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Localisation Inverse Problem and Dirichlet-to-Neumann Operator for Absorbing Laplacian Transport

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ABSTRACT

We study *Laplacian* transport by the *Dirichlet-to-Neumann* formalism in *isotropic* media $(\gamma = I)$. Our main results concern the solution of the *localisation inverse* problem of absorbing domains and its relative *Dirichlet-to-Neumann* operator $\Lambda_{\gamma=I,\partial\Omega}$. In this paper, we define explicitly operator $\Lambda_{\gamma=I,\partial\Omega}$, and we show that Green-Ostrogradski theorem is adopted to this type of problem in three dimensional case.

Keywords: Absorbing Laplacian Transport; Dirichlet-to-Neumann Operators; Inverse Problem

1. Laplacian Transport and Dirichlet-to-Neumann Operators

The theory of *Dirichlet-to-Neumann* operators is the basis of many research domains in analysis, particularly, those concerning *Laplacian* transports. It is also very important in mathematical-physics, geophysics, electrochemistry. Moreover, it is very useful in medical diagnosis, such as electrical impedance tomography:

In 1989, J. Lee and G. Uhlmann have introduced an example on the determination of conductivity matrix field in a bounded open domain, see e.g. [1]. This example is related to *measuring* the elliptic *Dirichlet-to-Neumann* map for associated conductivity equation, see e.g. [1].

The problem of electrical current flux is an example of so-called *diffusive Laplacian* transport. Besides the voltage-to-current problem, the motivation to study this kind of transport comes for instance, from the transfer across biological membranes, see e.g. [2,3].

Let some species of concentration $v(p)$, $x \in \mathbb{R}^d$, diffuse stationary in the *isotropic* bulk $(\gamma = I)$ from a (distant) source localised on the closed boundary $\partial\Omega$ towards a semipermeable compact interface ∂B of the *cell* $\overline{B} \subset \Omega$, where they disappear at a given rate $W \geq 0$. Then the steady field of concentrations (*Laplacian* transport with a diffusion coefficient $D \geq 0$) obeys the set of equations:

$$
\textbf{(P1)} \quad \begin{cases} \Delta v = 0,\ p \in \Omega \setminus \overline{B}, \\ v|_{\partial\Omega}(p) = f(p), \text{ the concentration at the source } \partial\Omega, \\ -D\,\partial_\nu v|_{\partial B}(\omega) = W\left(v - c^*\right)\Big|_{\partial B}(\omega), \text{ on the interface } \omega \in \partial B. \end{cases}
$$

Usually, one supposes that $v(p) = c^* \geq 0$, $p \in B$, is a constant concentration of the species inside the *cell* $\overline{B}$. This example motivates the following abstract stationary *diffusive Laplacian* transport problem with *absorption* on the surface ∂B:

$$
\textbf{(P2)} \quad \begin{cases} \Delta v = 0,\ p \in \Omega \setminus \overline{B}, \left[v(p) = \text{Const}, p \in \overline{B}\right], \\ v|_{\partial\Omega}(p) = f(p),\ p \in \partial\Omega, \\ \left(\alpha v + \partial_\nu v\right)|_{\partial B}(\omega) = h(\omega),\ \omega \in \partial B. \end{cases}
$$

This is the *Dirichlet-Neumann* problem for domain $\Omega \supset \overline{B}$ with the Robin [4] boundary condition on the absorbing surface ∂B. Varying $\alpha := WD^{-1}$ between $\alpha = 0$ and $\alpha = +\infty$, one recovers respectively the *Neumann* and the *Dirichlet* boundary conditions.

Now, we can associate with the problem **(P2)** a *Dirihlet-to-Neumann* operator

$$
\Lambda_{\gamma=I,\partial\Omega} : f \mapsto g := \partial_\nu v_f \Big|_{\partial\Omega}. \tag{1}
$$

Domain $\text{dom}\left(\Lambda_{I,\partial\Omega}\right)$ belongs to a certain *Sobolev* space of functions on the boundary $\partial\Omega$, which contains $v_f := v_f^{(\alpha,g)}$, the solutions of the problem **(P2)** for given f and for the Robin boundary condition on ∂B fixed by α and g.

The advantage of this approach is that as soon as the operator (1) is defined, one can apply it to study the mixed boundary value problem **(P2)**. This gives, in particularly, the value of the *particle* flux due to *Laplacian* transport across the membrane $\partial\Omega$. Moreover, the *total* current across the boundary $\partial\Omega$ can be defined (for given f) in term of *Dirihlet-to-Neumann* operator (1) as follows:

$$J_{\partial\Omega} := -D\int_{\partial\Omega} \mathrm{d}\sigma \Lambda_{\gamma=I,\partial\Omega} f, \tag{2}$$

where $\mathrm{d}\sigma$ designed the differential element relative to $\partial\Omega$.

There are at least two *inverse* problems derived from problem **(P2)**:

a) *geometrical inverse* problem: given *Dirichlet* data f and the corresponding (*measured*) *Neumann* data g, in (1), on the accessible outer boundary $\partial\Omega$, to reconstruct the shape of the *interior* boundary ∂B, see [5].

b) *localisation inverse* problem: concerns to localisate of the domain (*cell*) $\overline{B}$ with a given shape and the fixed parameters α and h, see [6].

