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Wormholes Supported by a Combination of Normal and Quintessential Matter in Einstein and Einstein-Maxwell Gravity

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ABSTRACT

It is shown in the first part of this paper that a combined model comprising ordinary and quintessential matter can support a traversable wormhole in Einstein-Maxwell gravity. Since the solution allows zero tidal forces, the wormhole is suitable for a humanoid traveler. The second part of the paper shows that the electric field can be eliminated (Einstein gravity), but only by tolerating enormous tidal forces. Such a wormhole would still be capable of transmitting signals.

Keywords: Wormholes; Quintessential Matter; Einstein-Maxwell Gravity

1. Introduction

Traversable wormholes, first conjectured by Morris and Thorne [1], are handles or tunnels in the spacetime topology connecting different regions of our Universe or of different universes altogether. Interest in traversable wormholes has increased in recent years due to an unexpected development, the discovery that our Universe is undergoing an accelerated expansion [2,3]. This acceleration is due to the presence of *dark energy*, a kind of negative pressure, implying that $\ddot{a} > 0$ in the Friedmann equation $\ddot{a}/a = -\frac{4\pi}{3}(\rho + 3p)$. In the equation of state $p = w\rho$, the range of values $-1 < w < -1/3$ results in $\ddot{a} > 0$. This range is referred to as *quintessence dark energy*. Smaller values of w are also of interest. Thus $w = -1$ corresponds to Einstein's cosmological constant [4]. The case $w < -1$ is referred to as *phantom energy* [5-10]. Here we have $\rho + p < 0$, in violation of the null energy condition. As a result, phantom energy could, in principle, support wormholes and thereby cause them to occur naturally.

Sections 2-4 discuss a combined model of quintessence matter and ordinary matter that could support a wormhole in Einstein-Maxwell gravity, once again suggesting that such wormholes could occur naturally. The theoretical construction by an advanced civilization is also an inviting prospect since the model allows the assumption of zero tidal forces. Section 4 considers the effect of eliminating the electric field. A wormhole solution can still be obtained but only by introducing a redshift function that results in enormous radial tidal forces, suggesting that some black holes may actually be wormholes fitting the conditions discussed in this paper and so may be capable of transmitting signals, a possibility that can in principle be tested.

2. The Model

Our starting point for a static spherically symmetric wormhole is the line element

$$ds^2 = -e^{\Phi(r)}dt^2 + e^{\Lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (1)$$

where $e^{\Lambda(r)} = 1/\left(1 - b(r)/r\right)$. Here $b = b(r)$ is the *shape function* and $\Phi = \Phi(r)$ is the *redshift function*, which must be everywhere finite to prevent an event horizon. For the shape function, $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole. Another requirement is the flare-out condition, $b'(r_0) < 1$ (in conjunction with $b(r) < r$), since it indicates a violation of the weak energy condition, a primary prerequisite for the existence of wormholes [1].

In this paper the model proposed for supporting the wormhole consists of a quintessence field and a second field with (possibly) anisotropic pressure representing normal matter. Here the Einstein field equations take on the following form (assuming $c = 1$):

$$G_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + \tau_{\mu\nu}\right), \quad (2)$$

where $\tau_{\mu\nu}$ is the energy momentum tensor of the quintessence-like field, which is characterized by a free pa-

rameter w_q such that $-1 < w_q < -1/3$. Following Kiselev [11], the components of this tensor satisfy the following conditions:

$$\tau_t^t = \tau_r^r = -\rho_q, \tag{3}$$

$$\tau_\theta^\theta = \tau_\phi^\phi = \frac{1}{2}\left(3w_q + 1\right)\rho_q. \tag{4}$$

Furthermore, the most general energy momentum tensor compatible with spherically symmetry is

$$T_\nu^\mu = \left(\rho + p_t\right)u^\mu u_\nu - p_t g_\nu^\mu + \left(p_r - p_t\right)\xi^\mu \xi_\nu \tag{5}$$

with $u^\mu u_\mu = -1$. The Einstein-Maxwell field equations for the above metric corresponding to a field consisting of a combined model comprising ordinary and quintessential matter are stated next [12,13]. Here E is the electric field strength, σ the electric charge density, and q the electric charge.

