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On Separation between Metric Observers in Segal's Compact Cosmos

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Abstract

A certain class K of GR homogeneous spacetimes is considered. For each pair E, $\tilde{E}$ of spacetimes from K, $\tilde{E} = g(E)$ where conformal transformation g is from $G = SU(2,2)$. Each E (being $U(2)$ or its double cover, as a manifold) is interpreted as related to an observer in Segal's universal cosmos. The definition of separation d between E and $\tilde{E}$ is based on the integration of the conformal factor of the transformation g. The integration is carried out separately over each region where the conformal factor is no less than 1 (or no greater than 1). Certain properties of $d = d(E, \tilde{E})$ are proven; examples are considered; and possible directions of further research are indicated.

Keywords

Separation between Spacetimes, Segal's Universal Cosmos, Conformal Group Action on *U*(2), DLF-Theory

1. Motivation and Introduction

The first author has been interested in GR ("GR" is for General Relativity) research for quite a while and he concentrated on a few most symmetric spacetimes ([1], [2], and more). Later (see [3], [4]) he has become a strong believer in Segal's Chronometric Theory (see [5], electronic archive arranged by Levichev), and he is attempting to modify Segal's Theory (see [6], a key publication). The collaboration of the two current authors is based on their mutual interest in Penrose-Hameroff approach to consciousness (see its update in [7], [8]). Specifically, we are putting forward an alternative definition of separation between space-times. In [9], the original definition was based on bringing up a Newtonian limit in GR. Our definition has been introduced in

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[10], [11], and we now present it in much more detail.

Recall the Lie group $U(2)$ as the totality of all two-by-two matrices z (with complex entries allowed) satisfying

$$z^* z = \mathbf{1}, \tag{1.1}$$

where z^* is the transpose and complex conjugate of z, and $\mathbf{1}$ is the unit matrix. Now, define the Lie group $\mathrm{G} = SU(2,2)$ as consisting of all four-by-four matrices g (with complex entries allowed) satisfying

$$g^* S g = S, \tag{1.2}$$

where S is the diagonal matrix $\operatorname{diag}\{1,1,-1,-1\}$. Recall the well-known linear-fractional G-action on $U(2)$:

$$g(z) = (Az + B)(Cz + D)^{-1}, \tag{1.3}$$

where a matrix g from G is determined by its 2×2 blocks A, B, C, D.

In Table I of [12], the matrices $\boldsymbol{L}_{ij}$ are chosen as basic vectors of the (fifteen-dimensional) Lie algebra $su(2,2)$, whereas L_{ij} are the corresponding vector fields on $U(2)$. The vector fields L_{ij} are determined by the G-action (1.3). As explained in [12], subscripts i, j take on −1, 0, 1, 2, 3, 4, and the convention $\boldsymbol{L}_{ji} = -\boldsymbol{L}_{ij}$ (resulting in $L_{ji} = -L_{ij}$) holds.

The Lorentzian inner product on $U(2)$ is introduced in such a way that left-invariant vector fields $X_0 = L_{-10}, X_1 = L_{14} - L_{23}, X_2 = L_{24} - L_{31}, X_3 = L_{34} - L_{12}$ form an orthonormal basis (following [12], we use +, −, −, − signature). The resulting product on $U(2)$ is bi-invariant (see [6]), and $\langle \mathbf{a}, \mathbf{b} \rangle$ below denotes the Lorentzian inner product of tangent vectors $\mathbf{a}, \mathbf{b}$ at a point z of $U(2)$. The spacetime thus obtained is denoted by E_0 (the meaning of the subscript will become clear in the next section). Transformations (1.3) are conformal in E_0 (a word of caution: this spacetime has been denoted as D in [6]). As it follows from Table I of [12], the vector fields $L_{-10}, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}$, generate isometries in E_0. The corresponding subgroup K in G consists of all matrices (1.2) with $B = C = 0$.

For what follows, it is instrumental to introduce a certain bi-invariant Riemannian inner product on $U(2)$. To do so, we recall that vector fields X_0, X_1, X_2, X_3 constitute a basis of the Lie algebra $u(2)$. This algebra is a direct sum of its center with $su(2)$. Namely, X_0 generates the center, whereas X_1, X_2, X_3 are basic vectors in $su(2)$. The Riemannian metric is determined by the demand on left-invariant vector fields X_0, X_1, X_2, X_3 to form an orthonormal basis. The corresponding Riemannian space is denoted E_R. Again, E_R is $U(2)$, as a manifold. Our $(\mathbf{a}, \mathbf{b})$ below denotes the Riemannian inner product of tangent vectors $\mathbf{a}, \mathbf{b}$ at a point z of $U(2)$. In the forthcoming sections the corresponding volume form on E_R will be instrumental. From Table I of [12], it follows that the group K acts as a group of both Lorentzian and Riemannian isometries.

