Journal of Modern Physics, 2013, 4, 572-578
http://dx.doi.org/10.4236/jmp.2013.45080 Published Online May 2013 (http://www.scirp.org/journal/jmp)

Localisation Inverse Problem of Absorbing Laplacian Transport

Ibrahim Baydoun
École Centrale Paris, Laboratoire de MSS-MAT, Grande Voie des Vignes,
Châtenay-Malabry Cedex, France
Email: ibrahim.baydoun@ecp.fr

Received December 24, 2012; revised February 6, 2013; accepted March 7, 2013

ABSTRACT

We study the *localisation inverse* problem corresponding to *Laplacian* transport of absorbing *cell*. Our main goal is to find sufficient *Dirichelet-to-Neumann* conditions insuring that this *inverse* problem is uniquely soluble. In this paper, we show that the *conformal mapping* technique is adopted to this type of problem in the two dimensional case.

Keywords: Laplacian Transport; Dirichlet-to-Neumann Operators; Inverse Problem; Conformal Mapping

1. Introduction

The theory of *Dirichlet-to-Neumann* operators is the basis of many research domains in analysis, particularly, those concerning *Laplacian* transports. It is also very important in mathematical-physics, geophysics, electrochemistry. Moreover, it is very useful in medical diagnosis, such as electrical impedance tomography, as showing in the following example:

Example 1. In 1989, J. Lee and G. Uhlmann have introduced an example on the determination of conductivity matrix field $\gamma(x) = \left[\gamma_{i,j}(x)\right]_{i,j=1}^{d}$, for x in a bounded open domain $\Omega \subset \mathbb{R}^d$, see e.g. [1]. This example is related to *measuring* of elliptic *Dirichlet-to-Neumann* map for associated conductivity equation, see e.g. [1]. Notice that the solution of this problem has a lot of practical applications in various domains overall in medicine, which is an important diagnostic tool, e.g. in the electrical impedance tomography; the tissue in the human body is an example of highly anisotropic conductor [2].

Under assumption that there are no sources or sinks of current the potential $v(x), x \in \Omega,$ for a given voltage $f(\omega), \omega \in \partial\Omega,$ on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$\textbf{(P1)} \quad \begin{cases} \operatorname{div}(\gamma\nabla v) = 0 \text{ in } \Omega, \\ v|_{\partial\Omega} = f \qquad \text{on } \partial\Omega. \end{cases}$$

Then the corresponding to **(P1)** *Dirichlet-to-Neumann* map (operator) $\Lambda_{\gamma,\partial\Omega}$ is (*formally*) defined by [3] as follows:

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \left.\frac{\partial v_f}{\partial \nu_\gamma}\right|_{\partial\Omega} := \nu \cdot \gamma \nabla v_f\big|_{\partial\Omega}. \qquad (1.1)$$

Here ν is the unit *outer-normal* vector to the boundary at $\omega \in \partial\Omega$ and the function $v := v_f$ is a solution of the Dirichlet problem **(P1)**.

Dirichlet-to-Neumann operator (1.1) is also called the voltage-to-current map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether on can determine the conductivity matrix γ by knowing electrical boundary measurements, *i.e.* the corresponding *Dirichlet-to-Neumann* operator? In general this operator does not determine the matrix γ *uniquely*, see e.g. [4].

The main question in this context is to find sufficient conditions insuring that the *inverse* problem is *uniquely* soluble.

The problem of electrical current flux in the form **(P1)** is an example of so-called *diffusive Laplacian* transport. Besides the voltage-to-current problem, the motivation to study this kind of transport comes for instance, from the transfer across biological membranes, see e.g. [5,6]:

Example 2. Let some species of concentration $C(x), x \in \mathbb{R}^d$, diffuse stationary in the *isotropic* bulk $(\gamma = I)$ from a (distant) source localised on the closed boundary $\partial\Omega$ towards a semipermeable compact interface ∂B of *cell* $\overline{B} \subset \Omega$, where they disappear at a given rate $W \geq 0$. Then the steady field of concentrations (*Laplacian* transport with a diffusion coefficient

$D \geq 0$) obeys the set of equations:

$$
\textbf{(P2)}\quad \begin{cases} \Delta C = 0, x \in \Omega \setminus \bar{B}, \\ C\big|_{\partial\Omega}(\omega_0) = f(\omega_0), \\ \text{the concentration at the source } \partial\Omega, \\ -D\partial_\nu C\big|_{\partial B}(\omega) = W\left(C - c^*\right)\big|_{\partial B}(\omega), \\ \text{on the interface } \omega \in \partial B. \end{cases}
$$

Here $c^* \geq 0$ is the constant concentration of the species inside *cell* $\bar{B}$.

Now, similar to (1.1), we can associate with the problem **(P2)** a *Dirihlet-to-Neumann* operator

$$
\Lambda_{I,\partial\Omega} : f \mapsto \partial_\nu u_f := \nu \cdot \nabla u_f \big|_{\partial\Omega}, \tag{1.2}
$$

with domain $\operatorname{dom}\left(\Lambda_{I,\partial\Omega}\right)$ belongs to a certain *Sobolev* space, see Section 2.

