

Nonlinear High Harmonics Generation in REB-Plasma System

Sherif M. Khalil^{1,2}, Mizna M. Al-Enazi¹

¹ Faculty of Science, Phys. Dept., Princess Nora Bent Abdurrahman University, Riyadh, KSA

² Plasma Physics & Nuclear Fusion Department, N.R.C., Atomic Energy Authority, Cairo, Egypt

E-mail: mohamed_khalil@post.com; skhalil@kacst.edu.sa

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Abstract

The interaction of a relativistic electron beam (REB) with inhomogeneous, magneto-active, relativistic warm plasma is theoretically investigated. The nonlinear formation of waves at second and triple frequency at the inlet of the beam into the plasma is investigated. Effects of external static or oscillating magnetic field are considered. Nonlinear effects associated with the generation of second and triple harmonics, play an important role in the process of energy transfer from the beam to the plasma as compared with linear stage.

Keywords: Nonlinear Generation - Relativistic Electron Beam - Magnetized, Relativistic, Warm Plasmas

1. Introduction

The recent and continuous development in the relativistic electron beam (REB) technology has shown the capability of generating powerful electron energy sources, making REB very useful to controlled thermonuclear fusion research in various ways.

Electron beams have many applications in areas like development of new methods in amplification and generation of electromagnetic waves, acceleration of charged particles in plasma, plasma generation, design of microwave tubes waveguides, explanation of natural phenomena that occur in space and solar plasma, material studies, compact torus formation, generation of x-ray and microwave, etc., The recent development of the relativistic electron beam (REB) technology has shown the capability of generating powerful electron energy sources, making the REB very useful for controlled thermonuclear fusion research. It is not surprising to find up till now a long list of studies and research on beam-plasma interaction and applications, which was also reviewed by many authors, e.g. [1-7].

Multiple harmonics generation by laser-plasma interaction has been widely investigated [8-12]. The generation of harmonics through a nonlinear mechanism driven by bunching at the fundamental has sparked interest as a path toward enhancing and extending the usefulness of an x-ray free-electron laser (FEL) facility [13]. Currently, high-order harmonic generation (HHG) is considered as

one of the more efficient technique for producing coherent short-wavelength radiation in a broad spectral range [14-18].

An interaction of an electron beam with inhomogeneous plasma is widely investigated in the near past by many authors, e.g., second harmonics generation and plasma heating by REB [19-25]. Wave excitation by REB is also used to minimize energy losses of waves propagating in waveguides filled with magnetized plasma [26-28].

In our investigation, we consider one-dimensional electrostatic oscillations when the directions of propagation, density gradient, and wave electric field coincide with x -axis. In our model we take into consideration the following assumptions

1) The hydrodynamic model applies to both the beam and the plasma.

2) Transient thin plasma layer of width (a) is considered so that the plasma density n_{op} unperturbed by the wave fields, is arbitrary function of x in the region $0 \leq x \leq a$ and equal to zero in the regions $x \leq 0$ and $x \geq a$.

3) The wavelength of the incident beam is large if compared with the width of the plasma transient layer ($a \ll \lambda$), i.e., $|ka/\sqrt{\epsilon_b} \ll 1|$ ϵ_b is the dielectric constant of beam.

4) Plasma electrons are relativistic, warm, collisionless and magnetized.

5) The relativistic electron beam is homogeneous,

magnetized and have arbitrary temperature compared to that of plasma electrons.