Notice that, as a group, $U(2)$ is not a direct product of its center with the subgroup $SU(2)$. The double cover $E^{(2)}$ of $U(2)$ is the direct product $S^1 \times S^3$ (with S^3 represented by $SU(2)$). The covering map sends (e^{it}, u) into the matrix $e^{it}u$ in $U(2)$. The corresponding Lorentzian metric on $E^{(2)}$ is of the form

$$(dt)^2 - (du)^2. \tag{1.4}$$

Here the variable t is along S^1 whereas $(du)^2$ is for the standard Riemannian metric on S^3. More details are given in our Appendix A, where u denotes a matrix from $SU(2)$. Our Appendix B is dedicated to a certain one-parameter group of transformations (1.3).

It is well-known ([12], [13]) that the (above introduced) covering map is a Lorentzian isometry. Infinitesimal G-action on $E^{(2)}$ is presented in Table I of [12]. It is known (see [13]) that action (1.3) can be lifted to a global conformal G-action on $E^{(2)}$. Using the corresponding commutative diagram, one can show that the lifted action of the group K is as follows:

$$(e^{it}, u) \to \left((\det A)e^{it}, (\det A)^{-1} A u D^{-1}\right). \tag{1.5}$$

Also, it is easily verifiable that for the Riemannian metric

$$(dt)^2 + (du)^2 \tag{1.6}$$

on $S^1 \times S^3$ the (above specified) covering map is a Riemannian isometry from $E^{(2)}$ onto E_R.

It makes sense to mention how a suitable version of the Einstein static universe, E_{uc}, can be introduced in the context of our work (the subscript *uc* is for universal cover). To be more precise, E_{uc} should be called *universal cosmos* ([14]) or *Segal's universal cosmos* ([13]). The universal cover E_{uc} of E_0 is $R^1 \times S^3$, topologically. The *G*-action on E_0 is canonically lifted to the G_{uc}-action on E_{uc} (the latter action preserves the causal structure of E_{uc}). In a cosmological model based on E_{uc}, there is a conformal invariant *R*, interpreted as the radius of a three-dimensional (physical) space S^3. I. Segal (in [14] and in other publications) has put this *R* for the (long wanted by Dirac and others) third fundamental constant additionally to the speed of light and to the Planck's constant. It is known that to model particles on E_{uc}, one can start with the world E_0, a compact one. The respective property is called automatic periodicity ([15], p. 202), and it allows us to only deal with compact spacetimes E_0 and $E^{(2)}$ (which explains our *compact Segal's cosmos* terminology).

More precisely, we deal with two classes of spacetimes: $\mathbf{K}_1$ and $\mathbf{K}_2$. Roughly speaking, the first class is obtained by application of all transformations (3) to E_0; details follow in the next section (where $\mathbf{K}_2$ will also be defined).

2. On the Notion of Separation between Spacetimes: The Main Definition and Related Properties

The separation (or *distance*) $d(x, y)$ will be defined for any pair *x*, *y* of spacetimes from $\mathbf{K}_1$ (or from $\mathbf{K}_2$).

As mentioned, the totality of all isometries in E_0 is the group K of all matrices (1.2) with $B = C = 0$. Each member *E* of the class $\mathbf{K}_1$ will be now put in correspondence with an element *x* of the homogeneous space G/K. Namely, each element (or *coset*) *x* of G/K is specified by an element *g* from G: $x = g\mathrm{K} = \{gk : k \in \mathrm{K}\}$. One and the same *x* can be determined by another element (say, g_1) from G: $g\mathrm{K} = g_1\mathrm{K}$. For such a pair (g, g_1), there exists such *k* from K, that $g_1 = gk$. When the subgroup K is viewed as an element of G/K, denote K as x_0. This x_0 we put into correspondence with E_0 (which has been described in Section 1). As a manifold, each element *x* of $\mathbf{K}_1$ is $U(2)$. In what follows, we use $\langle \mathbf{a}, \mathbf{b} \rangle$ (rather than $\langle \mathbf{a}, \mathbf{b} \rangle_0$) to denote the (Lorentzian) inner product of vectors $\mathbf{a}, \mathbf{b}$ from the tangent space $T(E_0)$ at *z*. This inner product has been introduced in our Section 1. To define spacetime *E* corresponding to a coset $x = g\mathrm{K}$, it is enough to specify the inner product $\langle .,. \rangle_E$, see (2.2) below. Such a transformation *g* is conformal in E_0. Namely, given vectors $\mathbf{a}, \mathbf{b}$ from the tangent space $T(E_0)$ at *z*, the inner product $\langle g_* \mathbf{a}, g_* \mathbf{b} \rangle$ at $g(z)$ of their images (under the tangent map g_*) satisfies

$$\langle g_* \mathbf{a}, g_* \mathbf{b} \rangle = h(z) \langle \mathbf{a}, \mathbf{b} \rangle. \tag{2.1}$$