The advantage of this approach is that as soon as the operator (1.2) is defined, one can apply it to study the mixed boundary value problem **(P2)**. This gives, in particularly, the value of the *particle* flux due to *Laplacian* transport across the membrane $\partial\Omega$. Moreover, the *total* current across the boundary $\partial\Omega$ can be defined (for given *f*) in term of *Dirihlet-to-Neumann* operator (1.2) as follows:

$$
J_{\partial\Omega} := -D \int_{\partial\Omega} \mathrm{d}\sigma \Lambda_{\gamma=I,\partial\Omega} f, \tag{1.3}
$$

where $\mathrm{d}\sigma$ designed the differential element relative to $\partial\Omega$.

The aim of the present paper is to show how can apply the theory of *Dirichlet-to-Neumann* operators on the *localisation* inverse problem in the framework of application outlined in problem **(P2)**, which consists in finding the sufficient (*Dirihlet-to-Neumann*) conditions to localise the position of *cell* $\bar{B}$ from the *experimentally measurable* macroscopic response parameters.

In Section 2, we introduce the *existence* and *uniqueness* for the solution of problem **(P2)**. In Section 3, we present our main results which consist in showing that *total* current (1.3), involving *Dirihlet-to-Neumann* operator (1.2), can resolve the *localisation inverse* problem in the two-dimensional case, when the compact $\Omega \subset \mathbb{R}^2$. We allow an explicit calculations for the solution of problem **(P2)** from *Dirichlet-Neumann* boundary conditions. Whereas for this solution, we use a method of *conformal mapping* for harmonic functions in doubly connected domains $\Omega \setminus B$.

2. Uniqueness of the Problem (P2)

We suppose that Ω and $B \subset \Omega$ be open bounded domains in $\mathbb{R}^d$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is $\partial\left(\Omega \setminus \bar{B}\right) = \partial\Omega \bigcup \partial B$ and $\partial\Omega \bigcap \partial B = \varnothing$.

Then the unit *outer-normal* to the boundary $\partial\left(\Omega \setminus \bar{B}\right)$ vector-field $\nu(x), x \in \partial\left(\Omega \setminus \bar{B}\right)$ is well-defined, and we consider the normal derivative in **(P2)** as the *interior* limit:

$$
\left(\partial_\nu u\right)\big|_{\partial B}(\omega) := \lim_{x \to \omega} \nu(\omega) \cdot (\nabla u)(x), x \in \Omega \setminus \bar{B}. \tag{2.1}
$$

The *existence* of the limit (2.1) as well as the restriction $u\big|_{\partial B}(\omega) := \lim_{x \to \omega} u(x)$ is insured since u has to be harmonic solution of problem **(P2)** for C^2-smooth boundaries $\partial\left(\Omega \setminus \bar{B}\right)$ [7].

Now, we introduce some indispensable standard notations and definitions, see [8]. Let $\mathcal{H}$ be Hilbert space $L^2(M)$ on domain $M \subset \mathbb{R}^d$ and $\partial\mathcal{H} := L^2(\partial M)$ denote the corresponding boundary space. We denote by $W_2^s(M)$ the *Sobolev* space of $\mathcal{H}$-functions, whose s-derivatives are also in $\mathcal{H}$, and similar, $W_2^s(\partial M)$ is the *Sobolev* space of $\partial\mathcal{H}$-functions on the C^2-smooth boundary ∂M.

Proposition 2.1 *Let* $f \in W_2^{1/2}(\partial\Omega)$ *for* C^2*-smooth boundaries* $\partial\left(\Omega \setminus \bar{B}\right)$. *Then the Dirichlet-Neumann problem* **(P2)** *has a unique* (*harmonic*) *solution in domain* $\Omega \setminus \bar{B}$.

Proof. For *existence* we refer to [7]. To prove the *uniqueness*, we consider the problem **(P2)** for $f = 0$ and $c^* = 0$. Then by Gauss-Ostrogradsky theorem, one gets that the corresponding solution u yields:

$$
\begin{aligned}
&\int_{\Omega \setminus \bar{B}} \mathrm{d}x \left(\nabla \overline{u(x)} \cdot \nabla u\right)(x) \\
&= \int_{\Omega \setminus \bar{B}} \mathrm{d}x \operatorname{div}\left(\overline{u(x)} (\nabla u)(x)\right) \\
&= \int_{\partial B} \mathrm{d}\sigma(\omega) \overline{u(\omega)} \left(\partial_\nu u\right)(\omega) \\
&= -WD^{-1} \int_{\partial B} \mathrm{d}\sigma(\omega) \left|u(\omega)\right|^2 \leq 0.
\end{aligned} \tag{2.2}
$$

The estimate (2.2) implies that $u\left(x \in \Omega \setminus \bar{B}\right) = \text{Const}$. Hence by the boundary condition one gets $\left(WD^{-1}u\right)\big|_{\partial B}(\omega) = 0$, and from $u\big|_{\partial\Omega}(x) = f\left(x \in \partial\Omega\right) = 0$, we obtain that for $WD^{-1} \geq 0$, the harmonic function $u(x) = 0$ for $x \in \Omega \setminus \bar{B}$. □

The next statement is a key for analysis of *inverse localisation* problems:

Proposition 2.2 *Consider two problems* **(P2)** *corresponding to a bounded domain* $\Omega \subset \mathbb{R}^2$ *with* C^2*-smooth boundary* $\partial\Omega$ *and to two subsets* B_1 *and* B_2 *with the same smoothness of the boundaries* $\partial B_1, \partial B_2$. *If for solutions* $u_f^{(1)}, u_f^{(2)}$ *of these problems one has*

$$
\partial_\nu u_f^{(1)}\big|_{\partial\Omega} = \partial_\nu u_f^{(2)}\big|_{\partial\Omega}, \tag{2.3}
$$

then $\partial B_1 = \partial B_2$.

