

# Schwarzschild Geodesics in Terms of Elliptic Functions and the Related Red Shift

Günter Scharf

Institut für Theoretische Physik, Universität Zürich, Switzerland

E-mail: [scharf@physik.unizh.ch](mailto:scharf@physik.unizh.ch)

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## Abstract

Using Weierstrassian elliptic functions the exact geodesics in the Schwarzschild metric are expressed in a simple and most transparent form. The results are useful for analytical and numerical applications. For example we calculate the perihelion precession and the light deflection in the post-Einsteinian approximation. The bounded orbits are computed in the post-Newtonian order. As a topical application we calculate the gravitational red shift for a star moving in the Schwarzschild field.

**Keywords:** Schwarzschild Geodesics, Red Shift

## 1. Introduction

Schwarzschild geodesics are elliptic functions, therefore, they should be written as such. For this purpose the Weierstrassian elliptic functions are most useful because they lead to simple expressions. The reason for this is that the solution of quartic or cubic equations can be avoided in this way.

In a recent paper [1] an analytic solution for the geodesic in the weak-field approximation was given. As pointed out in that paper the progress in the astronomical observations call for better analytical methods. In this respect it is desirable to have the exact geodesics in a form most suited for applications. For the orbits in polar coordinates (next section) this goal can be achieved by using Weierstrass'  $P$ -function for which many analytical and numerical methods are known [2]. Considering the motion in time (section 4) the related  $\zeta$ - and  $\sigma$ -functions of Weierstrass appear.

Jacobian elliptic functions have been used by Darwin [8] for the form of the orbits. After some transformation our result (2.13) agrees with his. But in his second paper he abandons the elliptic functions because they were "not so well adapted to a study of the time in those orbits". Obviously the Weierstrass functions are better suited for the problem. Indeed, expressing them by Theta functions one gets the natural expansion of the geodesics in powers of the Schwarzschild radius, this expansion involves elementary functions only. The Weierstrass functions

have also been used by Hagihara [11]<sup>1</sup>. But he has chosen the variables and constants of integration in a manner which leads to less explicit results. So it is difficult to derive the post-Newtonian corrections to the geodesics given here from his formulas. As a topical application we finally calculate the red shift for a star moving in the Schwarzschild field. The geodesics are also needed for the study of modifications of general relativity ([10], section 5.13).

## 2. The Orbits in Polar Coordinates $r = r(\varphi)$

We take the coordinates  $x^0 = ct$ ,  $x^1 = r$ ,  $x^2 = \vartheta$ ,  $x^3 = \varphi$  and write the Schwarzschild metric in the form

$$ds^2 = \frac{r-r_s}{r} c^2 dt^2 - \frac{r}{r-r_s} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (2.1)$$

where  $r_s = \frac{2GM}{c^2}$  is the Schwarzschild radius. We shall assume  $c = 1$  in the following. The geodesic equation

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \quad (2.2)$$

with the Christoffel  $\Gamma_{\beta\gamma}^\alpha$  leads to the following three differential equations

<sup>1</sup>I am indebted to C. Lämmerzahl and P. Fiziev for bringing this reference to my attention

$$\frac{d^2 t}{ds^2} + \nu' \frac{dt}{ds} \frac{dr}{ds} = 0 \quad (2.3)$$

$$\frac{d^2 r}{ds^2} + \frac{\nu'}{2} e^{2\nu} \left( \frac{dt}{ds} \right)^2 - \frac{\nu'}{2} \left( \frac{dr}{ds} \right)^2 - r e^\nu \left( \frac{d\varphi}{ds} \right)^2 = 0 \quad (2.4)$$

$$\frac{d^2 \varphi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\varphi}{ds} = 0. \quad (2.5)$$

Here we have used the standard representation

$$1 - \frac{r_s}{r} = e^\nu \quad (2.6)$$

and have chosen  $\vartheta = \pi/2$  as the plain of motion. The Christoffel symbols can be taken from the Appendix of [3].

Multiplying (2.3) by  $\exp \nu$  we find

$$\frac{\partial}{\partial s} \left( e^\nu \frac{dt}{ds} \right) = 0$$

so that

$$e^\nu \frac{dt}{ds} = \text{const.} = E$$

$$\frac{dt}{ds} = E e^{-\nu}. \quad (2.7)$$

Next multiplying (2.5) by  $r^2$  we get

$$r^2 \frac{d\varphi}{ds} = \text{const.} = L,$$

hence

$$\frac{d\varphi}{ds} = \frac{L}{r^2}.$$

For the constants of integration we use the notation of Chandrasekhar [4].

Finally, substituting (2.7) and (2.8) into (2.5) and multiplying by  $2 \exp(-\nu) \times dr/ds$  we obtain

$$\frac{d}{ds} \left[ e^{-\nu} \left( \frac{dr}{ds} \right)^2 - E^2 e^{-\nu} + \frac{L^2}{r^2} \right] = 0. \quad (2.9)$$

Consequently, the square bracket is equal to another constant  $= b$ . Then the resulting differential equation can be written as

$$\left( \frac{dr}{ds} \right)^2 = E^2 + e^\nu \left( b - \frac{L^2}{r^2} \right). \quad (2.10)$$

The constant  $b$  can be arbitrarily adjusted by rescaling the affine parameter  $s$ . Below we shall take  $b = -m^2$  where  $m$  is the rest mass of the test particle. This will enable us to include null geodesics (light rays) with  $m^2 = 0$ . Each geodesic is characterized by two constants of the motion: energy  $E$  and angular momentum  $L$ .