The everywhere positive function $h(z)$ is known as the *square of conformal coefficient*. Frequently, we will simply refer to this $h(z)$ as to a *conformal coefficient*. Given vectors $\mathbf{a}, \mathbf{b}$ from the tangent space $T(E)$ at *z*, their inner product *can be defined* as follows:

$$\langle \mathbf{a}, \mathbf{b} \rangle_E = \left[h\left(g^{-1}(z)\right) \right]^{-1} \langle \mathbf{a}, \mathbf{b} \rangle, \tag{2.2}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ is calculated at *z*. Notice, that *h* here is the (above mentioned) function on $U(2)$ determined by *g* It is easy to show that (2.2) is equivalent to the condition for *g* to be an isometry between $(E_0, \langle .,. \rangle)$ and $(E, \langle .,. \rangle_E)$:

$$\langle g_* \mathbf{a}, g_* \mathbf{b} \rangle_E = \langle \mathbf{a}, \mathbf{b} \rangle. \tag{2.3L}$$

Here the right hand side of (2.3L) is calculated in E_0 at *z*, and it defines the inner product $\langle g_* \mathbf{a}, g_* \mathbf{b} \rangle_E$ of vectors $g_* \mathbf{a}$ and $g_* \mathbf{b}$ at $g(z)$. To avoid verification of (2.2)-(2.3L) equivalence, we define the Lorentzian metric in *E* in terms of (2.3L). Similarly, we define the following Riemannian metric in *E*:

$$(g_* \mathbf{a}, g_* \mathbf{b})_E = (\mathbf{a}, \mathbf{b}). \tag{2.3R}$$

where the positive definite inner product in the right hand side of (2.3R) has been introduced in our Section 1.

Let us show that, given a coset *x* in G/K, (2.3L) correctly defines a Lorentzian metric on $U(2)$, whereas (2.3R) correctly defines a Riemannian metric on $U(2)$:

Scholium 2.1. The inner product (2.3L) (respectively, the inner product (2.3R)) is independent of the choice of g which represents a coset x.

Proof. If x is represented as g_1K, then $g_1 = gk$ where k is a certain element of the group K. Given such a representation, the analogue of (2.3R) is

$$\{g_{1*}\mathbf{a}, g_{1*}\mathbf{b}\}_E = \langle \mathbf{a}, \mathbf{b} \rangle \tag{2.4}$$

where the right hand side of (2.4) is calculated in E_0 at z, and it defines the inner product $\{g_{1*}\mathbf{a}, g_{1*}\mathbf{b}\}_E$ of vectors $g_{1*}\mathbf{a}$ and $g_{1*}\mathbf{b}$ at $g_1(z)$. We have to show that (2.4) introduces the same metric structure on $U(2)$ as (2.3L) does. To do so, we rewrite (2.3L) in the form of

$$\{(gk)_* \mathbf{a}, (gk)_* \mathbf{b}\}_E = \langle k_*\mathbf{a}, k_*\mathbf{b} \rangle \tag{2.5}$$

where the right hand side is calculated at $k(z)$, and the left hand side is calculated at $g(k(z))$.

However, $\langle k_*\mathbf{a}, k_*\mathbf{b} \rangle$ in (2.5) equals $\langle \mathbf{a}, \mathbf{b} \rangle$ in (2.4) since k is a Lorentzian isometry in E_0. Comparison of (2.4) with (2.5) finishes the proof. The verification process, that (2.3R) is independent of representative, copies the one for (2.3L). □

Let us notice (see [16]) that each $(E, \langle .,. \rangle_E)$ can be interpreted as a spacetime corresponding to a certain (*global*) observer. *A word of caution*: [16] treats the universal cover of E_0 whereas we only deal with compact spacetimes here.

Remark 2.2. We have thus defined the class $\mathbf{K}_1$ of spacetimes. Our class $\mathbf{K}_2$ can be similarly introduced in terms of the Lorentzian manifold $E^{(2)}$ (the 2-cover of E_0) with the (lifted) G-action on $E^{(2)}$.

Given a (1.3)-transformation g of (Lorentzian) E_0, define the following subsets of E_0:

$$T_g^+ = \{z : h_g(z) \geq 1\}, \tag{2.6}$$

$$T_g^- = \{z : h_g(z) \leq 1\}; \tag{2.7}$$

here $h_g(z)$ is the square of the conformal coefficient at z of the transformation g. A (non-negative) number d_g is defined as follows:

$$d_g = \ln\left\{\left[\int h_g^2\right] \Big/ \left[V\left(T_g^+\right)\right]\right\} + \ln\left\{\left[\int h_g^{-2}\right] \Big/ \left[V\left(T_g^-\right)\right]\right\}, \tag{2.8}$$

where $V(S)$ is for the volume of a set S in E_R (with the volume form introduced in our Section 1). Clearly, expressions inside the logarithms in (2.8) can be interpreted as corresponding cumulative distortions of the original metric structure in E_0. To be sure of convergence of all of the integrals involved, it is enough to mention that each of the two integrands is a continuous function over the corresponding region of integration, whereas each of the regions (2.6), (2.7) is a compact set.