Proof. By virtue of $u_f^{(1)}\big|_{\partial\Omega} = u_f^{(2)}\big|_{\partial\Omega} = f$ and by condition (2.3), the problem **(P2)** has two solutions for identical *external* (on $\partial\Omega$) and internal (on ∂B_1 and ∂B_2) Robin boundary conditions. Then by the standard arguments based on the Holmgren *uniqueness* theorem [9] for harmonic functions on $\mathbb{R}^2$, one obtains that $\partial B_1 = \partial B_2$. □

3. Inverse Problem: Conformal Mapping

Let Ω and $B \subset \Omega$ be respectively open bounded domains in $\mathbb{R}^2$ with C^2-smooth disjoint boundaries $\partial\Omega = C(O_0, R_0)$ and $\partial B = C(O, r_0)$, where $C(O_0, R_0)$ and $C(O, r_0)$ are two circles respectively with radius R_0 and r_0.

In the sequel, we denote by d_0 the distance between the two centers $d_0 = d_{O_0 \to O}$.

The solution of the *inverse localisation* problem is decomposed into two steps:

In the first step, we introduce the necessary *conformal mapping*, see [10]. There are two reasons that make interest for using this technique, indeed the convenable *conformal mapping* T:

(a) Transforms two non-concentric circles into two concentric circles.

(b) For any harmonic function u, $u \circ T$ still harmonic, see Proposition 3.1.

With this technique, we transform problem **(P2)** to another problem **(P2)***, whose **(P2)*** has as solution an harmonic function also, and as domains $T(\Omega)$ and $T(B)$ which are concentric.

Therefore, we can find easily the general form for the solution of **(P2)***, and consequently, we conclude the necessary coefficients for the general solution of problem **(P2)**, with which we will be able to resolve its *inverse* problem.

In the next step, we are interested by resolving the *localisation inverse* problem using the explicit formula of d_0, which will be calculated in terms of (*measurable*) *Dirichlet-Neumann* boundary hypothesis on $\partial\Omega$.

Proposition 3.1 *Let $T : N \to M$ be a conformal mapping defined by*:

$$z = (x_1, x_2) \to T(z) := \tilde{x}_1(x_1, x_2) + \mathrm{i}\tilde{x}_2(x_1, x_2).$$

If $u(x, y)$ is an harmonic function in M, then the composition

$$\tilde{u}(\tilde{x}_1, \tilde{x}_2) := (u \circ T)(x_1, x_2) = u\left[x_1(\tilde{x}_1, \tilde{x}_2), x_2(\tilde{x}_1, \tilde{x}_2)\right],$$

is an harmonic function in N.

In particular, by distinguishing explicitly the Laplacian in different coordinates, $\Delta_z := \partial_{x_1}^2 + \partial_{x_2}^2$ and $\Delta_T := \partial_{\tilde{x}_1}^2 + \partial_{\tilde{x}_2}^2$, one obtains:

$$\Delta_T \tilde{u}\left[\tilde{x}(x_1, x_2), \tilde{y}(x_1, x_2)\right] = \left|\partial_z T(z)\right|^{-2} \Delta_z u(x_1, x_2).$$

3.1. Necessary Conformal Mapping

Let $T : \mathbb{C} \mapsto \mathbb{C}$ be the *conformal mapping* defined by:[1]

$$\begin{aligned} &T(z) := \mathrm{i}(z + d\mathrm{i})(ez + \mathrm{i})^{-1} \\ &\text{that is} : T\left[z = (x_1, x_2)\right] = \tilde{x}_1 + \mathrm{i}\tilde{x}_2, \end{aligned} \tag{3.1}$$

where:

$$d = \frac{d_0^2 + r_0^2 - R_0^2 + \sqrt{\left(d_0^2 - r_0^2\right)^2 + R_0^4 - 2R_0^2\left(d_0^2 + r_0^2\right)}}{2d_0}$$

$$\text{and } e = \frac{d}{r_0^2}. \tag{3.2}$$

Remark 1 *We define the conformal mapping T relatively to the orthonormal reference with origin O and axis $Y'Y$, which is keen on the line OO_0 in the sense of the vector $\boldsymbol{OO}_0$.*

Corollary 3.2 *T transforms $C(O_0, R_0)$ and $C(O, r_0)$ from non-concentric circles to concentric circles $C(O, \tilde{R}_0)$ and $C(O, r_0)$, where $\tilde{R}_0$ is defined by*:

$$\tilde{R}_0 = \sqrt{\frac{R_0^2 + (d_0 + d)^2}{\left(R_0^2 + d_0^2\right)e^2 + 1 + 2d_0 e}}. \tag{3.3}$$

