n(z)/z = J_{n-1}(z) + J_{n+1}(z)$, we can write

$$\begin{aligned} \frac{J_1(z)}{z} &= \frac{1}{2}[J_0(z) + J_2(z)] \\ \frac{J_2(z)}{z^2} &= \frac{1}{24}[3J_0(z) + 4J_2(z) + J_4(z)] \\ \frac{J_3(z)}{z^3} &= \frac{1}{480}[10J_0(z) + 15J_2(z) + 6J_4(z) + J_6(z)] \end{aligned} \quad (13)$$

Clearly the real and imaginary parts of $G_n \equiv C_n + iS_n$ are analytic oscillating functions.

From [Appendix A](#), for the right side of the shutter $\chi > 0$, we have the exact Dirac shutter solution:

$$\begin{aligned} -2 \frac{\begin{pmatrix} \psi_1(\chi, \tau) \\ \psi_3(\chi, \tau) \end{pmatrix}}{\theta(\tau - \chi)} &= \begin{pmatrix} \gamma \\ N \end{pmatrix} \left\{ i\kappa \left(\frac{\sin[\Omega(\tau - \chi)]}{\Omega} - \frac{\chi}{\Omega} S_1 \right) \right. \\ &\quad \left. - \left(\cos[\Omega(\tau - \chi)] - \frac{\chi}{2} \frac{\sin[\Omega(\tau - \chi)]}{\Omega} + \frac{1}{\Omega} S_1 + \frac{\chi^2}{\Omega} S_2 \right) \right\} \\ &\quad - \begin{pmatrix} N \\ \gamma \end{pmatrix} \left\{ \cos[\Omega(\tau - \chi)] \pm i \frac{\sin[\Omega(\tau - \chi)]}{\Omega} \mp \chi \left(C_1 \pm \frac{i}{\Omega} S_1 \right) \right. \\ &\quad \left. - \frac{i}{\kappa} \left(-\left(\frac{\chi}{2} \pm i \right) e^{\mp i(\tau - \chi)} + \begin{pmatrix} G_1^* \\ G_1 \end{pmatrix}_{\kappa=0} + \chi^2 \begin{pmatrix} G_2^* \\ G_2 \end{pmatrix}_{\kappa=0} \right) \right. \\ &\quad \left. + \Omega \sin[\Omega(\tau - u)] + \frac{\chi}{2} \cos[\Omega(\tau - \chi)] - C_1 - \chi^2 C_2 \right. \\ &\quad \left. \pm i \left[\cos[\Omega(\tau - \chi)] - \frac{\chi}{2} \frac{\sin[\Omega(\tau - \chi)]}{\Omega} + \frac{1}{\Omega} S_1 + \frac{\chi^2}{\Omega} S_2 \right] \right\} \end{aligned} \quad (14)$$

Notice the function $\theta(\tau - \chi)$ which shows the relativistic condition that no wave function exists until $ct \geq z$. This property is missing in the Schrödinger solution.

3. Dirac Diffraction in Time

Given the Dirac wave function $\psi(z, t)$ in Equation (14), we can calculate the probability density ρ given by

$$\rho(\chi, \tau; \kappa) = |\psi|^2 = |\psi_1|^2 + |\psi_3|^2 \equiv \rho_1 + \rho_3 \quad (15)$$

In [Figure 2](#), for fixed values of $\kappa = 1$ ($v/c = 0.7$) and $\chi = 5$, we show a typical plot of the Dirac density $\rho(\tau)$. Surprisingly we find damped oscillations which resemble the Schrödinger diffraction in time oscillations. For this reason we call this plot a relativistic diffraction in time process. However, the Dirac oscillations are clearly different from the Schrödinger ones (see [Figure 1](#)). For ρ_1 notice the impressive double oscillations which are unique to the Dirac theory.

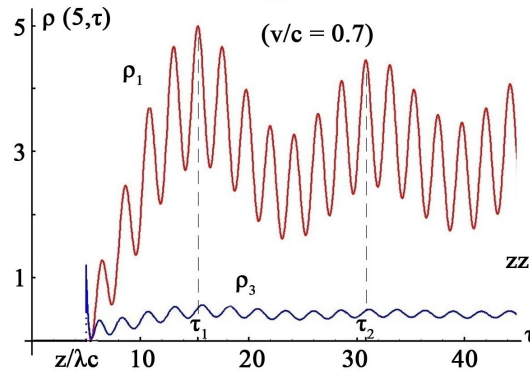


Figure 2. Dirac diffraction in time probability densities ρ_1 and ρ_3 .