To further deal with (2.8), we now proceed with more technicalities. Clearly, d_g can be viewed as $\ln\{(ac)/(bd)\}$ with $b = V(T_g^+)$, $d = V(T_g^-)$. The integration in a is over T_g^+, whereas in c the integration is over T_g^-.

Examples of integrals a, b, c, d evaluations are given in our Appendix C (see Theorem C.5). Notice that

$$d_g = 0 \text{ if and only if } g \text{ is an isometry of } E_0, \tag{2.9}$$

which follows from (2.8) because in this case $h_g(z) = 1$, a constant function on $U(2)$. As a result of $h_g(z) = 1$, each of the two terms in the sum (2.8) is zero.

Scholium 2.3. Given the (1.3)-transformation g and isometries k_1, k_2, the following holds:

$$d_w = d_g, \tag{2.10}$$

where $w = k_1^{-1} g k_2$.

Proof. To prove (2.10), we will now show that each of the four numbers (a, b, c, and d) remain the same when we switch from d_g to d_w. Namely:

$$T_w^+ = \{z : h_w(z) \geq 1\} = k_2^{-1}\left(\{k_2(z) : h_m(k_2(z)) \geq 1\}\right)$$
$$= k_2^{-1}\left(\{k_2(z) : h_g(k_2(z)) \geq 1\}\right), \text{ where } m = k_1^{-1}g; \text{ hence } T_w^+ = k_2^{-1}\left(T_g^+\right).$$

Similarly, $T_w^- = k_2^{-1}\left(T_g^-\right)$. Hence, $b = V\left(T_g^+\right) = V\left(T_w^+\right)$, and $d = V\left(T_g^-\right) = V\left(T_w^-\right)$, due to the K-invariance of the volume form.

Let us now use the variable $k_2(z)$ in the integral *a* of $(h_w)^2$ over T_w^+: the integrand is then $\left(h_m(k_2(z))\right)^2 = \left(h_g(k_2(z))\right)^2$, the region of integration is $k_2\left(T_w^+\right) = T_g^+$, and there is no extra factor in the integrand since k_2 is a transformation from the group K. Similarly, number *c* remains the same when we switch from d_g to d_w. □

Now, if two cosets are represented as $x = g_1\mathrm{K}$, $y = g_2\mathrm{K}$, define the distance $d(x, y)$ as

$$d(x, y) = d_g, \tag{2.11}$$

where $g = (g_1)^{-1} g_2$. The number $d(x, y)$ is independent of representatives since if *x* is represented by g_1k_1, and *y* is represented by g_2k_2, then for $w = (g_1k_1)^{-1} g_2k_2 = k_1^{-1}gk_2$, $d_w = d_g$ according to (2.10).

Corollary 2.4. In the above settings, $d(x, y) = d(x_0, q)$ where $x_0 = \mathrm{K}$ and $q = g_1^{-1}g_2\mathrm{K}$.

A word of caution: we use the term *distance* but we are not sure that the corresponding triangle inequality holds (even locally) for (2.11). However, we prove (below) that (2.11) is symmetric: $d(x, y) = d(y, x)$, and *G*-invariant:

$$d\left(f(x), f(y)\right) = d(x, y), \tag{2.12}$$

for arbitrary *f* from *G* (where we have in mind the canonical action of *G* in *G*/K).

As regards *G*-invariance, one can think of a possible relation of our definition (2.11) to the canonical inner product in the symmetric space *G*/K. This we do not discuss here.

Scholium 2.5. The distance (2.11) is symmetric: $d(x, y) = d(y, x)$.

Proof. As justified by our Corollary 2.4, assume that $x = \mathrm{K}$ and $y = g\mathrm{K}$. Define

$$T_m^+ = \{\tilde{z} : h_m(\tilde{z}) \geq 1\}, T_m^- = \{\tilde{z} : h_m(\tilde{z}) \leq 1\},$$

where $m = g^{-1}$, and where we use $\tilde{z}$ (rather than *z*, as before) to denote a matrix in $U(2)$. Tilde (below) indicates that computations are performed in *E* rather than in E_0.

For $d_m = \ln\left\{\left[\int (h_m)^2\right] / \left[\tilde{V}\left(T_m^+\right)\right]\right\} + \ln\left\{\left[\int (h_m)^{-2}\right] / \left[\tilde{V}\left(T_m^-\right)\right]\right\}$, the following is true: $\tilde{b} = \tilde{V}\left(T_m^+\right) = \tilde{V}\left(g\left(T_g^-\right)\right) = V\left(T_g^-\right) = d$ since, due to (2.3R), *g* is an isometry between the two Riemannian spaces. Similarly, $\tilde{d} = \tilde{V}\left(T_m^-\right) = \tilde{V}\left(g\left(T_g^+\right)\right) = V\left(T_g^+\right) = b$. The new integrand is then $\left(h_g(z)\right)^{-2}$, the new region of integration is T_g^-, and there is no extra factor in the integrand since *g* is an isometry between the two Riemannian spaces in question. We have thus proven that $\tilde{a} = c$. Similarly, $\tilde{c} = a$. We have thus proven the equality $\left(\tilde{a}/\tilde{b}\right)\left(\tilde{c}/\tilde{d}\right) = (a/b)(c/d)$, which results in $d(x, y) = d(y, x)$, the symmetry property of the distance between spacetimes. □