The main question in this context is to find sufficient conditions insuring that the *localization inverse* problem is uniquely soluble. Indeed:

First, we relate the above problems a) and b) with the *Dirichlet-to-Neumann* operator (1) by defining explicitly this operator, whose can define the *local* and *total* current across the *external* boundary $\partial\Omega$, which are useful to resolve a) and b).

Second, we study the *localisation inverse* problem in the framework of application outlined in the problem **(P2)**, which consist in finding sufficient (*Dirichlet-to-Neumann*) conditions to localise the position of the *cell* $\overline{B}$ from the *experimentally measurable* macroscopic response parameters.

In Section 2, we introduce the *existence* and *uniqueness* for the solution of problem **(P2)**. In Section 3, we introduce our first main result concerning the study of spherical case of problem **(P1)**, whose we give a general method to resolve the type of partial derivative system like **(P1)**, see proposition 3.2. Indeed, we allow an explicit calculations, based on Green-Ostrogradski theorem, for the solution of this problem.

In Section 4, it is our second main result which consist in showing that total current across the *external* boundary $\partial\Omega$, involving *Dirihlet-to-Neumann* operator (1), can resolve the *localisation inverse* problem in three dimensional case, when the compact $\Omega \subset \mathbb{R}^3$.

2. Uniqueness of the Problem (P2)

We suppose that Ω and $B \subset \Omega$ be open bounded domains in $\mathbb{R}^d$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is $\partial\left(\Omega\setminus\overline{B}\right) = \partial\Omega \cup \partial B$ and $\partial\Omega \cap \partial B = \varnothing$.

Then the unit *outer-normal* to the boundary $\partial\left(\Omega\setminus\overline{B}\right)$ vector-field $\nu(x)_{x\in\partial(\Omega\setminus B)}$ is well-defined, and we consider the normal derivative in **(P2)** as the *interior* limit:

$$\left(\partial_\nu u\right)\big|_{\partial B}(\omega) := \lim_{x\to\omega} \nu(\omega)\cdot(\nabla u)(x),\ x \in \Omega\setminus\overline{B}. \tag{3}$$

The *existence* of the limit (3) as well as the restriction $u|_{\partial B}(\omega) := \lim_{x\to\omega} u(x)$ is insured since u has to be harmonic solution of problem **(P2)** for C^2-smooth boundaries $\partial\left(\Omega\setminus\overline{B}\right)$ [7].

Now, we introduce some indispensable standard notations and definitions, see [8]. Let $\mathcal{H}$ be Hilbert space $L^2(M)$ on domain $M \subset \mathbb{R}^d$ and $\partial\mathcal{H} := L^2(\partial M)$ denote the corresponding boundary space. We denote by $W_2^s(M)$ the *Sobolev* space of $\mathcal{H}$-functions, whose s-derivatives are also in $\mathcal{H}$, and similar, $W_2^s(\partial M)$ is the *Sobolev* space of $\partial\mathcal{H}$-functions on the C^2-smooth boundary ∂M.

Proposition 2.1. *Let* $f \in W_2^{1/2}(\partial\Omega)$ *for* C^2*-smooth boundaries* $\partial\left(\Omega\setminus\overline{B}\right)$. *Then the Dirichlet-Neumann problem* **(P2)** *has a unique* (*harmonic*) *solution in domain* $\Omega\setminus\overline{B}$.

Proof. For *existence* we refer to [7]. To prove the *uniqueness*, we consider the problem **(P2)** for $f=0$ and $c^*=0$. Then by Gauss-Ostrogradsky theorem, one gets that the corresponding solution u yields:

$$\begin{aligned}
&\int_{\Omega\setminus\overline{B}} \mathrm{d}x\left(\nabla\overline{u(x)}\cdot\nabla u\right)(x) \\
&= \int_{\Omega\setminus\overline{B}} \mathrm{d}x\, div\left(\overline{u(x)}(\nabla u)(x)\right) \\
&= \int_{\partial B} \mathrm{d}\sigma(\omega)\overline{u(\omega)}\left(\partial_\nu u\right)(\omega) \\
&= -WD^{-1}\int_{\partial B} \mathrm{d}\sigma(\omega)\left|u(\omega)\right|^2 \le 0.
\end{aligned} \tag{4}$$

The estimate (4) implies that $u\left(x\in\Omega\setminus\overline{B}\right) = \text{Const}$. Hence by the boundary condition one gets $\left(WD^{-1}u\right)\big|_{\partial B}(\omega) = 0$, and from $u|_{\partial\Omega}(x) = f\left(x\in\partial\Omega\right) = 0$, we obtain that for $WD^{-1} \ge 0$, the harmonic function $u(x) = 0$ for $x \in \Omega\setminus\overline{B}$. □

The next statement is a key for analysis of *inverse localisation* problems:

Proposition 2.2. *Consider two problems* **(P2)** *corresponding to a bounded domain* $\Omega \subset \mathbb{R}^2$ *with* C^2*-smooth boundary* $\partial\Omega$ *and to two subsets* B_1 *and* B_2 *with the same smoothness of the boundaries* ∂B_1, ∂B_2. *If for solutions* $u_f^{(1)}$, $u_f^{(2)}$ *of these problems one has*

$$\partial_\nu u_f^{(1)}\Big|_{\partial\Omega} = \partial_\nu u_f^{(2)}\Big|_{\partial\Omega}, \quad (5)$$

then $\partial B_1 = \partial B_2$.