Taking the square root of (2.10) and dividing by (2.8) we get

$$\frac{dr}{d\varphi} = \sqrt{\frac{E^2 - m^2}{L^2} r^4 + \frac{m^2}{L^2} r_s r^3 - r^2 + r_s r} \equiv \sqrt{f(r)}. \quad (2.11)$$

Now  $\varphi = \varphi(r)$  can be written as an elliptic integral. However, it is better to consider the inverse  $r = r(\varphi)$  in terms of elliptic function by using a formula of Weierstrass ([5], p.452). Let the quartic  $f(r)$  be written as

$$f(r) = a_0 r^4 + 4a_1 r^3 + 6a_2 r^2 + 4a_3 r + a_4, \quad (2.12)$$

and let  $r_1$  be a zero  $f(r_1) = 0$ , then a solution of (2.11) is given by

$$r = r_1 + \frac{f'(r_1)}{4P(\varphi; g_2, g_3) - f''(r_1)/6}. \quad (2.13)$$

Here  $P(\varphi; g_2, g_3)$  is Weierstrass'  $P$ -function with invariants

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \quad (2.14)$$

$$g_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4. \quad (2.15)$$

In our case we have  $a_4 = 0$ . For the convenience of the reader we reproduce the short proof in the Appendix.

The result (2.13) is not yet the solution of our problem because it contains too many constants: the invariants  $g_2, g_3$  and the derivatives of  $f$  depend on  $E, L$ , but in addition the zero  $r_1$  appears. Of course one could calculate  $r_1$  as a function of  $E, L$  by solving the quartic equation  $f(r) = 0$ , but this gives complicated expressions. It is much better to use  $r_1$  and a second zero  $r_2$  as constants of integration instead of  $E, L$ . This is even desirable from the astronomers point of view because the zeros of derivative (2.11) are turning points of the geodesic, for example in case of a bounded orbit they can be identified with the perihelion and aphelion of the orbit. In order to express  $E, L$  by  $r_1, r_2$  we write our quartic in the form

$$f(r) = a_0 r (r - r_1)(r - r_2)(r - r_3) \quad (2.16)$$

and compare the coefficients of  $r^3, r^2, r$  with (2.12). This leads to

$$4a_1 = -a_0 (r_1 + r_2 + r_3) = \frac{m^2}{L^2} r_s$$

$$6a_2 = a_0 (r_1 r_2 + r_1 r_3 + r_2 r_3) = -1 \quad (2.17)$$

$$4a_3 = -a_0 r_1 r_2 r_3 = r_s.$$

Since

$$a_0 = \frac{E^2 - m^2}{L^2} \quad (2.18)$$

we can solve for

$$\frac{m^2}{L^2} r_s = \frac{r_1 + r_2 + r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}, \tag{2.19}$$

$$E^2 - m^2 = -\frac{m^2 r_s}{r_1 + r_2 + r_3}. \tag{2.20}$$

In addition we obtain the third zero

$$r_3 = r_s \frac{r_1 r_2}{r_1 r_2 - r_1 r_s - r_2 r_s}. \tag{2.21}$$

The relations (2.19-21) allow to express everything in terms of  $r_1, r_2$ . For the invariants we find

$$g_2 = \frac{1}{12} - \frac{m^2}{4L^2} r_s^2 = \frac{1}{12} - \frac{r_s}{4} \frac{r_1 + r_2 + r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3} \tag{2.22}$$

$$g_3 = \frac{1}{6^3} - \frac{m^2}{48L^2} r_s^2 - \frac{a_0}{16} r_s^2 = \frac{1}{6^3} - \frac{1}{48} \frac{(r_1 + r_2 + r_3) r_s}{r_1 r_2 + r_1 r_3 + r_2 r_3} + \frac{1}{16} \frac{r_s^2}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{2.23}$$

Here  $r_3$  has to be substituted by (2.21). For the derivatives  $f'(r_1)$ ,  $f''(r_1)$  which appear in our solution (2.13) we obtain

$$f'(r_1) = \frac{-r_1(r_1 - r_2)(r_1 - r_3)}{r_1 r_2 + r_1 r_3 + r_2 r_3} \tag{2.24}$$

$$f''(r_1) = -2 \frac{(r_1 - r_2)(r_1 - r_3)r_1 + (r_1 - r_2)(r_1 - r_3)}{r_1 r_2 + r_1 r_3 + r_2 r_3}. \tag{2.25}$$

With these substitutions the result (2.13) gives all possible geodesics in the form  $r = r(\varphi; r_1, r_2)$ . This will be discussed in the next section.

As a first check of the solution (2.13) we consider the Newtonian limit. Let the two zeros  $r_1, r_2$  be real and large compared to the Schwarzschild radius  $r_s$  in absolute value. Then neglecting  $O(r_s)$  in the invariants (2.22-23) the  $P$ -function becomes elementary ([2], p.652, equation 18.12.27):

$$P\left(\varphi; \frac{1}{12}, 6^{-3}\right) = -\frac{1}{12} + \frac{1}{4 \sin^2 \varphi/2} = -\frac{1}{12} + \frac{1}{2(1 - \cos \varphi)}. \tag{2.26}$$

The leading order in the derivatives of  $f$  is given by

$$f'(r_1) = -\frac{r_1}{r_2} (r_1 - r_2) \\ f''(r_1) = -2 \left( 3 \frac{r_1}{r_2} - 2 \right). \tag{2.27}$$

It is convenient to introduce the eccentricity  $\varepsilon$  by

$$\frac{r_1}{r_2} = \frac{1 - \varepsilon}{1 + \varepsilon}. \tag{2.28}$$

Using all this in (2.13) we find the wellknown conic

$$r = \frac{(1 + \varepsilon) r_1}{1 + \varepsilon \cos \varphi}. \tag{2.29}$$

Assuming both zeros  $r_1, r_2$  positive and  $r_1 < r_2$  we have  $\varepsilon < 1$  and the orbit is an ellipse with perihelion  $r_1$  and aphelion  $r_2$ . In the hyperbolic case  $\varepsilon > 1$  we see from (2.28) that if  $r_1$  is positive  $r_2$  must be negative. Then there is only one physical turning point  $r_1$  which is the point of closest approach. The latter always corresponds to  $\varphi = 0$ . The relativistic corrections to (2.29) are calculated in the following section.