$$\mathrm{e}^{-\Lambda}\left(\frac{\Lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G\rho + 8\pi G\rho_q + E^2, \tag{6}$$

$$\mathrm{e}^{-\Lambda}\left(\frac{\Phi'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi G p_r - 8\pi G\rho_q - E^2, \tag{7}$$

$$\begin{aligned}&\frac{1}{2}\mathrm{e}^{-\Lambda}\left(\frac{1}{2}\left(\Phi'\right)^2 + \Phi'' - \frac{1}{2}\Lambda'\Phi' + \frac{1}{r}\left(\Phi' - \Lambda'\right)\right)\\ &= 8\pi G\left(p_t + \frac{1}{2}\left(3w_q + 1\right)\rho_q\right) + E^2,\end{aligned} \tag{8}$$

$$\left(r^2 E\right)' = 4\pi r^2 \sigma \mathrm{e}^{\mu/2}. \tag{9}$$

Equation (9) can also be expressed in the form

$$E\left(r\right) = \frac{1}{r^2}\int_0^r 4\pi\left(r'\right)^2 \sigma \mathrm{e}^{\mu/2}\mathrm{d}r' = \frac{q\left(r\right)}{r^2}, \tag{10}$$

where $q\left(r\right)$ is the total charge on the sphere of radius r.

3. Solutions

We assume that for the normal-matter field we have the following equation of state for the radial pressure [14]:

$$p_r = m\rho, \quad 0 < m < 1. \tag{11}$$

For the lateral pressure we assume the equation of state

$$p_t = n\rho, \quad 0 < n < 1. \tag{12}$$

Generally, p_r is not equal to p_t, unless, of course, $m = n$.

Following Ref. [14], the factor $\sigma \mathrm{e}^{\mu/2}$ is assumed to have the form $\sigma_0 r^s$, where s is an arbitrary constant and σ_0 is the charge density at $r = 0$. As a result,

$$E\left(r\right) = 4\pi\sigma_0 \frac{r^{s+1}}{s+3}, \tag{13}$$

$$E^2\left(r\right) = 16\pi^2\sigma_0^2 \frac{r^{2s+2}}{\left(s+3\right)^2}, \tag{14}$$

and

$$q^2\left(r\right) = 16\pi^2\sigma_0^2 \frac{r^{2s+6}}{\left(s+3\right)^2}. \tag{15}$$

The next step is to obtain the shape function $b\left(r\right)$ by deriving a differential equation that can be solved for $\mathrm{e}^{-\Lambda(r)}$. The easiest way to accomplish this is to solve Equation (6) for $8\pi G\rho$ and substituting the resulting expression in Equation (7), which, in turn, is solved for $8\pi G\rho_q$. After substituting this expression in Equation (8) and making use of Equations (11) and (12), we obtain the simplified form

$$\left(\mathrm{e}^{-\Lambda}\right)' + \frac{\alpha \mathrm{e}^{-\Lambda}}{r} = \frac{\beta}{r} + rE^2\gamma. \tag{16}$$

Here α, β, and γ are dimensionless quantities given by the following:

$$\alpha = \frac{\alpha^*}{\frac{1}{2}\left(3w_q + 1\right)m/\left(m+1\right) + n/\left(m+1\right) + \frac{1}{4}r\Phi' + \frac{1}{2}}, \tag{17}$$

where

$$\begin{aligned}\alpha^* = &-n\Phi' r/\left(m+1\right) + \frac{1}{2}\left(3w_q + 1\right)\\ &+\frac{1}{2}\left(3w_q + 1\right)\Phi' r/\left(m+1\right)\\ &+r^2\frac{1}{4}\left(\Phi'\right)^2 + \frac{1}{2}r^2\Phi'' + \frac{1}{2}r\Phi',\end{aligned}$$

$$\beta = \frac{\frac{1}{2}\left(3w_q + 1\right)}{\frac{1}{2}\left(3w_q + 1\right)m/\left(m+1\right) + n/\left(m+1\right) + \frac{1}{4}r\Phi' + \frac{1}{2}} \tag{18}$$

and

$$\gamma = \frac{1 - \frac{1}{2}\left(3w_q + 1\right)}{\frac{1}{2}\left(3w_q + 1\right)m/\left(m+1\right) + n/\left(m+1\right) + \frac{1}{4}r\Phi' + \frac{1}{2}}. \tag{19}$$

Equation (16) is linear and would readily yield an exact solution provided that α and β are constants. This can only happen if $\Phi' = \eta/r$ for some constant η. In the first part of this paper we will assume that $\eta \equiv 0$, leading to the *zero-tidal-force* solution [1]. Whether occurring naturally or constructed by an advanced civilization, such a wormhole would be suitable for humanoid travelers.