As expected, for a relativistic solution, the Dirac density vanishes for times $0 \leq t \leq z/c$. The down oscillation, immediately following, $t = z/c$, is also a relativistic property.

4. The Klein-Gordon Shutter Problem

For relativistic particles with spin 0, the Klein-Gordon shutter problem is, by definition, the solution $\psi(z, t)$ of the equation:

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial (ct)^2} = \mu^2 \psi, \quad (16)$$

where $\mu \equiv 1/\lambda_c \equiv mc/\hbar$. The initial conditions correspond to a plane wave to the left of the shutter and zero to the right.

$$\psi(z, t) = e^{i(kz - \omega t)} \theta(-z), \quad (t \leq 0) \quad (17)$$

where $\omega = c(k^2 + \mu^2)^{1/2}$. Therefore at $t = 0$, when the shutter is suddenly opened, we have the initial conditions:

$$\psi(z, 0) = e^{ikz} \theta(-z), \quad \frac{\partial \psi(z, 0)}{\partial t} = -i\omega e^{ikz} \theta(-z). \quad (18)$$

Similar to the Dirac problem, in terms of the dimensionless variables: $\chi \equiv z/\lambda_c$, $\tau \equiv ct/\lambda_c$, $\kappa \equiv k\lambda_c$, and $\Omega \equiv \omega\lambda_c/c = \sqrt{1 + \kappa^2}$, we find the exact solution of this Klein-Gordon problem in [Appendix B](#). At a fixed distance $\chi > 0$, on the right side of the shutter, we have the exact Klein-Gordon shutter solution:

$$\psi(\chi, \tau; \kappa) = \frac{1}{2} \theta(\tau - \chi) \left\{ e^{i\Omega(\chi - \tau)} - \int_{\chi}^{\tau} du e^{i\Omega(u - \tau)} \left[\chi \frac{J_1(\sqrt{u^2 - \chi^2})}{\sqrt{u^2 - \chi^2}} + i\kappa J_0(\sqrt{u^2 - \chi^2}) \right] \right\} \quad (19)$$

or in simplified notation

$$\psi(\chi, \tau) = \frac{1}{2} \theta(\tau - \chi) \left\{ e^{i\Omega(\chi - \tau)} - \chi G_1^* - i\kappa G_0^* \right\}. \quad (20)$$

The presence of $\theta(\tau - \chi)$ means, as expected, that the wave function vanishes for $t < z/c$, where z is the distance from shutter to the particle detector.

Given the Klein-Gordon wave function $\psi(\chi, \tau)$, we have a charge density given by (charge $q = 1$),

$$\rho(\chi, \tau) = -\text{Im} \left[\psi^*(\chi, \tau) \frac{\partial \psi(\chi, \tau)}{\partial \tau} \right] \quad (21)$$

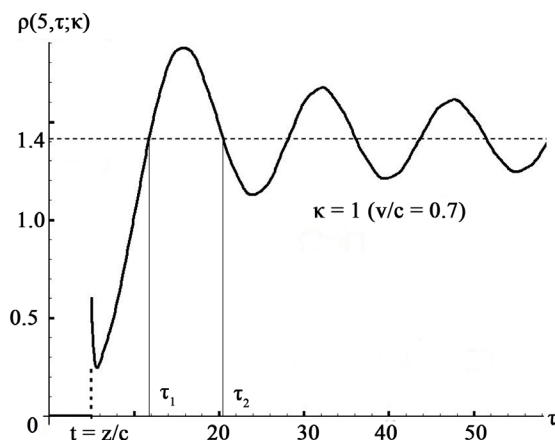


Figure 3. Typical Klein-Gordon diffraction in time for the charge density, $\chi = 5$.

In **Figure 3**, for fixed values of $\kappa = 1$ and $\chi = 5$, we show a typical plot of the charge density versus time for the Klein-Gordon solution. The impressive damped oscillations, shown in **Figure 3**, clearly resemble the optical Fresnel diffraction pattern by a straight edge. The double oscillation which is present in the Dirac solution is missing now.