3. Concluding Remarks and Future Research Insights

Examples of integrals a, b, c, d evaluations (in case of a certain one-parameter group of conformal transformations) are given in our Appendix C. It is of interest to know whether Theorem C.5 holds for other transformations from $G = SU(2,2)$. Evaluations in Appendix C indicate that definition (2.11) of distance between spacetimes seems to be quite a working one. As part of future research, it will be of interest to apply our definition in the case where the original spacetime is *F* (here we refer to the DLF-theory, [6]). In that case, the underlying manifold is (non-compact!) $U(1,1)$, rather than $U(2)$. Preliminary calculations indicate that a conformal coefficient might be unbounded. We will thus have to deal with improper 4D integrals, and the question of convergence will have to be studied first.

References

[1] Guts, A.K. and Levichev, A.V. (1984) On the Foundations of Relativity Theory. *Doklady Akademii Nauk SSSR*, **277**, 253-257. (In Russian)

[2] Levichev, A.V. (1989) On the Causal Structure of Homogeneous Lorentzian Manifolds. *General Relativity & Gravity*, **21**, 1027-1045. http://dx.doi.org/10.1007/BF00774087

[3] Levichev, A.V. (1993) The Chronometric Theory by I. Segal Is the Crowning Accomplishment of Special Relativity. *Izvestiya Vysshikh Uchebnykh Zavedenii Fizika*, **8**, 84-89. (In Russian)

[4] Levichev, A.V. (1995) Mathematical Foundations and Physical Applications of Chronometry. In: Hilgert, J., Hofmann, K. and Lawson, J., Eds., *Semigroups in Algebra, Geometry, and Analysis*, de Gryuter Expositions in Mathematics, Berlin, 77-103. http://dx.doi.org/10.1515/9783110885583.77

[5] http://dedekind.mit.edu/segal-archive/index.php

[6] Levichev, A.V. (2011) Pseudo-Hermitian Realization of the Minkowski World through DLF Theory. *Physica Scripta*, **83**, 1-9. http://dx.doi.org/10.1088/0031-8949/83/01/015101

[7] Hameroff, S. and Penrose, R. (2014) Consciousness in the Universe: A Review of the "Orch OR" Theory. *Physics of Life Reviews*, **11**, 39-78. http://dx.doi.org/10.1016/j.plrev.2013.08.002

[8] Hameroff, S. and Penrose, R. (2014) Reply to Criticism of the "Orch OR Qubit"—"Orchestrated Objective Reduction" Is Scientifically Justified. *Physics of Life Reviews*, **11**, 94-100. http://dx.doi.org/10.1016/j.plrev.2013.11.013

[9] Penrose, R. (1992) Gravity and Quantum Mechanics. In: Gleiser, R.J., Kozameh, C.N. and Moreschi, O.M., Eds., *General Relativity and Gravitation* 13. *Part* 1: *Plenary Lectures* 1992. *Proceedings of the* 13*th International Conference on General Relativity and Gravitation*, Cordoba, 28 June-4 July 1992, 179-189.

[10] Levichev, A. and Palyanov, A. (2014) On a Modification of the Theoretical Basis of the Penrose-Hameroff Model of Consciousness. In: *International Conference MM-HPC-BBB*-2014, Abstracts, Sobolev Institute of Mathematics SB RAS, Institute of Cytology and Genetics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, 49.

[11] Levichev, V. and Palyanov, A.Yu. (2014) On a Notion of Separation between Space-Times. In: *Geometry Days in Novosibirsk. Abstracts of the International Conference*, Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, 110.

[12] Segal, I.E., Hans, P.J., Bent, Ø., Paneitz, S.M. and Speh, B. (1981) Covariant Chronogeometry and Extreme Distances: Elementary Particles. *PNAS*, **78**, 5261-5265. http://dx.doi.org/10.1073/pnas.78.9.5261

[13] Werth, J.-E. (1986) Conformal Group Actions and Segal's Cosmology. *Reports on Mathematical Physics*, **23**, 257-268. http://dx.doi.org/10.1016/0034-4877(86)90023-6

[14] Segal, I.E. (1976) Mathematical Cosmology and Extragalactic Astronomy. Academic Press, New York.

[15] Branson, T.P. (1987) Group Representations Arising from Lorentz Conformal Geometry. *Journal of Functional Analysis*, **74**, 199-291. http://dx.doi.org/10.1016/0022-1236(87)90025-5

[16] Segal, I.E. (1984) Evolution of the Inertial Frame of the Universe. *Nuovo Cimento*, **79B**, 187-191. http://dx.doi.org/10.1007/BF02748970

[17] Kon, M. and Levichev, A. (2015) Towards Analysis in Space-Time Bundles Based on Pseudo-Hermitian Realization of the Minkowski Space. In Preparation.