### 3. Discussion of the Solution

The solution  $r = r(\varphi)$  (2.13) is an elliptic function of  $\varphi$  which implies that it is doubly-periodic ([2], p.629 or any book on elliptic functions). The values of the two half-periods  $\omega, \omega'$  depend on the three roots of the fundamental cubic equation

$$4e^3 - g_2 e - g_3 = 0. \tag{3.1}$$

Again it is not necessary to solve this equation because the solutions  $e_j, j=1,2,3$  can be easily obtained from the roots  $0, r_1, r_2, r_3$  of our quartic  $f(r) = 0$ . To see this we transform  $f(r)$  to Weierstrass' normal form as follows. First we set  $r = 1/x$  so that from (2.16) we get

$$f(r) = \frac{1}{x^4} (4a_3 x^3 + 6a_2 x^2 + 4a_1 x + a_0)$$

Next we remove the quadratic term by introducing

$$\frac{1}{r} = x = \frac{1}{a_3} \left( e - \frac{a_2}{2} \right). \tag{3.2}$$

This gives the normal form of Weierstrass

$$f(r) = \frac{a_3^2}{(e - \frac{1}{2} a_2)^4} (4e^3 - g_2 e - g_3), \tag{3.3}$$

with the above invariants (2.14-15). That means roots of  $f(r)$  are simply related to roots of (3.1) by the transformation

$$e_j = \frac{a_3}{r_j} + \frac{a_2}{2} = \frac{r_s}{4r_j} - \frac{1}{12}. \tag{3.4}$$

The cubic equation (3.1) with real coefficients has either three real roots or one real and two complex conjugated roots. The first case occurs if the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{3.5}$$

is positive, in the second case  $\Delta$  is negative. In terms of the roots  $\Delta$  is given by ([2], p.629, equation 18.1.8)

$$\Delta = 16(e_1 - e_2)^2 (e_2 - e_3)^2 (e_3 - e_1)^2. \tag{3.6}$$

The physically interesting orbits correspond to the first case of real roots. If we have two complex conjugated zeros  $r_2 = r_1^*$  then (2.28) implies that the eccentricity  $\varepsilon$  is imaginary. Such orbits have been discussed by Chandrasekhar ([4], p.111). Now we discuss the various cases.

### 3.1. Bound Orbits

In this case we have two positive turning points  $r_2 > r_1 > 0$ , consequently there are three real roots  $e_1 > 0 > e_2 > e_3$  given by

$$e_1 = \frac{r_s}{4r_3} - \frac{1}{12} = \frac{1}{6} - \frac{r_s}{4} \frac{r_1 + r_2}{r_1 r_2} \tag{3.7}$$

$$e_2 = -\frac{1}{12} + \frac{r_s}{4r_1}, \quad e_3 = -\frac{1}{12} + \frac{r_s}{4r_2}, \tag{3.8}$$

Our convention is chosen in agreement with [2]. The real half-period  $\omega$  of the  $P$ -function is given by ([2], p.549, equation 18.9.8)

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \frac{K(k^2)}{\sqrt{e_1 - e_3}} \tag{3.9}$$

$$\omega = \pi \left\{ 1 + \frac{3r_s}{4} \frac{r_1 + r_2}{r_1 r_2} + \frac{3r_s^2}{8(r_1 r_2)^2} \left[ (2r_1 + r_2)(2r_2 + r_1) + \frac{3}{8}(r_2 - r_1)^2 \right] \right\} = \pi \left\{ 1 + \frac{3}{2} \frac{r_s}{r_1} \frac{1}{1 + \varepsilon} + \frac{3}{18} \frac{r_s^2}{r_1^2} \frac{18 + \varepsilon^2}{(1 + \varepsilon)^2} + O(r_s^3) \right\} \tag{3.12}$$

The perihelion precession is given by  $\Delta\varphi = 2(\omega - \pi)$ . Then the order  $r_s$  in (3.12) is Einstein's result and the  $O(r_s^2)$  gives the correction to it. The accurate computation of the half-period is necessary to control the orbit in the large.

To compute the relativistic corrections for  $r(\varphi)$  from (2.13) we express the  $P$ -function by Theta functions ([5], p.464)

$$\begin{aligned} \mathcal{G}_1(z, q) &= 2q^{1/4} (\sin z - q^2 \sin 3z + q^6 \sin 5z - \dots) \\ \mathcal{G}_2(z, q) &= 2q^{1/4} (\cos z + q^2 \cos 3z + q^6 \cos 5z + \dots) \\ \mathcal{G}_3(z, q) &= 1 + 2q (\cos 2z + q^3 \cos 4z + q^8 \cos 6z + \dots) \\ \mathcal{G}_4(z, q) &= 1 - 2q (\cos 2z - q^3 \cos 4z + q^8 \cos 6z - \dots). \end{aligned} \tag{3.13}$$

Here  $q$  is the so-called Nome ([2], eq. 17.3.21)

$$q = \frac{k^2}{16} + 8 \left( \frac{k^2}{16} \right)^2 + \dots \tag{3.14}$$

These series are rapidly converging since  $k^2$  is small

$$P(\varphi) - \frac{f''(r_1)}{24} = \frac{1 + \varepsilon - 2\varepsilon \sin^2 \phi}{4(1 + \varepsilon) \sin^2 \phi} \left[ 1 + \frac{r_s}{r_1} \frac{1}{1 + \varepsilon \cos 2\phi} \left( -3 - \frac{\varepsilon}{2} (1 - \cos \phi) + 2\varepsilon \frac{3 + \varepsilon}{1 + \varepsilon} \sin^2 \phi \right) \right] + O\left(\frac{r_s}{r_1}\right) \tag{3.18}$$

where  $K(k^2)$  is the complete elliptic integral of the first kind with parameter

$$\begin{aligned} k^2 &= \frac{e_2 - e_3}{e_1 - e_3} = r_s \frac{r_2 - r_1}{r_1 r_2 - r_s (2r_1 + r_2)} \\ &= \frac{r_s}{r_1} \frac{2\varepsilon}{1 + \varepsilon} \left( 1 - r_s \frac{3 - \varepsilon}{1 + \varepsilon} \right)^{-1}. \end{aligned} \tag{3.10}$$