Remark 2 *Notice that when $d_0 \to 0$, the mapping T converges to the identity function.*

3.2. Problem (P2)*

Let F and $\tilde{f}$ be respectively the values of the normal derivative $\partial_\nu C\big|_{\partial B}$ and f with the new variables $(\tilde{x}_1, \tilde{x}_2)$, *i.e*:

$$\begin{cases} F(\tilde{x}_1, \tilde{x}_2) := \partial_\nu C(x_1, x_2)\big|_{\partial B}, \\ \tilde{f}(\tilde{x}_1, \tilde{x}_2) := f(x_1, x_2). \end{cases}$$

Then, if we make the substitution $C(x_1, x_2) = \tilde{C}(\tilde{x}_1, \tilde{x}_2)$, then by using the *conformal mapping* given in (3.1), we show from proposition 3.1 that the problem **(P2)** can be transformed to problem **(P2)***, which obeys the set of equations:

$$\textbf{(P2)*} \begin{cases} \Delta\tilde{C}(\tilde{x}_1, \tilde{x}_2) = 0, \\ (\tilde{x}_1, \tilde{x}_2) \in D(O, \tilde{R}_0) \setminus D(O, r_0), \\ \tilde{C}(\tilde{x}_1, \tilde{x}_2) = \tilde{f}(\tilde{x}_1, \tilde{x}_2) \\ \text{on } C(O, \tilde{R}_0) \\ -DF(\tilde{x}_1, \tilde{x}_2) = W\left[\tilde{C}(\tilde{x}_1, \tilde{x}_2) - c^*\right] \quad \text{on } C(O, r_0) \end{cases}$$

[1] i is the complex number.

In the sequel, we denote by $\tilde{\rho}$ and $\tilde{\varphi}$ the polar coordinates associated to the variables $(\tilde{x}_1,\tilde{x}_2)$.

Corollary 3.3 *The solution of the problem* **(P2)*** *gets the form*:

$$\begin{aligned}\tilde{C}(\tilde{\rho},\tilde{\varphi}) &= \tilde{\alpha}+\tilde{\beta}\ln\tilde{\rho}\\ &+\sum_{k=1}^{+\infty}\left[\tilde{c}_1(k)\tilde{\rho}^k+\tilde{c}_2(k)\tilde{\rho}^{-k}\right]\cos(k\tilde{\varphi})\\ &+\left[\tilde{c}_3(k)\tilde{\rho}^k+\tilde{c}_4(k)\tilde{\rho}^{-k}\right]\sin(k\tilde{\varphi}).\end{aligned} \tag{3.4}$$

Proof. As $\tilde{C}$ is harmonic, then the proof follows, see [10]. □

Hereinafter, we need to make explicit F in order to calculate the coefficients of the development (3.4).

Proposition 3.4 *The function* F *is given by*:

$$\begin{aligned}F(\tilde{\rho},\tilde{\varphi}) &= \frac{1}{r_0(1-ed)}\\ &\times\Big\{(1+ed)\tilde{\beta}-\left(er_0+dr_0^{-1}\right)\tilde{\beta}\sin\tilde{\varphi}\\ &+(1+ed)\sum_{k=1}^{+\infty}k\left[r_0^k\tilde{c}_1(k)-r_0^{-k}\tilde{c}_2(k)\right]\cos(k\tilde{\varphi})\\ &\quad+k\left[r_0^k\tilde{c}_3(k)-r_0^{-k}\tilde{c}_4(k)\right]\sin(k\tilde{\varphi})\\ &+\sum_{k=1}^{+\infty}\frac{k}{2}\left[r_0^{k-1}\tilde{c}_1(k)-r_0^{-k-1}\tilde{c}_2(k)\right]\left[\sin(k+1)\tilde{\varphi}-\sin(k-1)\tilde{\varphi}\right]\\ &+\frac{k}{2}\left[r_0^{k-1}\tilde{c}_3(k)-r_0^{-k-1}\tilde{c}_4(k)\right]\\ &\quad\cdot\left[\cos(k-1)\tilde{\varphi}-\cos(k+1)\tilde{\varphi}\right]\Big\}\end{aligned} \tag{3.5}$$

Proof. Since $F(\tilde{x}_1,\tilde{x}_2):=\partial_\nu C(x_1,x_2)\big|_{\partial B}$, we deduce from the definition of $\partial_\nu C(x_1,x_2)\big|_{\partial B}$ that:

$$\begin{aligned}F(\tilde{x}_1,\tilde{x}_2) &:= \partial_\nu C(x_1,x_2)\\ &:= r_0^{-1}\left(x_1\partial_{x_1}C+x_2\partial_{x_2}C\right).\end{aligned} \tag{3.6}$$

We need firstly to calculate $\partial_{x_1}C$ and $\partial_{x_2}C$ in terms of $(\tilde{x}_1,\tilde{x}_2)$. By virtue of the substitution $C(x_1,x_2)=\tilde{C}(\tilde{x}_1,\tilde{x}_2)$, we obtain the following differential relation:

$$\begin{aligned}\mathrm{d}C(x_1,x_2) &= \mathrm{d}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\\ &= \partial_{\tilde{x}_1}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\mathrm{d}\tilde{x}_1+\partial_{\tilde{x}_2}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\mathrm{d}\tilde{x}_2.\end{aligned}$$