Notice that the asymptotic behavior of ρ is not 1, as it occurs in the Schrödinger solution. In the particular case of $\chi = 5$ and $\kappa = 1$ ($v/c = 0.7$) shown in **Figure 3**, the stationary density is $\rho = 1.4$, which is the correct prediction for the shutter's initial conditions (18). In fact

$$\rho(\chi, 0) = -\text{Im} \left[\psi^*(\chi, 0) \frac{\partial \psi(\chi, 0)}{\partial \tau} \right] = \sqrt{1 + \kappa^2} \quad (22)$$

Therefore, for $\kappa = 1$ the predicted stationary density is 1.4.

5. Conclusions

We derived the exact solutions for the Klein-Gordon and the Dirac shutter problems. In agreement with Moshinsky we find that the relativistic solutions do not resemble the analytic expression that appears in the theory of diffraction in Optics. In spite of this, we prove that when the exact Dirac and Klein-Gordon densities are plotted versus time, the following happens: 1) both densities show transient oscillations which in some way resemble the optical diffraction pattern; 2) the Dirac density looks quite different from the Klein-Gordon one, which implies that transient time oscillations depend strongly on the particle spin.

For these reasons and in total disagreement with Moshinsky's conclusions [13], we claim that impressive diffractions in time oscillations do exist in the relativistic realm. For spin 0 and 1/2 particles, we prove that diffraction in time oscillations exists only for particles of rest mass different from zero; photons do not show such time oscillations.

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Appendix A

With the help of the dimensionless variables given by: $\chi \equiv z/\lambda_C$, $\tau \equiv ct/\lambda_C$, $\kappa \equiv k\lambda_C$, and $\Omega \equiv \omega\lambda_C/c = \sqrt{1+\kappa^2}$, the Dirac Equation (10) may be rewritten as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial \tau} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \chi} + i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix} = 0, \quad (23)$$

and the initial condition becomes:

$$\psi(\chi, 0) \equiv \begin{pmatrix} \psi_1(\chi, 0) \\ \psi_3(\chi, 0) \end{pmatrix} = \begin{pmatrix} N \\ \gamma \end{pmatrix} e^{i\kappa\chi} \theta(-\chi). \quad (24)$$

Taking the Laplace transform ($\tau \rightarrow s$) in Equation (23), and denoting $\mathcal{L}[\psi(\chi, \tau)] \equiv \phi(\chi, s) = (\phi_1(\chi, s), \phi_3(\chi, s))^T$, Equation (23) becomes a matrix differential equation

$$\frac{d\phi}{d\chi} + (\sigma_x s + \sigma_y) \phi = \begin{pmatrix} \gamma \\ N \end{pmatrix} e^{i\kappa\chi} \theta(-\chi) \quad (25)$$

which holds in the range $-\infty < \chi < \infty$. Due to the presence of the step function $\theta(-\chi)$, the origin $\chi = 0$ is a singular point where we demand that the function $\phi(\chi, s)$ must be continuous. We break the infinite range into the ranges ($\chi \leq 0$) and ($\chi \geq 0$). For the left side of the shutter, $\phi_<(\chi, s)$ denotes the solution of the differential equation:

$$\frac{d\phi_<}{d\chi} + (\sigma_x s + \sigma_y) \phi_< = \begin{pmatrix} \gamma \\ N \end{pmatrix} e^{i\kappa\chi}, \quad (\chi \leq 0) \quad (26)$$

and for the right side, $\phi_>(\chi, s)$ denotes the solution of

$$\frac{d\phi_>}{d\chi} + (\sigma_x s + \sigma_y) \phi_> = 0, \quad (\chi \geq 0). \quad (27)$$

Both functions $\phi_<$ and $\phi_>$ must be bounded ($\phi_<$ at $-\infty$) and ($\phi_>$ at $+\infty$), and must be continuous at the interface $\chi = 0$.