Appendix A: Parameterizations of $U(2)$ and $E^{(2)}$

The following presentation for $E^{(2)}$, the 2-cover of $U(2)$, has been widely used in the literature. Consider the direct sum $E^6 = E^2 \oplus E^4$ of two Euclidean spaces: E^2 with rectangular coordinates u_{-1}, u_0, and E^4 with rectangular coordinates u_1, u_2, u_3, u_4. Each "event" in $E^{(2)}$ is a 6-tuple $(u_{-1}, u_0, u_1, u_2, u_3, u_4)$, satisfying

$$u_{-1}^2 + u_0^2 = 1, \tag{A1}$$

and

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1. \tag{A2}$$

Clearly, $E^{(2)}$ is $S^1 \times S^3$, topologically. The earlier introduced e^{it} (see Section 2) is $u_{-1} + iu_0$, whereas the matrix u from $SU(2)$ is specified as follows:

$$u = \begin{bmatrix} u_4 + iu_3 & u_2 + iu_1 \\ iu_1 - u_2 & u_4 - iu_3 \end{bmatrix}. \tag{A3}$$

The covering map from $\mathrm{E}^{(2)}$ onto $U(2)$ takes the pair $(u_{-1} + iu_0, u)$ into the matrix $(u_{-1} + iu_0)u$, an element z of the group $U(2)$:

$$z = (u_{-1} + iu_0)u. \tag{A4}$$

Given a matrix z in $U(2)$, the factors $(u_{-1} + iu_0)$ and u are defined up to a sign, only. In terms of E^6, it is helpful to consider a pseudo-Euclidean metric

$$(du_{-1})^2 + (du_0)^2 - (du_1)^2 - (du_2)^2 - (du_3)^2 - (du_4)^2, \tag{A5L}$$

and an Euclidean metric

$$(du_{-1})^2 + (du_0)^2 + (du_1)^2 + (du_2)^2 + (du_3)^2 + (du_4)^2. \tag{A5R}$$

It is known (see [14], p. 40) that the restriction of (A5L) onto $E^{(2)} = S^1 \times S^3$ coincides with metric (1.4) of our Section 1. Similarly, the restriction of (A5R) onto $E^{(2)}$ coincides with metric (1.5).

Appendix B: The Case of a Certain One-Parameter Group of Conformal Transformations

This group consists of all (1.3)-transformations g of the form:

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{B.1}$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ s & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix},$$

where $c = \cosh\tau$, $s = \sinh\tau$, τ being a real parameter. This subgroup is contained in a (two-dimensional) subgroup A (from the Iwasawa decomposition $SU(2,2)$ = KAN). The $f(z)$ below is for the (positive) square root of the conformal factor $h(z)$. The latter has been defined by our (2.1). To simplify notation, we, sometimes, use the same symbol (like z or $\tilde{z}$, below) to denote both an element of $\mathrm{E}^{(2)}$ and a matrix in E_0. The statement and the proof of the following theorem presume usage of rectangular coordinates in Euclidean E^6: see Appendix A.

Theorem B.1. The image $\tilde{z}$ of z in $E^{(2)}$ and the conformal factor $h(z)$ at z (under the lift of the (B.1)—transformation g) are as follows:

$$\left.\begin{aligned} \tilde{u}_{-1} + i\tilde{u}_0 &= \left[cu_{-1} - su + i(cu_0 - su_1)\right] f(z), \\ \tilde{u}_2 + i\tilde{u}_1 &= \left[cu_2 - su_{-1} + i(cu_1 - su_0)\right] f(z), \\ \tilde{u}_4 + i\tilde{u}_3 &= (u_4 + iu_3) f(z); \end{aligned}\right\} \tag{B.2}$$

$$h(z) = (f(z))^2 = \left[(cu_{-1} - su_2)^2 + (cu_0 - su_1)^2\right]^{-1}. \tag{B.3}$$

Proof. Notice that due to (A3) and (A4) from Appendix A, the formulas (B.2) correctly define the transformation on the level of E_0 (when z and $\tilde{z}$ are matrices). To prove this first part, we use (B.1) in a straightforward way and (omitting routine details of the calculation) determine (B.2). At this stage of the proof we cannot be sure that $h(z)$ is the conformal coefficient. To prove that it is, apply the differential operator d to both sides of (B.2) in order to express

$$(d\tilde{u}_{-1})^2 + (d\tilde{u}_0)^2 + (d\tilde{u}_1)^2 - (d\tilde{u}_2)^2 - (d\tilde{u}_3)^2 - (d\tilde{u}_4)^2 \tag{B.4}$$

in terms of differentials du_{-1}, du_0, du_1, du_2, du_3, du_4. Comparison of the obtained expression with (A5L) verifies (B.3). □

Remark B.2. In the case considered, there is an alternative way to determine the conformal factor (B.3). It is as follows [17], Theorem 3: for a (1.3)-transformation g, the following equality holds for the conformal factor at z:

$$h(z) = det\left(\left[A - (Az + B)(Cz + D)^{-1} C\right] z (Az + B)^{-1}\right). \tag{B.5}$$

One can verify that (B.5), when applied in the (B.1)-case, results in (B.3).