Then by comparison, we conclude that:

$$\begin{aligned}\partial_{x_1}C &= \partial_{\tilde{x}_1}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\partial_{x_1}\tilde{x}_1+\partial_{\tilde{x}_2}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\partial_{x_1}\tilde{x}_2,\\ \partial_{x_2}C &= \partial_{\tilde{x}_1}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\partial_{x_2}\tilde{x}_1+\partial_{\tilde{x}_2}\tilde{C}(\tilde{x}_1,\tilde{x}_2)\partial_{x_2}\tilde{x}_2.\end{aligned} \tag{3.7}$$

Substitute Equation (3.7) in Equation (3.6), one obtains:

$$\begin{aligned}&F(\tilde{x}_1,\tilde{x}_2)\\ &= r_0^{-1}\left[\left(x_1\partial_{x_1}\tilde{x}_1+x_2\partial_{x_2}\tilde{x}_1\right)\partial_{\tilde{x}_1}\tilde{C}+\left(x_1\partial_{x_1}\tilde{x}_2+x_2\partial_{x_2}\tilde{x}_2\right)\partial_{\tilde{x}_2}\tilde{C}\right]\end{aligned} \tag{3.8}$$

Recall that Equation (3.1) gives us a relation between the variables (x_1,x_2) and $(\tilde{x}_1,\tilde{x}_2)$. Then, the quantities $\left(x_1\partial_{x_1}\tilde{x}_1+x_2\partial_{x_2}\tilde{x}_1\right)$ and $\left(x_1\partial_{x_1}\tilde{x}_2+x_2\partial_{x_2}\tilde{x}_2\right)$ in Equation (3.8) can be calculated in terms of $(\tilde{x}_1,\tilde{x}_2)$. Therefore we have:

$$\begin{aligned}F(\tilde{x}_1,\tilde{x}_2) &= \frac{1}{r_0(1-ed)}\\ &\cdot\Big\{\tilde{x}_1\left(1+ed-2e\tilde{x}_2\right)\partial_{\tilde{x}_1}C\\ &\quad+\left[e\left(\tilde{x}_1^2-\tilde{x}_2^2\right)+(1+ed)\tilde{x}_2-d\right]\partial_{\tilde{x}_2}\tilde{C}\Big\}\end{aligned}$$

Finally, if we replace $\tilde{x}_1$ and $\tilde{x}_2$ by their associated polar coordinates $(\tilde{\rho},\tilde{\varphi})$, then we can rewrite $F(\tilde{x}_1,\tilde{x}_2)$ in terms of $\tilde{\rho}$ and $\tilde{\varphi}$ (due to the relation between polar derivatives and cartesian derivatives) as follows:

$$\begin{aligned}F(\tilde{\rho},\tilde{\varphi}) &= \frac{1}{r_0(1-ed)}\\ &\cdot\left[(1+ed)r_0\partial_{\tilde{\rho}}\tilde{C}(\tilde{\rho},\tilde{\varphi})-\left(er_0^2+d\right)\sin\tilde{\varphi}\partial_{\tilde{\rho}}\tilde{C}(\tilde{\rho},\tilde{\varphi})\right].\end{aligned} \tag{3.9}$$

Therefore, it is enough to replace $\tilde{C}(\tilde{\rho},\tilde{\varphi})$ by its value given in Equation (3.4). □

In order to resolve problem **(P2)***, we need to make explicit $\tilde{f}$. So, the change of variables given in Equation (3.1), allows us to express $\tilde{f}$ with the following Fourier series:

$$\begin{aligned}\tilde{f}(\tilde{x}_1,\tilde{x}_2) &= \alpha^{\tilde{f}}\\ &+\sum_{k=1}^{+\infty}c_1^{\tilde{f}}(k)\cos(k\tilde{\varphi})+c_2^{\tilde{f}}(k)\sin(k\tilde{\varphi}),\end{aligned} \tag{3.10}$$

where:

$$\begin{cases}\alpha^{\tilde{f}} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}\tilde{f}(\tilde{x}_1,\tilde{x}_2)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}(x_1,x_2)f(x_1,x_2),\\ c_1^{\tilde{f}}(k)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}\tilde{f}(\tilde{x}_1,\tilde{x}_2)\cos(k\tilde{\varphi})\\ \qquad=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}(x_1,x_2)f(x_1,x_2)\cos\left[k\tilde{\varphi}(x_1,x_2)\right],\\ c_2^{\tilde{f}}(k)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}\tilde{f}(\tilde{x}_1,\tilde{x}_2)\sin(k\tilde{\varphi})\\ \qquad=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\mathrm{d}\tilde{\varphi}(x_1,x_2)f(x_1,x_2)\sin\left[k\tilde{\varphi}f(x_1,x_2)\right].\end{cases}$$

Remark 3 *Notice that the coefficients of* (16) *depend only of*:

1. $\tilde{\varphi}(x_1,x_2)$ given from the change of variables due to (3.1).

2. f which is the *external* condition boundary of problem **(P2)**.