Because the matrix

$$\sigma_x s + \sigma_y = \begin{pmatrix} 0 & s-i \\ s+i & 0 \end{pmatrix} \quad (28)$$

has eigenvalues given by: $\lambda_1 = \sqrt{s^2+1} = -\lambda_2$, with corresponding orthogonal eigenvectors given by:

$$u_1 = \begin{pmatrix} \sqrt{s-i} \\ \sqrt{s+i} \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\sqrt{s-i} \\ \sqrt{s+i} \end{pmatrix}, \quad (29)$$

then, taking into account the boundary conditions at $\pm\infty$, we have the general solutions for the matrix differential equations:

$$\phi_>(\chi, s) = A \begin{pmatrix} \sqrt{s-i} \\ \sqrt{s+i} \end{pmatrix} e^{-\chi\sqrt{s^2+1}} \quad (\chi \geq 0) \quad (30)$$

$$\phi_<(\chi, s) = B \begin{pmatrix} -\sqrt{s-i} \\ \sqrt{s+i} \end{pmatrix} e^{+\chi\sqrt{s^2+1}} - \frac{1}{s^2 + \Omega^2} \begin{pmatrix} i\kappa\gamma - (s-i)N \\ i\kappa N - (s+i)\gamma \end{pmatrix} e^{i\kappa\chi} \quad (\chi \leq 0) \quad (31)$$

where $\Omega^2 = 1 + \kappa^2$. The constants A and B are fixed from the condition at the interface:

$(\phi_<(\chi, s), \phi_3(\chi, s))^T = (\phi_>(\chi, s), \phi_3(\chi, s))^T$. We have then a set of two algebraic equations with solutions:

$$A = \frac{-1}{2} \frac{1}{s^2 + \Omega^2} \left[\frac{1}{\sqrt{s-i}} (i\kappa\gamma - (s-i)N) + \frac{1}{\sqrt{s+i}} (i\kappa N - (s+i)\gamma) \right] \quad (32)$$

$$B = \frac{-1}{2} \frac{1}{s^2 + \Omega^2} \left[\frac{1}{\sqrt{s+i}} (i\kappa N - (s+i)\gamma) - \frac{1}{\sqrt{s-i}} (i\kappa\gamma - (s-i)N) \right] \quad (33)$$

Substituting Equation (32) into Equation (30) and Equation (33) into Equation (31), we get the solution for Dirac shutter problem in the (χ, s) space. For $\chi \geq 0$, where the particle detector is located, we have the solution:

$$-2 \begin{pmatrix} \phi_1(\chi, s) \\ \phi_3(\chi, s) \end{pmatrix} = \frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \left\{ \begin{pmatrix} i\kappa\gamma \\ i\kappa N \end{pmatrix} - (s \mp i) \begin{pmatrix} N \\ \gamma \end{pmatrix} - \sqrt{s^2+1} \begin{pmatrix} \gamma \\ N \end{pmatrix} + \frac{\sqrt{s \mp i}}{\sqrt{s \pm i}} \begin{pmatrix} i\kappa N \\ i\kappa\gamma \end{pmatrix} \right\} \quad (34)$$

Notice the singular points at $s = \pm i\Omega$ (simple poles) and $s = \pm i$ (branch points), all of them locate in the imaginary axis. Therefore, by the Nyquist stability criterion, the time dependent solution $\psi(\chi, \tau)$ is an oscillatory bounded solution.

To simplify the final notation, we express the inverse Laplace transforms ($s \rightarrow \tau$) with the help of the integral-defined complex functions ($\tau \geq \chi$):

$$G_n(\chi, \tau; \kappa) \equiv \int_{\chi}^{\tau} du e^{i\Omega(\tau-u)} \frac{J_n(\sqrt{u^2 - \chi^2})}{(u^2 - \chi^2)^{n/2}} \equiv C_n + iS_n \quad n = 0, 1, 2, \dots \quad (35)$$

Using Laplace Transforms Tables [15] and the convolution theorem we find the following results, valid for $\chi \geq 0$,

$$\mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \right] = \theta(\tau - \chi) \left\{ \frac{\sin[\Omega(\tau - \chi)]}{\Omega} - \frac{\chi}{\Omega} S_1 \right\} \quad (36)$$

$$\mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} (s+i) \right] = \theta(\tau - \chi) \left\{ \cos[\Omega(\tau - \chi)] + i \frac{\sin[\Omega(\tau - \chi)]}{\Omega} - \chi \left(C_1 + \frac{i}{\Omega} S_1 \right) \right\}. \quad (37)$$