Returning to Equation (16) and using Equation (14), the integrating factor $\mathrm{e}^{\alpha \ln r} = r^\alpha$ yields the solution

$$\mathrm{e}^{-\Lambda}=\frac{\beta}{\alpha}+\gamma\left(16\pi^2\sigma_0^2\right)\frac{r^{2s+4}}{(s+3)^2(2s+4+\alpha)}+\frac{C}{r^{\alpha}}, \quad (20)$$

where C is an integration constant. From $\mathrm{e}^{-\Lambda}=1-b(r)/r$ in Section 2, we obtain the shape function $b(r)$

$$=r\left[1-\frac{\beta}{\alpha}-\gamma\left(16\pi^2\sigma_0^2\right)\frac{r^{2s+4}}{(s+3)^2(2s+4+\alpha)}-\frac{C}{r^{\alpha}}\right]. \quad (21)$$

4. Wormhole Structure

In Equation (20), C is an integration constant. So mathematically, $\mathrm{e}^{-\Lambda}$ is a solution for every C, leading to $b(r)$ in Equation (21). Physically, however, $b(r)$ is going to satisfy the requirements of a shape function only for a range of values of C. This problem can best be approached graphically by assigning some typical values to the various parameters and adjusting the value of C, as exemplified by **Figure 1**. First observe that if $\eta\equiv 0$, then $\alpha=\beta$. For the given values $w_q=-2/3$, $m=0.5$, $n=0.5$, $\sigma_0=1$, and $s=-3.8$, a suitable value for C is -0.19, as we will see. Substituting in Equation (21), we obtain

$$b(r)=127.62r^{-2.6}+0.19r^{1.75}. \quad (22)$$

To locate the throat $r=r_0$ of the wormhole, we define the function $B(r)=b(r)-r$ and determine where $B(r)$ intersects the r-axis, as shown in **Figure 2**. Observe that **Figure 2** indicates that for $r>r_0$, $B(r)<0$, so that $b(r)<r$ for $r>r_0$, an essential requirement for a shape function. Furthermore, $B(r)$ is a decreasing function near $r=r_0$; so $B'(r)<0$, which implies that $b'(r_0)<1$, the flare-out condition. With the flare-out condition now satisfied, the shape function has produced the desired wormhole structure. For completeness let us

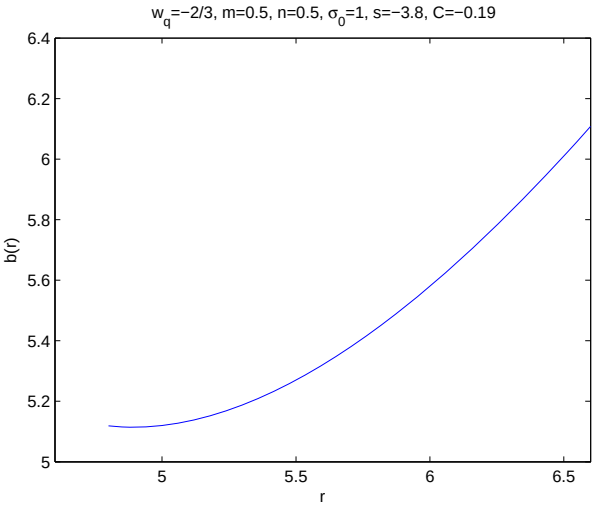

Figure 1. The shape function.

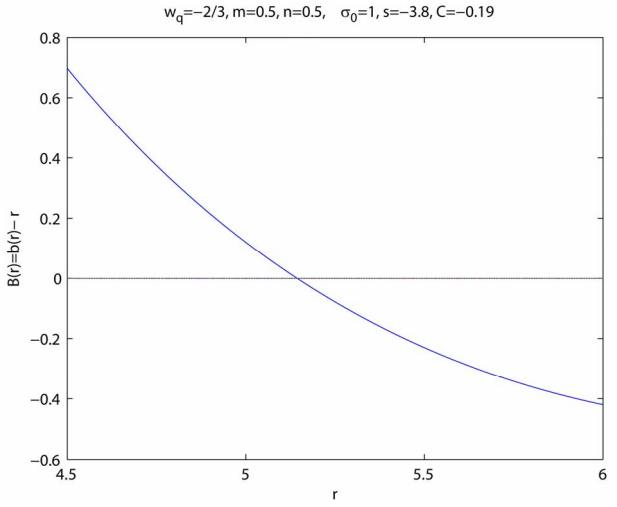

Figure 2. $B(r)=b(r)-r$ intersects the r-axis at $r=r_0$.

note that $r_0=5.143$ and $b'(r_0)=0.223$. (Suitable choices for C corresponding to other parameters will be discussed at the end of the section).