As a first application let us give the post-Einsteinian correction to the orbital precession. If  $k^2$  (3.10) is small we can use the expansion ([2], p.591, equation 17.3.11)

$$K(k^2) = \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \right] \tag{3.11}$$

From the roots  $e_1, e_3$  we find

$$\frac{1}{\sqrt{e_1 - e_3}} = 2 \left[ 1 + \frac{r_s}{2} \frac{2r_1 + r_2}{r_1 r_2} + \frac{3}{8} r_s^2 \left( \frac{2r_1 + r_2}{r_1 r_2} \right)^2 + O(r_s^3) \right].$$

This finally leads to the half-period

(3.10), they give the natural expansion in powers of the Schwarzschild radius  $r_s$ . Now the  $P$ -function is given in terms of Theta functions by ([2], eq. 18.10.5)

$$\begin{aligned} P(\varphi) &= e_2 + \frac{\pi^2}{4\omega^2} \left( \frac{\mathcal{G}'_1(0) \mathcal{G}_3(\phi)}{\mathcal{G}_3(0) \mathcal{G}_1(\phi)} \right)^2 \\ &= -\frac{1}{12} + \frac{r_s}{4r_1} + \frac{\pi^2}{4\omega^2} \frac{1}{\sin^2 \phi} \left[ 1 + 4q (\cos^2 \phi - 1) + O(q^2) \right], \end{aligned} \tag{3.15}$$

where

$$\phi = \frac{\pi}{2\omega} \varphi. \tag{3.16}$$

Using

$$\begin{aligned} f'(r_1) &= 2r_1 \left( \frac{\varepsilon}{1 + \varepsilon} - \frac{r_s}{r_1} \frac{3\varepsilon + \varepsilon^2}{(1 + \varepsilon)^2} \right) \\ f''(r_1) &= -2 \frac{1 - 5\varepsilon}{1 + \varepsilon} + 6 \frac{r_s}{r_1} \frac{1 - 4\varepsilon - \varepsilon^2}{(1 + \varepsilon)^2} \end{aligned} \tag{3.17}$$

this leads to

Substituting this into (2.13) gives the desired orbit to  $O(r_s)$

$$\frac{r(\phi)}{r_1} = \frac{1+\varepsilon}{1+\varepsilon \cos 2\phi} + \frac{r_s}{r_1} \varepsilon \frac{2 \sin^2 \phi}{1+\varepsilon \cos 2\phi} \left\{ \frac{1}{1+\varepsilon \cos 2\phi} \left[ 3 + \frac{\varepsilon}{2}(1-\cos \phi) - 2\varepsilon \frac{3+\varepsilon}{1+\varepsilon} \sin^2 \phi \right] - \frac{3+\varepsilon}{1+\varepsilon} \right\}. \quad (3.19)$$

It is important to insert the period  $\omega$  in  $\phi$  (3.16) according to (3.12) in order to describe the perihelion precession correctly.

If the two roots  $r_1 = r_2$  coincide, it follows from (2.24) that  $f'(r_1) = 0$ . According to (2.13) we then have circular motion  $r = r_1$ . If all three zeros coincide  $r_1 = r_2 = r_3$  then (2.21) gives  $r_3 = 3r_s$  which is the innermost circular orbit.

### 3.2. Unbound Orbits

In this case there is only one physical point, the point of closest approach  $r_1$ . The other root  $r_2$  is negative, therefore, it is better to use the eccentricity  $\varepsilon$  (2.27) as the second basic quantity. With  $r_3$  given by (2.21) we then have

$$r_1 > r_3 > 0 > r_2 = \frac{1+\varepsilon}{1-\varepsilon} r_1, \quad (3.20)$$

because  $\varepsilon \geq 1$ . The periodicity of (2.13) in  $\phi$  is now realized by a jump to an unphysical branch with  $r < 0$ . In reality a comet moves on one branch only, but it is a tricky problem to decide on which one. This is due to the fact that the period differs a little from  $2\pi$  as in the bounded case. Consequently, neighboring physical branches  $r > 0$  are a little rotated against each other and the distinction between them is not easy. The quantity of physical interest is the direction  $\varphi_\infty$  of the asymptote. It follows from the original equation (2.11) by integrating the inverse over  $r$  from  $r_1$  to  $\infty$

$$\varphi_\infty = \int_{r_1}^{\infty} \frac{dr}{\sqrt{f(r)}}. \quad (3.21)$$

This is an elliptic integral which can be transformed to Legendre's normal form

$$\varphi_\infty = \frac{\mu}{\sqrt{a_0}} \int_0^{\Phi_2} \frac{d\Phi}{\sqrt{1-k^2 \sin^2 \Phi}} \quad (3.22)$$

by the transformation ([6], vol.II, p.308)

$$\sin^2 \Phi = \frac{r_3 - r_2}{r_1 - r_2} \frac{r - r_1}{r - r_3}, \quad \sin^2 \Phi_2 = \frac{r_3 - r_2}{r_1 - r_2}. \quad (3.23)$$

The parameter  $k^2$  in (3.22) is given by

$$k^2 = \frac{r_3}{r_1} \frac{r_1 - r_2}{r_3 - r_2}, \quad (3.24)$$

and

$$\mu = \frac{2}{\sqrt{r_1(r_3 - r_2)}}. \quad (3.25)$$

The integral (3.22) is an incomplete elliptic integral of the first kind

$$\varphi_\infty = \frac{\mu}{\sqrt{a_0}} F(\Phi_2, k^2) \quad (3.26)$$

which has the expansion ([6], vol.II, p.313)

$$F(\Phi_2, k^2) = \Phi_2 + \frac{k^2}{4} \left( \Phi_2 - \frac{1}{2} \sin 2\Phi_2 \right) + O(k^4). \quad (3.27)$$