Proof. By virtue of $u_f^{(1)}\Big|_{\partial\Omega} = u_f^{(2)}\Big|_{\partial\Omega} = f$ and by condition (5), the problem **(P2)** has two solutions for identical *external* (on $\partial\Omega$) and internal (on ∂B_1 and ∂B_2) Robin boundary conditions. Then by the standard arguments based on the Holmgren uniqueness theorem [9] for harmonic functions on $\mathbb{R}^2$, one obtains that $\partial B_1 = \partial B_2$. □

3. Dirichlet-to-Neumann Operators for Absorbing Laplacian Transport

Here, we consider the spherical shell of the problem **(P1)** so that $\Omega = B(O_0, R_0)$ and the absorbing cell is also a ball $B = B(O, r_0 < R_0)$, whose we denote by d_0 the distance between the two centers $d_0 = d_{O_0 \to O}$.

Hereafter, we denote the previous hypothesis by spherical case.

In the sequel, we resolve the problem **(P1)** in order to calculate explicitly *Dirichlet-to-Neumann* operator relative to this case.

Before resolving problem **(P1)**, we need the following theorem which the key of the solution:

Theorem 3.1. (Gauss-Ostrogradski)

Let $\boldsymbol{V}$ a field vector across the domain $\Psi \subset \mathbb{R}^3$, having as border $\partial\Psi$.

$$\int_{M\in\partial\Psi} \boldsymbol{V}\cdot\boldsymbol{n}_M \,\mathrm{d}\sigma = \int_\Psi div\boldsymbol{V}\mathrm{d}\Psi, \quad (6)$$

whose $div\boldsymbol{V}$ designated the divergence of field vector $\boldsymbol{V}$. $\mathrm{d}\sigma$ and $\mathrm{d}\Psi$ designated respectively the differential elements relative to $\partial\Psi$ and Ψ. $\boldsymbol{n}_M$ designated the unit *outer-normal* vector on $\partial\Psi$ at arbitrary point M.

Remark 1. *Let the orthonormal reference with origin* O *and axis* $Y'Y$, *which is keen on the line* OO_0 *in the sense of the vector* $\boldsymbol{OO}_0$.

On the other hand, since for all $l \in \mathbb{N}$, spherical harmonic function $Y_{l,m=0}(\theta,\varphi)$ is independent of φ if $m = 0$, then we note:

$$Y_l(\theta) := Y_{l,m=0}(\theta,\varphi).$$

Since v_f is harmonic function, then it takes the following form, see [10]:

$$v_f(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_l r^l + b_l r^{-l-1}\right)Y_{l,m}(\theta,\varphi). \quad (7)$$

Therefore, we need to calculate the coefficients of (7) from the condition boundaries. Indeed, since the radius of points of ∂B are equal to constant r_0, then the condition boundary on ∂B implies easily, by identification, the following system:

$$\left(S_f\right)\ \begin{cases} a_0 = -b_0\left(DW^{-1}r_0^{-2} + r_0^{-1}\right) + c^* Y_{0,0}^{-1}(\varphi,\theta) & if l = 0, \\ a_l = b_l\left[D(l+1)r_0^{-l-2} + Wr_0^{-l-1}\right]\left(Dlr_0^{l-1} - Wr_0^l\right)^{-1} & if l \neq 0. \end{cases}$$

But, on the boundary $\partial\Omega$, the radius aren't equal, and depend of spherical angle θ. Then, for this reason, we use Gauss-Ostrogradski theorem's, whose we show that it is useful to find another relation between the coefficients of (7) like $\left(S_f\right)$. Consequently, we get for each $l \in \mathbb{N}$, a system of two equations with two unknowns a_l and b_l, which it is sufficient to calculate a_l and b_l:

Proposition 3.2. *The condition boundary on* $\partial\Omega$ *implies*:

$$\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_l(\theta)\left[f(\varphi,\theta)r^{l+1}(\theta) - a_l r^{2l+1}(\theta)Y_l(\theta)\right] = b_l, \forall l \in \mathbb{N}. \quad (8)$$

Proof. Let $l_0 \in \mathbb{N}$, we construct the following vector field $\boldsymbol{V}_{l_0}$ by:

$$\boldsymbol{V}_{l_0} = H_{l_0}(r,\varphi,\theta)\boldsymbol{e}_\varphi, \quad (9)$$

whose, $H_{l_0}(r,\varphi,\theta)$ is a primitive relative to φ for the following function:

$$h_{l_0}(r,\varphi,\theta) = Y_{l_0}(\theta)\frac{\sin\theta}{r}\left[\partial_r\left(\frac{v(r,\varphi,\theta) - a_{l_0}r^{l_0}Y_{l_0}(\theta)}{b_{l_0}r^{-l_0-1}}\right)\right].$$