Next, we use the relation

$$-\frac{d}{d\chi} \frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} = \frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \sqrt{s^2+1} \quad (38)$$

to obtain

$$\mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \sqrt{s^2+1} \right] = \theta(\tau - \chi) \left\{ \cos[\Omega(\tau - \chi)] - \frac{\chi}{2} \frac{\sin[\Omega(\tau - \chi)]}{\Omega} + \frac{1}{\Omega} S_1 + \frac{\chi^2}{\Omega} S_2 \right\} \quad (39)$$

Finally, using the identities

$$\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \frac{\sqrt{s-i}}{\sqrt{s+i}} = \frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \frac{\sqrt{s^2+1}}{s+i} \quad (40)$$

and

$$\frac{1}{(s^2 + \Omega^2)(s+i)} = \frac{1}{\Omega^2 - 1} \left(\frac{-s+i}{s^2 + \Omega^2} + \frac{1}{s+i} \right) \quad (41)$$

we obtain

$$\begin{aligned} \square^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \frac{\sqrt{s-i}}{\sqrt{s+i}} \right] &= \frac{\theta(\tau - \chi)}{\Omega^2 - 1} \left\{ - \left(\frac{\chi}{2} + i \right) e^{-i(\tau - \chi)} + G_1^*(\kappa = 0) + \chi^2 G_2^*(\kappa = 0) \right. \\ &+ \Omega \sin[\Omega(\tau - u)] + \frac{\chi}{2} \cos[\Omega(\tau - \chi)] - C_1 - \chi^2 C_2 \\ &\left. + i \left[\cos[\Omega(\tau - \chi)] - \frac{\chi}{2} \frac{\sin[\Omega(\tau - \chi)]}{\Omega} + \frac{1}{\Omega} S_1 + \frac{\chi^2}{\Omega} S_2 \right] \right\} \quad (42) \end{aligned}$$

Therefore, for $\chi \geq 0$, the Dirac final solution is given by:

$$\begin{aligned}
 -2 \begin{pmatrix} \psi_1(\chi, \tau)_> \\ \psi_3(\chi, \tau)_> \end{pmatrix} &= \mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \begin{pmatrix} i\kappa\gamma \\ i\kappa N \end{pmatrix} \right] - \mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} (s \mp i) \begin{pmatrix} N \\ \gamma \end{pmatrix} \right] \\
 &\quad - \mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \sqrt{s^2+1} \begin{pmatrix} \gamma \\ N \end{pmatrix} \right] + \mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s^2 + \Omega^2} \frac{\sqrt{s \mp i}}{\sqrt{s \pm i}} \begin{pmatrix} i\kappa N \\ i\kappa\gamma \end{pmatrix} \right]
 \end{aligned} \tag{43}$$

where each inverse Laplace transform has been previously calculated. We claim that Equation (43) is the exact Dirac wave function for the shutter problem, valid for $\chi \geq 0$ and $\Omega \neq 1$ ($k \neq 0$).

Appendix B

In a similar way to the Dirac solution, the Klein-Gordon shutter problem can be written in terms of dimensionless variables:

$$\frac{\partial^2 \psi}{\partial \chi^2} - \frac{\partial^2 \psi}{\partial \tau^2} = \psi \tag{44}$$

with initial conditions given by:

$$\psi(\chi, 0) = e^{i\kappa\chi} \theta(-\chi), \quad \frac{\partial \psi(\chi, 0)}{\partial \tau} = -i\Omega e^{i\kappa\chi} \theta(-\chi). \tag{45}$$

Taking the Laplace transform ($\tau \rightarrow s$) of Equation (44) we find the differential equations:

$$\frac{d^2 \phi_<}{d\chi^2} - (s^2 + 1)\phi_< = -(s - i\Omega)e^{i\kappa\chi}, \quad (\chi \leq 0) \tag{46}$$

and

$$\frac{d^2 \phi_>}{d\chi^2} - (s^2 + 1)\phi_> = 0. \quad (\chi \geq 0) \tag{47}$$

Here both functions $\phi_<$ and $\phi_>$ must be bounded: ($\phi_<$ at $-\infty$) and ($\phi_>$ at $+\infty$). The two functions and their first derivatives must be continuous at the interface $\chi = 0$.