For small  $r_s/r_1$  we find

$$k^2 = \frac{2\varepsilon}{\varepsilon+1} \frac{r_s}{r_1} + O(r_s r_1)^2$$

$$\frac{\mu}{\sqrt{a_0}} = 2 + \frac{3-\varepsilon}{1+\varepsilon} \frac{r_s}{r_1} + O(r_s r_1)^2$$

$$\sin^2 \Phi_2 = \frac{\varepsilon+1}{2\varepsilon} \left( 1 - \frac{r_3}{r_1} \right).$$

This gives

$$\cos 2\Phi_2 = -\frac{1}{\varepsilon} - \frac{\varepsilon-1}{\varepsilon} \frac{r_3}{r_1}$$

and

$$\varphi_\infty = 2\Phi_2 + \frac{r_s}{r_1} \left( \frac{3}{\varepsilon+1} \Phi_2 - \frac{\varepsilon}{2\varepsilon+2} \sin 2\Phi_2 \right). \quad (3.28)$$

It is convenient to calculate

$$\cos \varphi_\infty = -\frac{1}{\varepsilon} - \frac{r_s}{r_1} \left( \frac{\varepsilon-1}{2\varepsilon} + 3 \frac{\sqrt{\varepsilon^2-1}}{\varepsilon(\varepsilon+1)} \Phi_2 \right) + O(r_s^2). \quad (3.29)$$

The leading order is the Newtonian asymptote of the hyperbola.

### 3.3. Null Geodesics

For  $m^2 = 0$  there is only one constant of integration in the quartic (2.11)

$$f(r) = \frac{r^4}{d^2} - r^2 + r_s r$$

which is the so-called impact parameter

$$d = \frac{L}{E}. \quad (3.30)$$

Now it is necessary to calculate the roots of  $f(r) = 0$ . This is easily done by means of a power series expansion

$$r = c_0 d + c_1 r_s + c_2 r_s^2 + \dots$$

We find

$$\begin{aligned} r_1 &= d \left( 1 - \frac{\delta}{2} - \frac{3}{8} \delta^2 + O(\delta^3) \right) \\ r_2 &= -d \left( 1 + \frac{\delta}{2} - \frac{3}{8} \delta^2 + O(\delta^3) \right) \\ r_3 &= d(\delta + \delta^3), \end{aligned} \quad (3.31)$$

where

$$\delta = \frac{r_s}{d} \quad (3.32)$$

and we have ordered the zeros in the same way as in (3.20). Then as in the last subsection we can calculate the direction of the asymptote (3.21) which now is equal to

$$\varphi_\infty = \mu d F(\Phi_1, k^2) \quad (3.33)$$

with  $\Phi_1$  given by (3.23)

$$\sin^2 \Phi_1 = \frac{1}{2} + \frac{3}{4} \delta + O(\delta^3) \quad (3.34)$$

and  $k^2$  by (3.24)

$$k^2 = 2\delta \left( 1 - \delta + \frac{25}{8} \delta^2 \right) \quad (3.35)$$

and  $\mu$  by (3.25)

$$\mu = \frac{2}{d} \left( 1 - \frac{\delta}{2} + \frac{9}{8} \delta^2 \right). \quad (3.36)$$

We want to calculate the light deflection in the post-Einsteinian approximation. Using again the expansion (3.27) we have

$$\varphi_\infty = 2 \left( 1 - \frac{\delta}{2} + \frac{9}{8} \delta^2 \right) \left[ \Phi_1 + \frac{\delta}{2} (1 - \delta) \left( \Phi_1 - \frac{1}{2} \sin 2\Phi_1 \right) \right].$$

From (3.34) we obtain

$$\Phi_1 = \frac{\pi}{4} + \frac{3}{4} \delta + O(\delta^3).$$

Then up to  $O(\delta^2)$  we find

$$\varphi_\infty = \frac{\pi}{2} + \delta + \left( \frac{3}{4} + \frac{3}{16} \pi \right) \delta^2. \quad (3.37)$$

The deflection angle is given by

$$\Delta\varphi = 2 \left( \varphi_\infty - \frac{\pi}{2} \right) = 2\delta + \left( \frac{3}{2} + \frac{3}{8} \pi \right) \delta^2. \quad (3.38)$$

Instead of the impact parameter  $d$  in  $\delta$  (3.32) we would like to use the distance of closest approach  $r_1$

(2.24) in the form

$$\delta_1 = \frac{r_s}{r_1}. \quad (3.39)$$

The two are related by

$$\delta = \delta_1 - \frac{1}{2} \delta_1^2 + O(\delta_1^3)$$

which leads to

$$\Delta\varphi = 2\delta_1 + \left( \frac{1}{2} + \frac{3}{8} \pi \right) \delta_1^2. \quad (3.40)$$

The first term  $2\delta_1$  is Einstein's result.

#### 4. The Motion in Time $t = t(\varphi)$

By dividing (2.7) by (2.10) we find

$$\begin{aligned} \frac{dt}{dr} &= E \frac{r}{r-r_s} \left[ E^2 - \left( 1 - \frac{r_s}{r} \right) \left( m^2 + \frac{L^2}{r^2} \right) \right]^{-1/2} \\ &= \frac{E}{L} \frac{r^3}{(r-r_s) \sqrt{f(r)}}. \end{aligned} \quad (4.1)$$

We choose  $t=0$  at the point of closest approach  $r=r_1$  and get

$$t = \frac{E}{L} \int_{r_1}^r \frac{dx}{\sqrt{f(x)}} \left( x^2 + r_s x + r_s^2 + \frac{r_s^3}{x-r_s} \right). \quad (4.2)$$

This is a sum of elliptic integrals of first, second and third kind. The coordinate time is given as a function of  $r$  by calculating these. However, we want  $t$  as a function of  $\varphi$  and, therefore, use the substitution (6.7) of the Appendix again

$$\begin{aligned} x &= r_1 + \frac{f'(r_1)}{4P(\varphi) - f''(r_1)/6} \\ \frac{dx}{\sqrt{f(x)}} &= d\varphi, \end{aligned} \quad (4.3)$$

where the last relation follows from (2.11).