Calculate the flux of field $\boldsymbol{V}_{l_0}$ across the domain $B(O_0,R_0)\setminus B(O,r_0)$ using Gauss-Ostrogradski theorem 3.1:

$$\int_{B(O_0,R_0)\setminus B(O,r_0)} \mathrm{d}v\, div\left(\boldsymbol{V}_{l_0}\right) = \int_{\partial\left[B(O_0,R_0)\setminus B(O,r_0)\right]} \boldsymbol{V}_{l_0}\cdot\boldsymbol{n}\mathrm{d}\sigma, \quad (10)$$

where:

$$\begin{cases} \boldsymbol{n} & \text{unit outer-normal vector of } \partial\left[B(O_0,R_0)\setminus B(O,r_0)\right], \\ \mathrm{d}\sigma & \text{areal differential elemen trelative to } \partial\left[B(O_0,R_0)\setminus B(O,r_0)\right], \\ \mathrm{d}v = r^2\sin\theta\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta & \text{volume differential elemen trelative to } B(O_0,R_0)\setminus B(O,r_0), \\ div(\boldsymbol{v}) & \text{divergence of vector } \boldsymbol{v}. \end{cases}$$

1. Calculate $\int_{B(O_0,R_0)\setminus B(O,r_0)} \mathrm{d}v\,div\left(\boldsymbol{V}_{l_0}\right)$:

In domain $B\left(O_0,R_0\right)\setminus B\left(O,r_0\right)$, we have:

$$\begin{cases} \text{Radius } r \text{ varies between } r_0 \text{ and } r\left(\theta\right) := r_{M\in\partial B(O_0,R_0)}. \\ \text{Angle } \varphi \text{ varies between 0 and } 2\pi. \\ \text{Angle } \theta \text{ varies between 0 and } \pi. \end{cases}$$

On the other hand, $div\left(\boldsymbol{V}_{l_0}\right)$ can be calculated from (9) by:

$$div\left(\boldsymbol{V}_{l_0}\right) = \frac{1}{r\sin\theta}\partial_\varphi H\left(r,\varphi,\theta\right) = \frac{Y_{l_0}\left(\theta\right)}{r^2}\partial_r\left(\frac{v\left(r,\varphi,\theta\right)-a_{l_0}r^{l_0}Y_{l_0}\left(\theta\right)}{b_{l_0}r^{-l_0-1}}\right).$$

Then, we deduce that:

$$\int_{B(O_0,R_0)\setminus B(O,r_0)} \mathrm{d}v\,div\left(\boldsymbol{V}_{l_0}\right) = \int_{r_0}^{r(\theta)}\int_0^{2\pi}\int_0^{\pi} r^2\sin\theta\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta\, div\left(\boldsymbol{V}_{l_0}\right) = \int_{r_0}^{r(\theta)}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)\partial_r\left(\frac{v\left(r,\varphi,\theta\right)-a_{l_0}r^{l_0}Y_{l_0}\left(\theta\right)}{b_{l_0}r^{-l_0-1}}\right).$$

Therefore, from Fubini's theorem of multiple integrals, we obtain:

$$\begin{aligned} &\int_{B(O_0,R_0)\setminus B(O,r_0)} \mathrm{d}v\,div\left(\boldsymbol{V}_{l_0}\right) \\ &= \int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)\int_{r_0}^{r(\theta)}\partial_r\left[\frac{v\left(r,\varphi,\theta\right)-a_{l_0}r^{l_0}Y_{l_0}\left(\theta\right)}{b_{l_0}r^{-l_0-1}}\right]\mathrm{d}r \\ &= \int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)\left[\frac{v\left(r,\varphi,\theta\right)-a_{l_0}r^{l_0}Y_{l_0}\left(\theta\right)}{b_{l_0}r^{-l_0-1}}\right]\Bigg|_{r_0}^{r(\theta)} \\ &= \frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)\left[v\left(r\left(\theta\right),\varphi,\theta\right)r^{l_0+1}\left(\theta\right)-a_{l_0}r^{2l_0+1}\left(\theta\right)Y_{l_0}\left(\theta\right)\right] \\ &\quad -\frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)r_0^{l_0+1}\left[v\left(r_0,\varphi,\theta\right)-a_{l_0}r_0^{l_0}Y_{l_0}\left(\theta\right)\right]. \end{aligned} \tag{11}$$

Moreover, condition boundary on $\partial\Omega$ implies that:

$$v\left(r\left(\theta\right),\varphi,\theta\right) = v\left[r_{M\in\partial B(O_0,R_0)},\varphi,\theta\right] = v\left[M\in\partial B\left(O_0,R_0\right)\right] = f\left(\varphi,\theta\right).$$