To the right of $r=r_0$, $b(r)$ keeps rising, but at $r=6.6$, $b'(r)$ is still less than unity. So at $r_1=6.6$, the interior shape function, Equation (22), can be joined smoothly to the exterior function

$$b_{ext}(r)=5.123\sqrt{r}-7.054.$$

To check this statement, observe that

$$b_{int}(6.6)=b_{ext}(6.6)=6.107,$$

while

$$b'_{int}(6.6)=b'_{ext}(6.6)=0.997.$$

To the right of $r=r_1$, $b(r)/r\to 0$ as $r\to\infty$, so that after adjusting the constant redshift function, the wormhole spacetime is asymptotically flat. (The components $g_{\hat{\theta}\hat{\theta}}$ and $g_{\hat{\phi}\hat{\phi}}$ are already continuous for the exterior and interior components, respectively [15-17]).

Returning to Equation (21), an example of an anisotropic case is $m=0.6$, $n=0.3$, $w_q=-2/3$, $\sigma_0=1$, and $s=-3.8$; a suitable choice for C is -0.12. The result is

$$b(r)=160.92r^{-2.6}+0.12r^2.$$

Here $r_0=5.58$ and $b'(r_0)=0.48$.

An example of a value of w_q closer to -1, the lower end of the quintessence range, is the following: $w_q=-0.8$, $m=n=0.5$, $\sigma_0=1$, and $s=-3.5$. Letting $C=-0.04$, the shape function is

$$b(r)=429.52r^{-2}+0.04r^{13/6}.$$

This time $r_0=10.32$ and $b'(r_0)=0.53$.

5. Could the Electric Field Be Eliminated?

The purpose of this section is to study conditions under which a combined model of quintessential and ordinary matter may be sufficient without the electric field E.

If E is eliminated, then the assumption of zero tidal forces becomes too restrictive. So we assume that $\Phi' = \eta/r$ for some nonzero constant η. This, in turn, means that

$$e^{\Phi} = A_1 r^{\eta}. \tag{23}$$

Now Equation (16) yields

$$e^{-\Lambda} = \frac{\beta}{\alpha} + \frac{A_2}{r^{\alpha}}. \tag{24}$$

Both A_1 and A_2 are positive integration constants. (The reason that A_2 has to be positive is that β is close to zero whenever w_q is close to $-1/3$); α and β now become (for $\eta \neq 0$)

$$\alpha = \frac{\frac{1}{2}\left(3w_q+1\right)+\frac{\eta^2}{4}+\eta\left[\frac{1}{2}\left(3w_q+1\right)-n\right]\Big/\left(m+1\right)}{\frac{\eta}{4}+\frac{1}{2}+\left[\frac{1}{2}\left(3w_q+1\right)m+n\right]\Big/\left(m+1\right)} \tag{25}$$

and

$$\beta = \frac{\frac{1}{2}\left(3w_q+1\right)}{\frac{\eta}{4}+\frac{1}{2}+\left[\frac{1}{2}\left(3w_q+1\right)m+n\right]\Big/\left(m+1\right)}. \tag{26}$$

The last two equations are similar to those in Ref. [12], which deals with galactic rotation curves.

As noted in Section 2, the shape function $b = b(r)$ is obtained from $e^{-\Lambda(r)}$, so that

$$b(r) = r\left(1-e^{-\Lambda(r)}\right) = r\left(1-\frac{\beta}{\alpha}-\frac{A_2}{r^{\alpha}}\right). \tag{27}$$