The last integral  $O(r_s^3)$  in (4.2) is a small correction and we neglect it at the moment. Then integrals of the following form remain to be calculated

$$J_n(\varphi) = \int_0^\varphi \frac{du}{(P(u) - P(v))^n} \quad (4.4)$$

where we have set

$$P(v) = \frac{f''(r_1)}{24}. \quad (4.5)$$

Such integrals are known ([7], vol.4, p.109-110)

$$J_1(\varphi) = \frac{1}{P'(v)} \left[ 2\zeta(v)\varphi + \log \frac{\sigma(v-\varphi)}{\sigma(v+\varphi)} \right] \quad (4.6)$$

$$J_2(\varphi) = -\frac{1}{P'^2(v)} \left[ \zeta(\varphi+v) + \zeta(\varphi-v) + 2P(v)\varphi + P''(v)J_1(\varphi) \right]. \quad (4.7)$$

These results are easily verified by differentiating and using addition formulas. Of course  $J_0(\varphi)$  is just the polar angle  $\varphi$ . Then (4.2) leads to the desired result for  $t(\varphi)$ :

$$t(\varphi) = \frac{E}{L} \{ \tau_0\varphi + \tau_1 J_1(\varphi) + \tau_2 J_2(\varphi) \} + O(r_s^3). \quad (4.8)$$

where

$$\tau_0 = r_1^2 + r_s r_1 + r_s^2 \quad (4.9)$$

$$\tau_1 = \left( \frac{r_1}{2} + \frac{r_s}{4} \right) f'(r_1) \quad (4.10)$$

$$\tau_2 = \frac{f'^2(r_1)}{16}. \quad (4.11)$$

Again we evaluate this for bounded orbits in the post-Newtonian approximation by means of the expansion in Theta functions.

The quantity  $v$  in (4.8-11) is given as the zero of (3.18). Introducing

$$V = \frac{\pi}{2\omega} v \quad (4.12)$$

we find

$$\cos 2V = -\frac{1}{\varepsilon} - \frac{r_s}{r_1} \frac{3\varepsilon + 1}{2\varepsilon} \equiv -\beta. \quad (4.13)$$

Since  $\varepsilon < 1$ ,  $V$  is complex:

$$2V = \pi + i \log \left( \beta + \sqrt{\beta^2 - 1} \right) \equiv \pi + 2ib \quad (4.14)$$

Using ([2], eq.18.10.6)

$$\begin{aligned} P'(v) &= -\frac{\pi^3}{4\omega^3} \frac{\mathcal{G}_2(V)\mathcal{G}_3(V)\mathcal{G}_4(V)\mathcal{G}_1^3(0)}{\mathcal{G}_2(0)\mathcal{G}_3(0)\mathcal{G}_4(0)\mathcal{G}_1^3(V)} \\ &= -\frac{\pi^3}{4\omega^3} \frac{\cos V}{\sin^3 V} + O(q^2) \end{aligned}$$

we obtain

$$\begin{aligned} P'(v) &= i \frac{\pi^3}{2\omega^3} \frac{(\beta^2 - 1)^{1/2}}{(\beta + 1)^2} + O(r_s^2) \\ &= i \frac{\varepsilon}{2} \frac{\sqrt{1 - \varepsilon^2}}{(1 + \varepsilon)^2} \left( 1 + \frac{r_s}{r_1} \frac{3\varepsilon^2 + 4\varepsilon - 5}{1 - \varepsilon^2} \right). \end{aligned} \quad (4.15)$$

Similarly we calculate  $\zeta(v)$  from ([2], eq.18.10.7)

$$\zeta(v) = \frac{\eta v}{\omega} + \frac{\pi}{2\omega} \frac{\mathcal{G}'_1(V)}{\mathcal{G}_1(V)} \quad (4.16)$$

where

$$\eta = \zeta(\omega) = -\frac{\pi^2}{12\omega} \frac{\mathcal{G}'''_1(0)}{\mathcal{G}'_1(0)} \quad (4.17)$$

and  $\sigma(z)$  from ([2], eq.18.10.8)

$$\sigma(z) = \frac{2\omega}{\pi} \exp \left( \frac{\eta z^2}{2\omega} \right) \frac{\mathcal{G}_1(Z)}{\mathcal{G}'_1(0)}, \quad Z = \frac{\pi z}{2\omega}. \quad (4.18)$$

This implies

$$\begin{aligned} \log \frac{\sigma(v-\varphi)}{\sigma(v+\varphi)} &= -2 \frac{\eta}{\omega} v\varphi + \log \frac{\sin(V-\Phi)}{\sin(V+\Phi)} + O(q^2) \\ &= -2 \frac{\eta}{\omega} v\varphi + \log \frac{\cos(ib-\Phi)}{\cos(ib+\Phi)} \\ &\equiv -2 \frac{\eta}{\omega} v\varphi + 2i\alpha(\varphi), \end{aligned} \quad (4.19)$$

where  $\alpha$  is given by

$$\alpha(\varphi) = \arctan(\tan \Phi \tanh b). \quad (4.20)$$

Then  $J_1(\varphi)$  (4.6) is equal to

$$J_1(\varphi) = -\frac{2\omega^2}{\pi^2} (\beta + 1)\varphi + \frac{4\omega^3}{\pi^3} \frac{(\beta + 1)^2}{\sqrt{\beta^2 - 1}} \alpha(\varphi) + O\left(\frac{r_s}{r_1}\right)^2.$$

To expand this in the post-Newtonian order we first calculate  $\alpha(\varphi)$  from

$$\tan \alpha = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan \Phi + \frac{r_s}{r_1} \varepsilon \frac{1+3\varepsilon}{1-\varepsilon^2} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan \Phi.$$