So, by replacing $v\left(r\left(\theta\right),\varphi,\theta\right)$ by its value $f\left(\varphi,\theta\right)$ in (11), we deduce:

$$\begin{aligned} &\int_{B(O_0,R_0)\setminus B(O,r_0)} \mathrm{d}v\,div\left(\boldsymbol{V}_{l_0}\right) \\ &= \frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)\left[f\left(\varphi,\theta\right)r^{l_0+1}\left(\theta\right)-a_{l_0}r^{2l_0+1}\left(\theta\right)Y_{l_0}\left(\theta\right)\right] \\ &\quad -\frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)r_0^{l_0+1}\left[v\left(r_0,\varphi,\theta\right)-a_{l_0}r_0^{l_0}Y_{l_0}\left(\theta\right)\right]. \end{aligned} \tag{12}$$

But, we can prove that:

$$\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}\left(\theta\right)r_0^{l_0+1}\left[v\left(r_0,\varphi,\theta\right)-a_{l_0}r_0^{l_0}Y_{l_0}\left(\theta\right)\right] = b_{l_0}. \tag{13}$$

<u>Indeed:</u> from (7), we have that

$$r_0^{l_0+1}\left[v\left(r_0,\varphi,\theta\right)-a_{l_0}r_0^{l_0}Y_{l_0}\left(\theta\right)\right] = \sum_{\substack{l=0\\ l\neq l_0}}^{+\infty}\sum_{m=-l}^{m=l}\left(a_l r_0^{l+l_0+1}+b_l r_0^{-l+l_0}\right)Y_{l,m}\left(\varphi,\theta\right)+b_{l_0}Y_{l_0}\left(\theta\right).$$

Multiplying by $\sin\theta$ the previous equation, and integrating it on domain $\{(\varphi,\theta):\varphi\in[0,2\pi[,\theta\in[0,\pi[\}$, we obtain:

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)r_0^{l_0+1}\left[C(r_0,\varphi,\theta)-a_{l_0}r_0^{l_0}Y_{l_0}(\theta)\right]\\&=b_{l_0}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)+\sum_{\substack{l=0\\l\neq l_0}}^{+\infty}\sum_{m=-l}^{m=l}\left(a_l r_0^{l+l_0+1}+b_l r_0^{-l+l_0}\right)\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)Y_{l,m}(\varphi,\theta).\end{aligned}\tag{14}$$

On the other hand, spherical harmonic functions form a basis for the Hilbert space $L^2[S(O,1)]$ following inner product:

$$\langle f,g\rangle:=\int_{S(O,1)}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\, f\bar{g};\ \forall f,g\in L^2[S(O,1)].$$

Consequently, we deduce, since

$$\left\langle Y_{l,m},Y_{\tilde{l},\tilde{m}}\right\rangle=\delta_{(l,m)}(\tilde{l},\tilde{m});\quad \forall(l,m),(\tilde{l},\tilde{m})\in\mathbb{N},$$

that:

$$\left\langle Y_{l_0},Y_{l,m}\right\rangle:=\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)Y_{l,m}(\varphi,\theta)=\delta_{l_0}(l);\quad\forall l,l_0\in\mathbb{N}.\tag{15}$$

Here, δ designed Dirac function.

So, by inserting (15) in (14), we deduce above equality (13) as follows:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)r_0^{l_0+1}\left[v(r_0,\varphi,\theta)-a_{l_0}r_0^{l_0}Y_{l_0}(\theta)\right]&=\sum_{\substack{l=0\\l\neq l_0}}^{+\infty}\left(a_l r_0^{l+l_0+1}+b_l r_0^{-l+l_0}\right)\delta_{l_0}(l)+b_{l_0}\delta_{l_0}(l_0)\\&=\sum_{\substack{l=0\\l\neq l_0}}^{+\infty}\left(a_l r_0^{l+l_0+1}+b_l r_0^{-l+l_0}\right)0+b_{l_0}1=b_{l_0}.\end{aligned}$$

We continue the proof by inserting (13) in (12):

$$\int_{B(O_0,R_0)\setminus B(O,r_0)}\mathrm{d}v\,div\left(\boldsymbol{V}_{l_0}\right)=\frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)\left[f(\varphi,\theta)r^{l_0+1}(\theta)-a_{l_0}r^{2l_0+1}(\theta)Y_{l_0}(\theta)\right]-1.\tag{16}$$

2. Calculate $\int_{\partial[B(O_0,R_0)\setminus B(O,r_0)]}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}\mathrm{d}\sigma$:

knowing that,

$$\partial\left[B(O_0,R_0)\setminus B(O,r_0)\right]=S(O_0,R_0)\cup S(O,r_0),$$

then:

$$\begin{aligned}&\int_{\partial[B(O_0,R_0)\setminus B(O,r_0)]}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}\mathrm{d}\sigma\\&=\int_{M\in S(O_0,R_0)}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}_M\mathrm{d}\sigma+\int_{M\in S(O,r_0)}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}_M\mathrm{d}\sigma.\end{aligned}\tag{17}$$