To meet the condition $b(r_0) = r_0$, we must have

$$1-\frac{\beta}{\alpha}-\frac{A_2}{r_0^{\alpha}} = 1.$$

Solving for r_0, we obtain the radius of the throat:

$$r_0 = \left(-\frac{\alpha}{\beta}A_2\right)^{1/\alpha}. \tag{28}$$

Since $A_2 > 0$, α and β must have opposite signs. From $b(r) = r\left(1-\beta/\alpha - A_2/r^{\alpha}\right)$, we have

$$b'(r_0) = 1-\frac{\beta}{\alpha}-A_2\left(1-\alpha\right)r_0^{-\alpha}$$

and, after substituting Equation (28),

$$b'(r_0) = 1-\frac{\beta}{\alpha}-A_2\left(1-\alpha\right)\left(-\frac{\beta}{\alpha A_2}\right),$$

which simplifies to $b'(r_0) = 1-\beta$. It follows immediately that if $\beta < 0$, then $b'(r_0) > 1$, so that the flare-out condition cannot be met. To get a value for β between 0 and 1, the exponent η in the redshift function, Equation (23), has to be negative and sufficiently large in absolute value. Such a value will cause α to be negative, which can best be seen from a simple numerical example: for convenience, let us choose $w_q = -1$, the lower end of the quintessence range, and $m = n = 0.1$. Then we must have $\eta < -6$. The result is a large positive numerator in Equation (25) because the last term is positive and $\eta^2/4$ is large. So α and β have opposite signs, as expected. (Observe that for the isotropic case, if $w_q = -1$, then the values of α and β are independent of m and n).

Continuing the numerical example, if we let $A_2 = 1$ and $\eta = -7$, then $\beta = 0.8$, $\alpha = -14.6$, and

$$b(r) = 1.055r - r^{15.6}.$$

From Equation (28), $r_0 = 0.820$, while

$$b'(r_0) = 1-\beta = 0.2.$$

As we have seen, $b'(r_0)$ is independent of A_2. So we are free to choose a smaller value in Equation (28) to obtain a larger throat size.

We conclude that we can readily find an interior wormhole solution around $r = r_0$ without E, provided that we are willing to choose a sufficiently large (and negative) value for η, resulting in what may be called an unpalatable shape function: $\Phi = \ln A_1 + \eta \ln r$. At the throat, $|\Phi'| = |\eta/r_0|$, which indicates the presence of an enormous radial tidal force, even for large throat sizes. (Recall that from Ref. [1], to meet the tidal constraint, we must have roughly $|\Phi'| < \left(10^8\,\mathrm{m}\right)^{-2}$. Such a wormhole would not be suitable for a humanoid traveler, but it may still be useful for sending probes or for transmitting signals.

The enormous tidal force is actually comparable to that of a solar-mass black hole of radius 2.9 km near the event horizon, making the solution physically plausible: since we have complete control over $b'(r_0)$ and r_0, we are not only able to satisfy the flare-out condition but we can place the throat wherever we wish. Moreover, the assumption $w_q = -1$ is equivalent to Einstein's cosmological constant, the best model for dark energy [18]. Also physically desirable is the assumption of isotropic pressure, *i.e.*, $m = n$ in the respective equations of state. As we have seen, in the isotropic case our conclusions are independent of m and n. So by placing the throat just outside the event horizon of a suitable black hole, it is possible in principle to construct a "transmission station" for transmitting signals to a distant advanced civilization and, conversely, receiving them. If such a

wormhole were to exist, it would be indistinguishable from a black hole at a distance. This suggests a possibility in the opposite direction: A black hole could conceivably be a wormhole fitting our description. The easiest way to test this hypothesis is to listen for signals, artificial or natural, emanating from a (presumptive) black hole.

6. Conclusions

This paper discusses a class of wormholes supported by a combined model consisting of quintessential matter and ordinary matter, first in Einstein-Maxwell gravity and then in Einstein gravity, that is, in the absence of an electric field. To obtain an exact solution, it was necessary to assume that the redshift function has the form $e^{\Phi(r)} = A_1 r^{\eta}$ for some constant η. In the Einstein-Maxwell case, this constant could be taken as zero, thereby producing a zero-tidal-force solution, which, in turn, would make the wormhole traversable for humanoid travelers. Without the electric field E, the exponent η has to be nonzero and leads to a less desirable solution with large tidal forces. Concerning the exact solution, it is shown in Ref. [19] that the existence of an exact solution implies the existence of a large set of additional solutions, suggesting that wormholes of the type discussed in this paper could occur naturally.

It is argued briefly in the Einstein case with a quintessential-dark-energy background that some black holes may actually be wormholes with enormous tidal forces, a hypothesis that may be testable.

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