Proposition 3.5 *The coefficients of Equation* (3.4) *are given by*:

$$\tilde{\alpha}=\alpha^{\tilde{f}}-\tilde{\beta}\ln\tilde{R}_0 \text{ and } \tilde{\beta}=\frac{Q_1}{Q_2}, \text{whose}: \quad (3.11)$$

$$Q_1=-2WD^{-1}r_0(1-ed)\left(\alpha^{\tilde{f}}-c^*\right)+\tilde{R}_0r_0^{-2}c_2^{\tilde{f}}(1)+\frac{(1-ed)(1+ed)^{-1}D^{-1}\tilde{R}_0W\left(1+\tilde{R}_0^2r_0^{-2}\right)c_2^{\tilde{f}}(1)}{r_0+\tilde{R}_0^2r_0^{-1}+D^{-1}Wr_0(1+ed)^{-1}(1-ed)\left(r_0-\tilde{R}_0^2r_0^{-1}\right)-\tilde{R}_0r_0^{-1}c_2^{\tilde{f}}(1)},$$

$$Q_2=2(1+ed)+2D^{-1}r_0(1-ed)W\ln\left(r_0\tilde{R}_0^{-1}\right)+\frac{\left(er_0+dr_0^{-1}\right)(1+ed)^{-1}\left(1+\tilde{R}_0^2r_0^{-2}\right)}{r_0+\tilde{R}_0^2r_0^{-1}+D^{-1}Wr_0(1-ed)(1+ed)^{-1}\left(r_0-\tilde{R}_0^2r_0^{-1}\right)-\tilde{R}_0r_0^{-1}c_2^{\tilde{f}}(1)}.$$

Proof. First boundary condition of problem **(P2)*** implies: $\tilde{C}\big|_{C(O,\tilde{R}_0)}(\tilde{x}_1,\tilde{x}_2)=\tilde{f}(\tilde{x}_1,\tilde{x}_2)$. Then, by replacing $\tilde{C}(\tilde{x}_1,\tilde{x}_2)$ and $\tilde{f}(\tilde{x}_1,\tilde{x}_2)$ by their values given respectively in (3.4) and (3.10), we obtain after identification that:

$$\begin{cases}\tilde{\alpha}+\tilde{\beta}\ln\tilde{R}_0=\alpha^{\tilde{f}}\\ \text{identification between the constants,}\\ \tilde{c}_2(1)+\tilde{c}_1(1)\tilde{R}_0^2=c_1^{\tilde{f}}(1)\tilde{R}_0\\ \text{identification between the coefficients of }\cos(\tilde{\varphi}),\\ \tilde{c}_4(1)+\tilde{c}_3(1)\tilde{R}_0^2=c_2^{\tilde{f}}(1)\tilde{R}_0\\ \text{identification between the coefficients of }\sin(\tilde{\varphi}).\end{cases}$$

Afterwards, from the second boundary condition in problem **(P2)***, we have $-DF(\tilde{\rho},\tilde{\varphi})=W\left[\tilde{C}\big|_{C(O,r_0)}(\tilde{\rho},\tilde{\varphi})-c^*\right]$. Then, by the similar manner, we deduce from (3.5) and (3.4) that:

$$\begin{cases}(1+ed)\tilde{\beta}+\frac{1}{2}\left[\tilde{c}_3(1)-r_0^{-2}\tilde{c}_4(1)\right]\\ =-D^{-1}r_0(1-ed)W\left(\tilde{\alpha}+\tilde{\beta}\ln r_0-c^*\right),\\ (1+ed)\left[\tilde{c}_1(1)r_0-r_0^{-1}\tilde{c}_2(1)\right]\\ =-D^{-1}r_0(1-ed)W\left[\tilde{c}_1(1)r_0+r_0^{-1}\tilde{c}_2(1)\right],\\ \left(er_0^2+d\right)\tilde{\beta}+(1+ed)\left[r_0^2\tilde{c}_3(1)-\tilde{c}_4(1)\right]\\ =D^{-1}r_0(1-ed)W\left[r_0^2\tilde{c}_3(1)+\tilde{c}_4(1)\right].\end{cases}$$

Finally, one has a system of six equations with six unknowns $\tilde{\alpha},\tilde{\beta},\tilde{c}_1(1),\tilde{c}_2(1),\tilde{c}_3(1)$ and $\tilde{c}_4(1)$. The solution of this system ends the proof. □

Remark 4 *Notice that the proof of Proposition* 3.5 *show us the advantage of conformal mapping technique. Indeed, the identification between Fourier series on the boundary conditions of problem* **(P2)*** *is easily calculated because its boundaries are two concentric circles, and consequently its radius are constant. But, it is not the case in problem* **(P2)**, *because here its boundaries are non-concentric, whose its radius depend of polar angle.*

Remark 5 *For the inverse problem, we will just need the explicit value of* $\tilde{\alpha}$ *and* $\tilde{\beta}$, *that why we didn't make explicit the other coefficients values for the Fourier series of* $\tilde{C}$.

3.3. Localisation Inverse Problem

For resolving the *inverse* problem, we need the following:

(i) First, we aim to calculate the *total* flux $J_{\partial\Omega}$ across the *external* boundary $\partial\Omega$. For that, we need to express the solution C of problem **(P2)** in terms of Fourier series.