It is of interest to determine all *fixed points* (that is, matrices $z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$ with $g(z) = z$ property) of the transformation (B.1).

Scholium B.3. The totality of all fixed points of (B.1) is a pair of circles. One of the circles is given by equations $z_3 = 1$, $z_1 = z_4 = 0$. The other circle is given by equations $z_3 = -1$, $z_1 = z_4 = 0$.

Proof. As it follows from (B.1), the totality of all fixed points is the solution set of

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} c + sz_3 & sz_4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 \\ cz_3 + s & z_4 \end{bmatrix}, \tag{B.6}$$

equality of two matrices. Comparison of first entries in second rows results in

$$s(z_3)^2 = s. \tag{B.7}$$

Since g is not an identity transformation, $(z_3)^2 = 1$. If $z_3 = 1$, then comparison of first entries in the first rows results in $(c + s) z_1 = z_1$, that is, $z_1 = 0$. Comparison of second entries in second rows results in $z_4 = 0$. Now, if $z_3 = -1$ in (B.7), then, again, $z_1 = z_4 = 0$. □

Our next goal is to prove that each fixed point (of a given (B.1)—transfor-mation) is an extreme point of the conformal coefficient $h(z)$: maximum is reached at each point of one circle whereas minimum is reached at each point of the other circle.

Scholium B.4. If $(u_1)^2 + (u_2)^2 < 1$ at z_0, then z_0 is not a point of extremum for $h(z)$.

Proof. If $(u_1)^2 + (u_2)^2 < 1$ at z_0, then u_1, u_2 can be chosen as two (of the total of four) free real variables at the vicinity of z_0. If z_0 is a point of extremum for $h(z)$, then each of the two partial derivatives of h (w.r.t. u_1, and w.r.t. u_2) vanish at z_0. However, that would have resulted in vanishing of $h(z)$ at z_0. □

Corollary B.5. If $z = (u_{-1}, u_0, u_1, u_2, u_3, u_4)$ is an extreme point for h, then $u_3 = u_4 = 0$ (that is, $z_1 = z_4 = 0$).

Corollary B.6. At the point of extremum for the conformal coefficient, either $u_{-1}u_2 + u_0 u_1 = 1$, or $u_{-1}u_2 + u_0 u_1 = -1$ (that is, $z_3 = -1$, or $z_3 = 1$).

Proof follows from the expression $h(z) = \left[c^2 + s^2 - 2sc(u_{-1}u_2 + u_0 u_1)\right]^{-1}$ which holds at every point z where $u_3 = u_4 = 0$. □

Corollary B.7. An extreme value of $h(z)$ is reached at z_0 if and only if z_0 is a fixed point of (B.1). One can verify that the two extreme values are e^{τ} and $e^{-\tau}$ where τ is the (non-zero) value of the parameter in (B.1).

Appendix C: Evaluations of Integrals (2.8) for the Case of Appendix B Transformations in $E^{(2)}$

We start with the form

$$h=\left(cu_{-1}-su_2\right)^2+\left(cu_0-su_1\right)^2 \tag{C.1}$$

on the torus $T = S^1 \times S^3$, see our Theorem B.1. Now, T^+ is for the part of T where $h\geq 1$, T^- is for the part of T where $h\leq 1$. Introduce

$$I_k^+=\int_{T^+}h^k, \tag{C.2}$$

$$I_k^-=\int_{T^-}h^k; \tag{C.3}$$

where in both cases we have in mind the volume form which has been introduced on T in Section 2.

A word of caution: the function (C.1) is the inverse of the conformal coefficient (B.3). Nevertheless, the findings (which follow) of this Appendix C are relevant to the Appendix A content since k in (C.2), (C.3) can be any integer.

The majority of these Appendix C findings are due to V. V. Ivanov (Sobolev Institute of Mathematics, Novosibirsk, Russia).