2.1 Showing that:

$$\int_{S(O,r_0)}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}_{C(O,r_0)}\mathrm{d}\sigma=0.\tag{18}$$

Indeed: unit *outer-normal* vector $\boldsymbol{n}_M$ relative to domain $B(O_0,R_0)\setminus B(O,r_0)$ at arbitrary point $M\in S(O,r_0)$ is $-\boldsymbol{e}_r$. This implies:

$$\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}=H(r,\varphi,\theta)\boldsymbol{e}_\varphi\cdot(-\boldsymbol{e}_r)=0.$$

2.2 Showing that:

$$\int_{S(O_0,R_0)}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}_{\partial B(O_0,R_0)}\mathrm{d}\sigma=0.\tag{19}$$

Indeed: the symmetry of the shape implies that unit *outer-normal* vector of $S(O_0,R_0)$ relative to domain $B(O_0,R_0)\setminus B(O,r_0)$ is below in plan generated by the two vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ which are orthogonal to field vector $\boldsymbol{V}_{l_0}$ directed by $\boldsymbol{e}_\varphi$. So, we obtain:

$$\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}_M=0,\quad\forall M\in S(O_0,R_0).$$

Then, by inserting (18) and (19) in (9), we deduce that:

$$\int_{\partial[B(O_0,R_0)\setminus B(O,r_0)]}\boldsymbol{V}_{l_0}\cdot\boldsymbol{n}\mathrm{d}\sigma=0.\tag{20}$$

3. Boundary Equation

Finally, by inserting (16) and (20) in (10), we obtain that:

$$=\frac{1}{b_{l_0}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_{l_0}(\theta)\left[f(r,\varphi,\theta)r^{l_0+1}(\theta)-a_{l_0}r^{2l_0+1}(\theta)Y_{l_0}(\theta)\right]-1=0.$$

The previous equation ends the proof since it is true for any $l_0 \in \mathbb{N}$. □

Proposition 3.3. *If* $f \in W_2^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$, *then problem* **(P1)** *have unique solution with the form* (7), *whose the coefficients are given by*:

- $$a_0 = \frac{\left(DW^{-1}r_0^{-2} + r_0^{-1}\right)\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_0(\theta) f(\varphi,\theta) r(\theta) - c^* Y_0^{-1}(\theta)}{\left(DW^{-1}r_0^{-2} + r_0^{-1}\right)\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_0^2(\theta) r(\theta) - 1}$$
- $$b_0 = \frac{\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_0(\theta) r(\theta)\left[f(\varphi,\theta) - c^*\right]}{1 - \left(DW^{-1}r_0^{-2} + r_0^{-1}\right)\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_0^2(\theta) r(\theta)}$$
- $$a_l = \frac{r_0^l\left[D(l+1)r_0^{-l-1} + Wr_0^{-l}\right]\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_l(\theta) r^{l+1}(\theta) f(\varphi,\theta)}{r_0^{2l}\left(Dl - Wr_0\right) + \left[D(l+1)r_0^{-1} + W\right]\int_0^{2\pi}\int_0^{\pi} \sin\theta r^{2l+1}(\theta) Y_l^2(\theta) \mathrm{d}\varphi \mathrm{d}\theta}$$
- $$b_l = \frac{r_0^{l+2}\left(Dlr_0^{l-1} - Wr_0^l\right)\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\varphi \mathrm{d}\theta \sin\theta Y_l(\theta) r^{l+1}(\theta) f(\varphi,\theta)}{r_0^{2l+1}\left(Dl - Wr_0\right) + \left[D(l+1) + Wr_0\right]\int_0^{2\pi}\int_0^{\pi} \sin\theta r^{2l+1}(\theta) Y_l^2(\theta) \mathrm{d}\varphi \mathrm{d}\theta}.$$

where, $r(\theta) = d_0\cos\theta + \sqrt{d_0^2\cos^2\theta + R_0^2 - d_0^2}$ is the distance $d_{O\to M}$ between arbitrary point M on sphere $\partial\Omega_0 = S(O_0, R_0)$ and the center O.

Proof. It is enough to resolve for any $l \in \mathbb{N}$, the system of two unknowns a_l et b_l given by the two boundary conditions (8) and (S_f). □

Since the solution of problem **(P1)** is given from proposition 3.3, then we can deduce its relative *Dirichlet-to-Neumann* operator:

Corollary 3.4. *The Dirichlet-to-Neumann operator* (1) *is defined by*

$$\Lambda_{I,\partial\Omega} : f \in W_2^{1/2}(\partial\Omega) \mapsto \frac{\partial v_f}{\partial\nu}\Big|_{\partial\Omega} := \nabla v_f(r,\varphi,\theta)\cdot \boldsymbol{n}_{\partial\Omega}\Big|_{\partial\Omega}, \text{ where}: \tag{21}$$

$$\frac{\partial v_f}{\partial\nu}\Big|_{\partial\Omega}(\varphi,\theta) = R_0^{-1}\left[r(\theta) - d_0\cos\theta\right]\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\left[la_l r^{l-1}(\theta) - (l+1)b_l r^{-l-2}(\theta)\right]Y_{l,m}(\varphi,\theta)$$
$$- R_0^{-1}d_0\sin\theta\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\left[a_l r^l(\theta) + b_l r^{-l-1}(\theta)\right]\partial_\theta Y_{l,m}(\varphi,\theta).$$

Here, the coefficients $\{a_l, b_l\}_{l\in\mathbb{N}}$ are given by proposition 3.3.