(ii) Second, we aim to find an equation for d_0.

In the sequel, we denote by ρ and φ the polar coordinates associated to the initial variables (x_1,x_2).

(i) The solution of problem **(P2)** gets the following form:

$$\begin{aligned}C(\rho,\varphi)&=\alpha+\beta\ln\rho\\ &+\sum_{k=1}^{+\infty}\left[c_1(k)\rho^k+c_2(k)\rho^{-k}\right]\cos(k\varphi)\\ &+\left[c_3(k)\rho^k+c_4(k)\rho^{-k}\right]\sin(k\varphi)\end{aligned} \quad (3.12)$$

where $\alpha,\beta,c_1(k),c_2(k),c_3(k)$ and $c_4(k)$ for all $k\in\mathbb{N}$, are the Fourier coefficients of C.

Corollary 3.6 *The total flux* $J_{\partial\Omega}$ *and* $J_{\partial B}$ *satisfy the following*:

$$J_{\partial\Omega}=J_{\partial B}=-2\pi D\beta \quad (3.13)$$

Proof. Since the differential element $\mathrm{d}l$ at boundary ∂B are respectively equal to $r_0\mathrm{d}\varphi$, then by inserting (3.12) in (1.2), we deduce that:

$$J_{\partial B}:=\int_{M\in\partial B}\mathrm{d}l\boldsymbol{j}_M\cdot\boldsymbol{n}_M=-2\pi D\beta,$$

where $\boldsymbol{j}_M:=-D\nabla C(M)$ and $\boldsymbol{n}_M=\boldsymbol{e}_r$ designed respectively *local* current and *outer-normal* vector at arbitrary point M.

On the other hand, by Gauss-Ostrogradsky theorem, one gets:

$$J_{\partial\Omega} - J_{\partial B} := \int_{M \in \partial\Omega \cup \partial B} \mathrm{d}\boldsymbol{l}\boldsymbol{j}_M \cdot \boldsymbol{n}$$
$$= -D \int_{\Omega \setminus \bar{B}} \mathrm{d}S \nabla \cdot \nabla C = -D \int_{\Omega \setminus \bar{B}} \mathrm{d}S \Delta C = 0,$$

where $\mathrm{d}S$ is the areal differential element. Therefore, (19) is deduced. □

(ii) We now state the following corollary and proposition which will be important to find a relation between the coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ involving d_0.

Corollary 3.7

$$-D\beta = r_0 W\left(\alpha + \beta \ln r_0 - c^*\right). \tag{3.14}$$

Proof. Substitute $C(\rho,\varphi)$ in the second boundary condition of problem **(P2)** (at the interface ∂B of *cell* $\bar{B}$) by its value given in Equation (3.12), therefore the proof follows by identification (between two Fourier series). □

Proposition 3.8

$$\alpha + \beta \ln r_0 = \tilde{\alpha} + \tilde{\beta} \ln r_0. \tag{3.15}$$

Proof. Since $C(x_1,x_2) = \tilde{C}(\tilde{x}_1,\tilde{x}_2)$, one gets also $C(\rho,\varphi) = \tilde{C}(\tilde{\rho},\tilde{\varphi})$. But, we have $T[\partial B = C(O,r_0)] = C(O,r_0)$, then $C(r_0,\varphi) = \tilde{C}(r_0,\tilde{\varphi})$.

On the other hand, $(\varphi,\tilde{\varphi})$ varies in $[0,2\pi[\times[0,2\pi[$. So, by applying the double integral on the domain $[0,2\pi[\times[0,2\pi[$ for the equality $C(r_0,\varphi) = \tilde{C}(r_0,\tilde{\varphi})$, and by replacing $C(r_0,\varphi)$ and $\tilde{C}(r_0,\tilde{\varphi})$ by their values given respectively in Equations (3.4) and (3.12), we deduce that:

$$\int_0^{2\pi}\int_0^{2\pi} \alpha + \beta \ln r_0 + \sum_{k=1}^{+\infty} \left[r_0^k c_1(k) + r_0^{-k} c_2(k)\right]\cos(k\varphi) + \left[r_0^k c_3(k) + r_0^{-k} c_4(k)\right]\sin(k\varphi)\,\mathrm{d}\varphi\mathrm{d}\tilde{\varphi}$$
$$= \int_0^{2\pi}\int_0^{2\pi} \tilde{\alpha} + \tilde{\beta} \ln r_0 + \sum_{k=1}^{+\infty} \left[r_0^k \tilde{c}_1(k) + r_0^{-k} \tilde{c}_2(k)\right]\cos(k\tilde{\varphi}) + \left[r_0^k \tilde{c}_3(k) + r_0^{-k} \tilde{c}_4(k)\right]\sin(k\tilde{\varphi})\,\mathrm{d}\varphi\mathrm{d}\tilde{\varphi}.$$

Then Equation (3.15) follows from Fubini's theorem. □

Finally, we aim to find the equation satisfied by d_0.