Parameterize T as follows:

$$\left.\begin{aligned}&u_{-1}=\cos\left(\phi-\psi\right),u_0=\sin\left(\phi-\psi\right),u_1=\varrho\cos\left(\psi\right),u_2=\varrho\sin\left(\psi\right),\\&u_3=\sqrt{1-\varrho^2}\cos\left(\xi\right),\quad u_4=\sqrt{1-\varrho^2}\cos\left(\xi\right).\end{aligned}\right\} \tag{C.4}$$

In terms of these parameters, (C.1) becomes

$$h=h\left(\varrho,\phi\right)=c^2-2cs\varrho\sin\left(\phi\right)+s^2\varrho^2. \tag{C.5}$$

The integrals (C.2), (C.3) are reduced as follows:

$$I_k^{\pm}=4\pi^2J_k^{\pm}, \tag{C.6}$$

where

$$J_k^{\pm}=\iint\limits_{\Omega^{\pm}}\varrho h^k\left(\varrho,\phi\right)\mathrm{d}\varrho\mathrm{d}\phi. \tag{C.7}$$

Here we consider the rectangle $0\leq\rho\leq 1, 0\leq\phi\leq 2\pi$, and $\Omega^+=\left\{\left(\rho,\phi\right):h\geq 1\right\}$, $\Omega^-=\left\{\left(\rho,\phi\right):h\leq 1\right\}$. Notice that the integrals (C.6) and (C.7) are independent of the sign of $s=\sinh\left(\tau\right)$ which allows us to stay with $s>0$, only. The next step is to interpret ρ, ϕ as polar coordinates on the x, y plane:

$$x=\rho\cos\left(\phi\right), y=\rho\sin\left(\phi\right). \tag{C.8}$$

Our function (C.5) becomes

$$h\left(x,y\right)=\left(sx\right)^2+\left(c-sy\right)^2, \tag{C.9}$$

whereas Ω^+, Ω^- are to be converted into D^+, D^- with their union being the unit disc D centered at the origin (0,0) of the x, y plane. Finally, introduce coordinates r, α:

$$x=r\sin\left(\alpha\right), y=c/s-r\cos\left(\alpha\right). \tag{C.10}$$

r being the distance between $P=\left(0,c/s\right)$ and $Q=\left(x,y\right)$, whereas the angle α, in radians, is an angle between vectors $\left\{0,-1\right\}$ and PQ. Expression (C.9) becomes

$$h\left(r,\alpha\right)=s^2r^2. \tag{C.11}$$

Introduce an (acute) angle ω which is determined by any of the relations

$$\omega\cos\left(\omega\right)=1/c,\quad\sin\left(\omega\right)=s/c,\quad\tan\left(\omega\right)=s. \tag{C.12}$$

Omitting a few more (straightforward) technicalities, we obtain

$$J_k^{\pm} = \left(\pm 2/s^2\right)\int_0^{\omega}\int_1^{H_{\pm}(\alpha)} r^{2k+1}\mathrm{d}r\mathrm{d}\alpha. \quad \text{(C.13,C14)}$$

The upper limits $H_+(\alpha)$, $H_-(\alpha)$ are as follows:

$$H_{\pm}(\alpha) = c\cos(\alpha) \pm \sqrt{c^2\cos^2\alpha - 1}. \quad \text{(C.15,C.16)}$$

Let us conclude in terms of the following statements.

Theorem C.1. For k not equal −1, the integrals (C.2), (C.3) can be evaluated as follows:

$$I_k^{\pm} = \frac{\pm 4\pi^2}{(k+1)s^2}\int_0^{\omega}\left(H^{2k+2} - 1\right)\mathrm{d}\alpha. \quad \text{(C.17)}$$

For $k = -1$

$$I_{-1}^{\pm} = \frac{\pm 8\pi^2}{s^2}\int_0^{\omega}\ln\left(H_{\pm}(\alpha)\right)\mathrm{d}\alpha. \quad \text{(C.18)}$$

Theorem C.2. For every integer k,

$$I_{-k-2}^{+} = I_k^{-}, \quad I_{-k-2}^{-} = I_k^{+}. \quad \text{(C.19)}$$

Theorem C.3. For a nonnegative k, each of the integrals (C.17) is a finite linear combination of integrals A_m, B_m where

$$\left.\begin{aligned} A_m &= \int_0^{\omega} c^{2m}\cos^{2m}(\alpha)\mathrm{d}\alpha, \\ B_m &= \int_0^{\omega} c^{2m-1}\cos^{2m-1}(\alpha)\sqrt{c^2\cos^2(\alpha) - 1}\mathrm{d}\alpha. \end{aligned}\right\} \quad \text{(C.20)}$$

Remark C.4. Each of the integrals (C.20) is an elementary one and it can be expressed as a polynomial in s and ω.

Recall notations a, b, c, d of Section 2 (see the line prior to Formula (2.9)) for the integrals which are of our utmost interest.

Theorem C.5. The integrals a, b, c, d, are as follows:

$$a = \left\{4\pi^2/\left[3\left(s^2\right)\right]\right\}\left\{32A_3 - 48A_2 + 18A_1 - 2A_0 + 32B_3 - 32B_2 + 6B_1\right\},$$

$$b = 4\pi^2\left[\pi/2 + \omega + (s - \omega)/\left(s^2\right)\right],$$

$$c = \left\{4\pi^2/\left[3\left(s^2\right)\right]\right\}\left\{32A_3 - 48A_2 + 18A_1 - 2A_0 - 32B_3 + 32B_2 - 6B_1\right\},$$

$$d = 4\pi^2\left[\pi/2 - \omega - (s - \omega)/\left(s^2\right)\right].$$