Introducing

$$\psi = 2 \arctan \left( \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan \Phi \right) \quad (4.21)$$

we get

$$\alpha = \frac{\psi}{2} + \frac{r_s}{r_1} \frac{\varepsilon}{4} \frac{1+3\varepsilon}{1-\varepsilon^2} \sin \psi.$$

As before in (3.19) we do not expand  $\Phi$  in (4.21). However, if one does so one finds a contribution  $O(r_s/r_1)$

$$\psi = 2 \arctan \left( \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan(\varphi/2) \right) - \frac{3}{2} \frac{r_s}{r_1} \frac{\sqrt{1-\varepsilon}}{\sqrt{1+\varepsilon}} \frac{\varphi}{1 + \varepsilon \cos \varphi} \quad (4.22)$$

where the first term, say  $\psi_N$ , is the parameter which appears in Newtonian mechanics (Kepler's equation, see

below (4.29)). Now the expansion of  $J_1(\varphi)$  is given by

$$\begin{aligned}
 J_1(\varphi) &= J_1^0(\varphi) + \frac{r_s}{r_1} J_1^1(\varphi) \\
 J_1^0(\varphi) &= 2 \frac{1+\varepsilon}{\varepsilon} \left( \psi \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} - \varphi \right) \\
 J_1^1(\varphi) &= \frac{1}{\varepsilon} \left[ -\varphi(7+3\varepsilon) + \varepsilon \right] \frac{1+3\varepsilon}{1-\varepsilon^2} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \sin \psi \\
 &\quad + 2\psi \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \frac{5-4\varepsilon-3\varepsilon^2}{1-\varepsilon}.
 \end{aligned}
 \tag{4.23}$$

For  $\varepsilon \rightarrow 0$  we have the simple finite limit

$$J_1(\varphi) \Big|_{\varepsilon=0} = \frac{2\omega}{\pi} \left( 1 + 3 \frac{r_s}{r_1} \right) (2\Phi - \sin 2\Phi)$$

$$J_2^0(\varphi) = \frac{4(1+\varepsilon)^2}{\varepsilon(1-\varepsilon)} \left[ \varphi - \frac{(1+\varepsilon)\sin 2\Phi}{1+\varepsilon \cos 2\Phi} + \frac{2\varepsilon-1}{2(1+\varepsilon)} J_1^0(\varphi) \right]
 \tag{4.25}$$

$$\begin{aligned}
 J_2^1(\varphi) &= \frac{2(1+\varepsilon)^2}{\varepsilon(1-\varepsilon)} \left[ \varphi \frac{15-12\varepsilon-11\varepsilon^2}{1-\varepsilon^2} - \frac{\sin 2\Phi}{1+\varepsilon \cos 2\Phi} \left( \frac{1+4\varepsilon+3\varepsilon^2}{1+\varepsilon \cos 2\Phi} + \frac{17-13\varepsilon-12\varepsilon^2}{1-\varepsilon} \right) \right. \\
 &\quad \left. + J_1^0(\varphi) \frac{6-15\varepsilon+8\varepsilon^2-\varepsilon^3}{(1+\varepsilon)^2(1-\varepsilon)} - J_1^1(\varphi) \frac{1-2\varepsilon}{1+\varepsilon} \right].
 \end{aligned}
 \tag{4.26}$$

Again we do not expand  $\Phi$  (3.16) in order to keep the perihelion precession as precise as possible. The limit for  $\varepsilon \rightarrow 0$  is equal to

$$J_2(\varphi) \Big|_{\varepsilon=0} = 2 \frac{\omega}{\pi} \left( 1 + 6 \frac{r_s}{r_1} \right) (6\Phi - 4 \sin 2\Phi + \sin 2\Phi \cos 2\Phi).$$

In the final result (4.8) for the time

$$t(\varphi) = t_0(\varphi) + \frac{r_s}{r_1} t_1(\varphi)
 \tag{4.27}$$

the pre-factor  $E/L$  also gives a correction:

$$\frac{E}{L} = \sqrt{\frac{2}{r_s r_1 (1+\varepsilon)}} \left( 1 - \frac{r_s}{r_1 (1+\varepsilon)} \right)$$

$$t_1(\varphi) = t_0(\varphi) \frac{\varepsilon}{1+\varepsilon} + \varepsilon \sqrt{\frac{2r_1^3}{r_s(1+\varepsilon)}} \left[ \frac{1}{1+\varepsilon} J_1^1(\varphi) - \frac{1}{2} \frac{\varepsilon+5}{(1+\varepsilon)^2} J_1^0(\varphi) + \frac{\varepsilon}{4(1+\varepsilon^2)} J_2^1(\varphi) - \frac{\varepsilon}{2} \frac{3+\varepsilon}{(1+\varepsilon)^3} J_2^0(\varphi) \right].
 \tag{4.30}$$

As in (3.19) the post-Newtonian correction vanishes for circular motion  $\varepsilon = 0$ .

### 5. Gravitational Red Shift

The study of Schwarzschild geodesics is relevant for the investigation of the recently discovered S-stars near the

Galactic Center ([1] and references given there). These stars move in the strong gravitational field of the central black hole so that general relativistic effects are observable and the Schwarzschild metric  $g_{\mu\nu}$  is a fairly good description of the situation. The measurable quantity of interest is the red shift of spectral lines in the light emitted by the moving star. Therefore we finally consider

which also follows directly from the definition (4.4).