In fact, by inserting (3.13) and (3.14) in Equation (3.15), we obtain:

$$r_0^{-1} W^{-1} J_{\partial\Omega} + 2\pi c^* - = 2\pi\left(\tilde{\alpha} + \tilde{\beta} \ln r_0\right). \tag{3.16}$$

4. Conclusions

From Proposition 3.5, we remark that $\tilde{\alpha}$ and $\tilde{\beta}$ are calculated in terms of e,d and $\tilde{R}_0$, which their formulae given in (3.2) and (3.3) depend of d_0 and the coefficients of (3.10).

Then, Equation (3.16) becomes an equation of the only unknown d_0 involving the parameters $J_{\partial\Omega}$ (1.3) and $f := C|_{\partial\Omega}$ (see Remark 3), which are the *Dirichlet-to-Neumann* hypothesis of problem **(P2)** on the *external* boundary, and we can found them from an *experimental measures*.

To summarize, we have found an equation for d_0, which is the distance between the center O of *cell* $\bar{B}$ and the center O_0 of Ω, so it remains to find the position of the center O. In fact:

Let $M_{\max}$ and $M_{\min}$ be two points at the *external* boundary $\partial\Omega$ whose the norm of the *local* current $\boldsymbol{j}$ reaches respectively its maximum and minimum values, see **Figure 1**. Then, from the symmetry of the shape, we deduce that the center O of *cell* $\bar{B}$ is localized at the line passed by the points $M_{\max}$, $M_{\min}$ and O_0, exactly between $M_{\max}$ and O_0 where the distance d_0 between O and O_0 is given by Equation (3.16).

By conclusion, we can now answer the question posed in the introduction about the *uniqueness* of the *inverse localisation* problem associated to **(P2)**, and we can conclude that the *total* flux (1.3) is sufficient to resolve the *localisation inverse* problem, in two-dimensional case, if the shape is regular. But, it is not enough in other type of *inverse* problem like *geometrical inverse* problem, see [11].

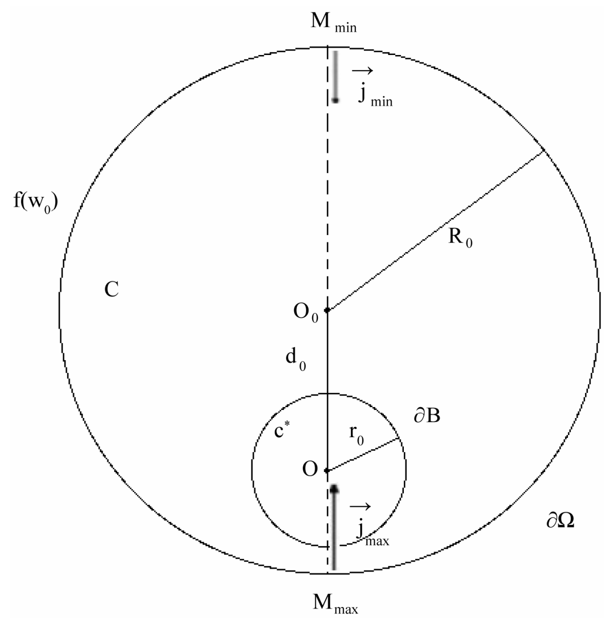

Figure 1. Position of cell $\bar{B}$.

5. Acknowledgements

I want to thank the CNRS Federation "Francilienne de Mecanique, Materiaux Structures et Procedes" CNRS FR2609, in particularly Prof. Didier Clouteau, for financial support in this study.

REFERENCES

[1] J. Lee and G. Uhlmann, *Communications on Pure and Applied Mathematics*, Vol. 42, 1989, pp. 1097-1112. doi:10.1002/cpa.3160420804

[2] D. C. Barber and B. H. Brown, *Journal of Physics E*: *Scientific Instruments*, Vol. 17, 1984, pp. 723-733. doi:10.1088/0022-3735/17/9/002

[3] M. E. Taylor, "Pseudodifferential Operators," Princeton University Press, Princeton, 1996.

[4] A. Greenleaf and G. Uhlmann, *Duke Mathematical Journal*, Vol. 108, 2001, pp. 599-617. doi:10.1215/S0012-7094-01-10837-5

[5] B. Sapoval, *Physical Review Letters*, Vol. 73, 1994, pp. 3314-3316. doi:10.1103/PhysRevLett.73.3314

[6] D. S. Grebenkov, M. Filoche and B. Sapoval, *Physical Review E*, Vol. 73, 2006, Article ID: 021103. doi:10.1103/PhysRevE.73.021103

[7] M. E. Taylor, "Partial Differential Equations II: Qualitative Studies of Linear Equations," Springer-Verlag, Berlin, 1996. doi:10.1007/978-1-4757-4187-2

[8] J. K. Hunter and B. Nachtergaele, "Applied Analysis," World Scientific, Singapore City, 2001.

[9] D. Tataru, *Communications in Partial Differential Equations*, Vol. 20, 1995, pp. 855-884.

[10] I. Baydoun, "Opérateurs de Dirichlet-Neumann et Leurs Applications: Transport Laplacien, Problème Inverse et Opérateur de Dirichlet-Neumann," Éditions Universitaires Européennes, 2012.

[11] I. Baydoun and V. A. Zagrebnov, *Theoretical and Mathematical Physics*, Vol. 168, 2011, pp. 1180-1191. doi:10.1007/s11232-011-0097-8