Proof. Since we have that $\boldsymbol{e}_z = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta$, then unit *outer-normal* vector $\boldsymbol{n}_M$ for arbitrary point $M \in \partial\Omega = S(O_0, R_0)$ is given by:

$$\boldsymbol{n}_M = R_0^{-1}\left(r_M\boldsymbol{e}_r - d_0\boldsymbol{e}_z\right) = R_0^{-1}\left\{\left[r(\theta) - d_0\cos\theta\right]\boldsymbol{e}_r + d_0\sin\theta\boldsymbol{e}_\theta\right\},$$

and consequently, this implies:

$$\partial_\nu v_f(r,\varphi,\theta)\Big|_{\partial\Omega} := \nabla v_f(r,\varphi,\theta)\cdot\boldsymbol{n}_M\Big|_{M\in\partial\Omega} = \nabla v_f(r,\varphi,\theta)\cdot R_0^{-1}\left\{\left[r(\theta) - d_0\cos\theta\right]\boldsymbol{e}_r + d_0\sin\theta\boldsymbol{e}_\theta\right\}\Big|_{\partial\Omega}$$
$$= R_0^{-1}\left[r(\theta) - d_0\cos\theta\right]\partial_r v_f(r,\varphi,\theta) - R_0^{-1}d_0\sin\theta\partial_\theta v_f(r,\varphi,\theta)\Big|_{\partial\Omega}.$$

But, we have from the proof of proposition 3.2 that $r(M \in \partial\Omega_0) = r(\theta)$. Then:

$$\partial_\nu v_f(r,\varphi,\theta)\Big|_{\partial\Omega} = R_0^{-1}\left[r(\theta) - d_0\cos\theta\right]\partial_r v_f(r(\theta),\varphi,\theta) - R_0^{-1}d_0\sin\theta\partial_\theta v_f(r(\theta),\varphi,\theta).$$

Consequently, it is enough to replace $v_f(r,\varphi,\theta)$ in the previous equation by its value given in (7). □

Remark 2. *For general properties of Dirichlet-to-Neumann operators, mainly existence and uniqueness,* *we refer to* [10], *chapter* 4.

Remark 3. *Notice that definition of Dirichlet-to-Neumann operator* (21) *implies that it has as eigenfunctions the spherical harmonic function* $Y_{l,m}(\theta,\varphi)\big|_{l\in\mathbb{N},m\le|l|}$, *and a discrete spectrum* $\sigma_{\Lambda_{I,\partial\Omega}} := \{\lambda_{l,m}\}_{l=0,|m|\le l}^{\infty}$, *whose* $\lim_{l\to\infty}\lambda_{l,m}=\infty$.

Corollary 3.5. *Dirichlet-to-Neumann operator* (21) *is unbounded*, *non-negative*, *self-adjoint*, *first-order elliptic pseudo-differential operator with compact resolvent on the Hilbert space* $L^2(\partial\Omega,\sin\theta\mathrm{d}\varphi\mathrm{d}\theta)$.

Proof. For the proof, we refer to [10], chapter 4. □

Remark 4. *Corollary* 3.4 *implies using Hille-Yosida*'*s theorem that Dirichlet-to-Neumann operator* (21) *can be generate certain semigroup* $\left\{S(t):=\mathrm{e}^{-t\Lambda_{I,\partial\Omega}}\right\}_{t>0}$. *Moreover*, *we can prove using Arzela-Ascoli*'*s criterion that this semigroup is contractant holomorphic in the both Banach space* $C(\partial\Omega)$ *and* $L^2(\partial\Omega,\mathrm{d}\theta\sin\theta\mathrm{d}\varphi)$.

Proof. For a complete proof, see [10], chapter 4. □

4. Localisation Inverse Problem

We are interested by resolving the *localisation inverse* problem of **(P1)** using the explicit formula of d_0, which will be calculated in terms of measurable *Dirichlet-to-Neumann* boundary hypothesis on external boundary $\partial\Omega$.

For resolving this problem, we need the following:

i) First, we aim to calculate the total flux $J_{\partial\Omega}$ across *external* boundary $\partial\Omega$.

ii) Second, we aim to find an equation involving the distance $d_0 := d_{O\to O_0}$ between O center of cell $\bar{B}$ and O_0 center of Ω.