To expand

$$\begin{aligned}
 J_2(\varphi) &= -\frac{1}{P'^2(v)} \left[ 2 \left( \frac{\eta}{\omega} + P(v) \right) \varphi \right. \\
 &\quad \left. - \frac{\pi}{\omega \cos 2\Phi + \beta} + P''(v) J_1(\varphi) \right] \\
 &= J_2^0(\varphi) + \frac{r_s}{r_1} J_2^1(\varphi)
 \end{aligned}$$

we need

$$\begin{aligned}
 P''(v) &= 6P^2(v) - \frac{g_2}{2} \\
 &= \frac{\varepsilon}{2} \frac{2\varepsilon-1}{(1+\varepsilon)^2} \left( 1 + \frac{r_s}{2r_1} \frac{11-18\varepsilon-5\varepsilon^2}{(2\varepsilon-1)(1+\varepsilon)} \right).
 \end{aligned}$$

This finally leads to

which follows from (2.19-21). In  $t_0$  the terms proportional to  $\varphi$  cancel

$$t_0(\varphi) = \sqrt{\frac{2r_1^3}{r_s(1+\varepsilon)}} \frac{1+\varepsilon}{1-\varepsilon} \left( \frac{\psi}{\sqrt{1-\varepsilon^2}} - \frac{\varepsilon \sin 2\Phi}{1+\varepsilon \cos 2\Phi} \right).
 \tag{4.28}$$

Approximating  $2\Phi$  by  $\varphi$  this is in agreement with Kepler's equation

$$t_0(\varphi) = \sqrt{\frac{2r_1^3}{r_s(1-\varepsilon)^3}} (\psi - \varepsilon \sin \psi).
 \tag{4.29}$$

The post-Newtonian corrections in (4.27) come from various places. To show this we write the result in the form



this.

Let  $\nu_1$  be the frequency of a given atomic line from the star and  $\nu_0$  the frequency of the same line observed in the rest frame of the galaxy. If  $dx^\mu/dt$  is the velocity of the star, the two frequencies are related by ([9] p.83, equ. 3.5.6)

$$\frac{\nu_1}{\nu_0} = \frac{\left( g_{\alpha\beta}(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2}}{g_{00}(X)^{1/2}}. \tag{5.1}$$

We assume that the observer at  $X$  is far away from the center such that the denominator can be approximated by 1. For a star moving in the plane  $\mathcal{G} = \pi/2$  we have

$$g_{\alpha\beta}(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = e^\nu - e^\nu \left( \frac{dr}{dt} \right)^2 - r^2 \left( \frac{d\varphi}{dt} \right)^2. \tag{5.2}$$

From (2.10) and (2.7) we find

$$\frac{dr}{dt} = e^\nu \sqrt{1 - \frac{e^\nu}{E^2} \left( m^2 + \frac{L^2}{r^2} \right)}$$

and (2.8) gives

$$\frac{d\varphi}{dt} = \frac{L e^\nu}{E r^2}.$$

Substituting all this into (5.1) we see that  $L$  drops out and we end up with the simple result

$$\frac{\nu_1}{\nu_0} = \frac{m}{E} e^\nu = \frac{m}{E} \left( 1 - \frac{r_s}{r(\varphi)} \right). \tag{5.3}$$

By (2.20) we can express  $E$  by the perihelion  $r_1$  and aphelion  $r_2$

$$E^2 = m^2 \left( 1 - \frac{r_s}{r_1 + r_2 + r_3} \right)$$

where  $r_3$  is the small correction (2.21). Then we finally get

$$\frac{\nu_1}{\nu_0} = \left( 1 - \frac{r_s}{r(\varphi)} \right) \left( 1 - \frac{r_s}{r_1 + r_2 + r_3} \right)^{-1/2}. \tag{5.4}$$

The lowest order  $O(r_s)$  is equal to

$$\frac{\nu_1}{\nu_0} = 1 - \frac{r_s}{r(\varphi)} + \frac{r_s}{2(r_1 + r_2)} + O(r_s^2). \tag{5.5}$$

Since the last term is always smaller than the second one we indeed have red shift  $\lambda_1 > \lambda_0$ . Of course, it is maximal at the perihelion where  $r = r_1$  is minimal. The total observed red shift is obtained by multiplying (5.5) with the Doppler factor  $(1 + v_r)^{-1}$  where  $v_r$  is the component of the relative velocity along the direction from the observer to the star ([9], p.30).

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### 7. References

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## Appendix: Integration of the Differential Equation

We closely follow Whittaker and Watson ([5], p.452). With the notation of the paper (2.11), let

$$\varphi = \int_{r_1}^x \frac{dx}{\sqrt{f(x)}} \quad (1)$$

where  $r_1$  is any zero,  $f(r_1) = 0$ . By Taylor's theorem, we have

$$f(x) = 4A_3(x-r_1) + 6A_2(x-r_1)^2 + 4A_1(x-r_1)^3 + A_0(x-r_1)^4,$$

where

$$A_0 = a_0, \quad A_1 = a_0 r_1 + a_1$$

$$A_2 = a_0 r_1^2 + 2a_1 r_1 + a_2,$$

$$A_3 = a_0 r_1^3 + 3a_1 r_1^2 + 3a_2 r_1 + a_3.$$

Introducing the new integration variable

$$s = (x-r_1)^{-1}, \quad s_1 = (r-r_1)^{-1}, \quad (2)$$

we have

$$\varphi = \int_{s_1}^{\infty} [4A_3 s^3 + 6A_2 s^2 + 4A_1 s + A_0]^{-1/2} ds.$$

To remove the second term in the cubic we set

$$s = \frac{1}{A_3} \left( z - \frac{A_2}{2} \right), \quad s_1 = \frac{1}{A_3} \left( z_1 - \frac{A_2}{2} \right) \quad (3)$$

and we get

$$\varphi = \int_{z_1}^{\infty} [4z^3 - (3A_2^2 - 4A_1 A_3)z - (2A_1 A_2 A_3 - A_2^3 - A_0 A_3^2)]^{-1/2} dz \quad (4)$$

The coefficients of  $z$  and  $z^0$  are just the invariants  $g_2, g_3$  (2.14-15) of the original quartic.

Now the inversion of the integral gives Weierstrass'  $P$ -function

$$z_1 = P(\varphi; g_2, g_3). \quad (5)$$

From (6.2) and (6.3) we have

$$r = r_1 + \frac{A_3}{z_1 - A_2/2} \quad (6)$$

and hence

$$r = r_1 + \frac{f'(r_1)}{4P(\varphi) - f''(r_1)/6}. \quad (7)$$