Taking into account the boundary conditions at $\pm\infty$, the solutions of Equations (46) and (47) are:

$$\phi_<(\chi, s) = Ae^{\chi\sqrt{s^2+1}} + \frac{1}{s + i\Omega} \exp(i\kappa\chi) \quad (\chi \leq 0) \tag{48}$$

$$\phi_>(\chi, s) = Be^{-\chi\sqrt{s^2+1}} \quad (\chi \geq 0) \tag{49}$$

where $\Omega = \sqrt{1 + \kappa^2}$. The constants A and B are fixed from the conditions at the interface: $\phi_<$ and $\phi_>$, and their first derivatives, $d\phi_</d\chi$ and $d\phi_>/d\chi$, must be continuous at $\chi = 0$. We have then a set of two coupled algebraic equations with solutions given by:

$$A = \frac{-1}{2} \frac{1}{s + i\Omega} \left(1 + \frac{iK}{\sqrt{s^2+1}} \right) \tag{50}$$

$$B = \frac{1}{2} \frac{1}{s + i\Omega} \left(1 - \frac{iK}{\sqrt{s^2+1}} \right) \tag{51}$$

Substituting Equation (50) into Equation (48) and Equation (51) into Equation (49) we have the solutions:

$$\phi_<(\chi, s) = \frac{-1}{2} \frac{1}{s + i\Omega} \left(1 + \frac{iK}{\sqrt{s^2+1}} \right) \exp(+\chi\sqrt{s^2+1}) + \frac{1}{s + i\Omega} \exp(i\kappa\chi), \tag{52}$$

$$\phi_>(\chi, s) = \frac{1}{2} \frac{1}{s + i\Omega} \left(1 - \frac{iK}{\sqrt{s^2+1}} \right) \exp(-\chi\sqrt{s^2+1}). \tag{53}$$

Finally, we need to invert the Laplace transforms ($s \rightarrow \tau$). We find in Laplace Transforms Tables [15] the following results valid for $\chi \geq 0$:

$$\mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{(s+i\Omega)\sqrt{s^2+1}} \right] = \theta(\tau-\chi) \int_{|\chi|}^{\tau} du e^{-i\Omega(\tau-u)} J_0(\sqrt{u^2-\chi^2}) \quad (54)$$

$$\mathcal{L}^{-1} \left[\frac{e^{-\chi\sqrt{s^2+1}}}{s+i\Omega} \right] = \theta(\tau-\chi) \left[e^{-i\Omega(\tau-|\chi|)} - \chi \int_{\chi}^{\tau} du e^{-i\Omega(\tau-u)} \frac{J_1(\sqrt{u^2-\chi^2})}{\sqrt{u^2-\chi^2}} \right] \quad (55)$$

We have then the final solutions, for $\chi \geq 0$:

$$\psi_{>}(\chi, \tau) = \frac{1}{2} \theta(\tau-\chi) \left\{ e^{-i\Omega(\tau-\chi)} - \int_{\chi}^{\tau} du e^{-i\Omega(\tau-u)} \left[\chi \frac{J_1(\sqrt{u^2-\chi^2})}{\sqrt{u^2-\chi^2}} + i\kappa J_0(\sqrt{u^2-\chi^2}) \right] \right\} \quad (56)$$

and for $\chi \leq 0$ we get the incident and reflected wave:

$$\psi_{<}(\chi, \tau) = e^{i(\kappa\chi-\Omega\tau)} - \frac{1}{2} \theta(\tau+\chi) \left\{ e^{-i\Omega(\chi+\tau)} + \int_{-\chi}^{\tau} du e^{-i\Omega(\tau-u)} \left[\chi \frac{J_1(\sqrt{u^2-\chi^2})}{\sqrt{u^2-\chi^2}} + i\kappa J_0(\sqrt{u^2-\chi^2}) \right] \right\} \quad (57)$$

We claim that Equations (56) and (57) are the exact Klein-Gordon wave functions for the shutter problem. It's no surprising to find the Bessel functions $J_0(\sqrt{u^2-\chi^2})$ and $J_1(\sqrt{u^2-\chi^2})/\sqrt{u^2-\chi^2}$, they are just the Green's function and its derivative respectively for the Klein-Gordon equation [16].