Proposition 4.1. *The total flux* $J_{\partial\Omega}$ *and* $J_{\partial B}$ *satisfy the following*:

$$J_{\partial\Omega} = J_{\partial B} = 2\sqrt{\pi}b_0 D. \tag{22}$$

Proof. Since the differential element at the boundary ∂B and unit *outer-normal* vector $\boldsymbol{n}_M$ at arbitrary point $M\in\partial B$ are respectively equal to $r_0\sin\theta\mathrm{d}\varphi\mathrm{d}\theta$ and $\boldsymbol{e}_r$, then we deduce from (7) that:

$$J_{\partial B} := -D\int_{M\in\partial B}\nabla v\cdot\boldsymbol{n}_M\mathrm{d}\sigma = 2\sqrt{\pi}b_0 D.$$

On the other hand, by Gauss-Ostrogradsky theorem, one gets:

$$\begin{aligned} J_{\partial\Omega} - J_{\partial B} &:= \int_{\partial\Omega\cup\partial B}\mathrm{d}\sigma\boldsymbol{j}\cdot\boldsymbol{n} \\ &= -D\int_{\Omega\backslash B}\mathrm{d}\sigma\nabla\cdot\nabla v \\ &= -D\int_{\Omega\backslash B}\mathrm{d}V\Delta v = 0, \end{aligned}$$

where $\mathrm{d}V$ is the volume differential element. Therefore, (22) is deduced. □

Since we have, from proposition 4.1, that *total* flux $J_{\partial\Omega}$ across *external* boundary $\partial\Omega$ depended only of one coefficient b_0 of development (7), whose b_0 depended of distance $d_0 := d_{O\to O_0}$, then an equation of $d_0 := d_{O\to O_0}$ can be find easily. Indeed:

Corollary 4.2. *The distance* $d_0 := d_{O\to O_0}$ *verifies the following equation*:

$$\begin{aligned} &1-\left(\frac{D}{Wr_0^2}+\frac{1}{r_0}\right)\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_0^2(\theta)\sqrt{d_0^2\cos^2\theta+R_0^2-d_0^2} \\ &=\frac{2\sqrt{\pi}D}{J_{\partial\Omega}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_0(\theta)\sqrt{d_0^2\cos^2\theta+R_0^2-d_0^2}\left[f(\varphi,\theta)-c^*\right]. \end{aligned} \tag{23}$$

Proof. It is enough to insert (22) in the expression of b_0 given in proposition 3.2 in order to substitute b_0, after replacing $r(\theta)$ by its value in term of d_0:

$$\begin{aligned} &1-\left(\frac{D}{Wr_0^2}+\frac{1}{r_0}\right)\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_0^2(\theta)\left[d_0\cos\theta+\sqrt{d_0^2\cos^2\theta+R_0^2-d_0^2}\right] \\ &=\frac{2\sqrt{\pi}D}{J_{\partial\Omega}}\int_0^{2\pi}\int_0^{\pi}\mathrm{d}\varphi\mathrm{d}\theta\sin\theta Y_0(\theta)\left[d_0\cos\theta+\sqrt{d_0^2\cos^2\theta+R_0^2-d_0^2}\right]\left[f(\varphi,\theta)-c^*\right]. \end{aligned}$$

Consequently, the fact that $\int_0^{\pi}\mathrm{d}\theta\sin\theta\cos\theta=0$ ends the proof. □

5. Conclusions

(23) is an equation of the only unknown d_0 involving the parameters $J_{\partial\Omega}$ and $f := v\big|_{\partial\Omega}$, which are the *Dirichlet-to-Neumann* hypothesis of problem **(P1)** on the *external* boundary, and we can found them from an *experimental measures*.

To summarize, we have found an equation for d_0 which is the distance between the center O of the *cell* $\bar{B}$ and the center O_0 of Ω, so it remains to find the position of the center O. In fact:

Let M_{max} and M_{min} be two points at the *external* boundary $\partial\Omega$ whose the norm of the local current $\boldsymbol{j}$ reaches respectively its maximum and minimum values, see **Figure 1**. Then, from the symmetry of the shape, we deduce that the center O of the *cell* $\overline{B}$ is localized at the line passed by the points M_{max}, M_{min} and O_0, exactly between M_{max} and O_0 where the distance d_0 between O and O_0 is given by Equation (23).

By conclusion, we can now answer the question posed in the introduction about the *uniqueness* of the *inverse localisation* problem for **(P1)**, and we can conclude that *total* flux $J_{\partial\Omega}$ (2), involving *Dirihlet-to-Neumann* operator (1), is sufficient to resolve the *localisation inverse* problem, in three-dimensional case, if the shape is regular. But, it is not enough in other type of inverse problem like geometrical inverse problem, see [5].

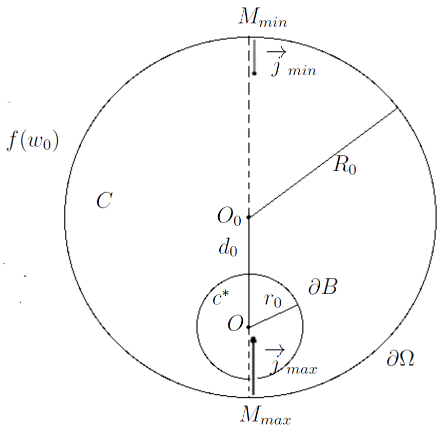

Figure 1. Position of the *cell overline B*.

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