6) Effects of external static or oscillating magnetic field

$$\vec{H}_{ext} = \begin{cases} \vec{e}_z H_0 \\ \vec{e}_z H_0 e^{-i\omega_m t} \end{cases}$$

2. Basic Equations and Linear Theory

According to above assumption we can use the following equations:

Equation of motion with relativistic effect,

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} \right) m \vec{V} = -e \left(\vec{E} + \frac{1}{c} \vec{V} \times (\vec{H} + \vec{H}_{ext}) \right) - \frac{1}{n} \vec{\nabla} P_s \quad (1)$$

where, $s = p, b$ for plasma and beam, respectively, $P_s = K T_s n_s$ is the pressure with gradient $\vec{\nabla} P_s = m V_{Ts}^2 \vec{\nabla} n_s$, n_s is the plasma/beam density, K is Boltzmann constant, $V_{Ts} = \sqrt{K T_s / m_s}$ is the thermal velocity, $m = \frac{m_0}{\sqrt{1 - \beta^2}}$ is the relativistic mass while m_0 and m are the rest mass and the mass of electrons respectively, $\beta^2 = \frac{V_0^2}{C^2}$ and V_0 the initial electron velocity. All other terms have their usual meaning.

Continuity equation

$$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{V}_s) = 0 \quad (2)$$

Poisson's equation

$$\frac{\partial E}{\partial x} = -4\pi e \sum_s n_s \quad (3)$$

As plasma is characterized by a collective process, and according to perturbation theory, we can express the density and velocity of the plasma electrons and beam as:

$$n_s = \sum_{s,\beta} n_s^{(\beta)}, \quad \vec{V}_s = \sum_{s,\beta} \vec{V}_s^{(\beta)}, \quad \beta = 0, 1, 2, 3, \dots$$

The superscript 0 indicate unperturbed quantities, while 1, 2, 3 indicate perturbation of first, second, third order, and so on. The same could be applied to the electric field as:

$$\vec{E} = \sum_{\beta} \vec{E}_{\beta} \quad (4)$$

We also assume that

$$\left(\left| n_s^{(\beta)} \right|, \left| v_s^{(\beta)} \right| \right) \gg \left(\left| n_s^{(\beta+1)} \right|, \left| v_s^{(\beta+1)} \right| \right)$$

$$\frac{\partial v_b^{(0)}}{\partial x} = \frac{\partial v_b^{(0)}}{\partial t} = \frac{\partial n_b^{(0)}}{\partial t} = 0 \quad \text{and} \quad v_b^{(0)} \gg 0$$

2.1. Cold Plasma - Cold Beam

For cold magnetized non-relativistic plasma and cold non-magnetized relativistic beam, and on the basis of above assumptions, the first perturbed densities and velocities of beam and plasma reads:

$$v_b^{(1)} = \frac{-e}{m\alpha\gamma v_b^{(0)}} e^{\frac{i\omega_1}{v_b^{(0)}}x} \int E_1 e^{\frac{-i\omega_1}{v_b^{(0)}}x} dx \quad (5)$$

$$n_b^{(1)} = \frac{en_b^{(0)}}{m\alpha\gamma v_b^{(0)2}} e^{\frac{i\omega_1}{v_b^{(0)}}x} \int e^{\frac{-i\omega_1}{v_b^{(0)}}x} \left[E_1 + \frac{i\omega_1}{v_b^{(0)}} e^{\frac{i\omega_1}{v_b^{(0)}}x} \int E_1 e^{\frac{-i\omega_1}{v_b^{(0)}}x} dx' \right] dx \quad (6)$$

$$v_p^{(1)} = \frac{-ie}{m\tilde{\omega}_1^2} E_1 \quad (7)$$

$$n_p^{(1)} = \frac{-e}{m\omega_1 \tilde{\omega}_1} \frac{\partial}{\partial x} (n_p^{(0)} E_1) \quad (8)$$

where, $\alpha = 1 + \gamma^2 \left(\frac{v_b^{(0)}}{c} \right)^2$, $\tilde{\omega}_1 = \omega_1 - i\omega_c$, ($\omega_c = \frac{-eH_0}{mc}$)

is the electron cyclotron frequency, while ω_1 is the fundamental frequency).

Using (6) and (8) into (3), and after lengthy but easy calculations we obtain the following differential equation for the fundamental electric field E_1 :

$$\left(v_b^{(0)} \frac{\partial}{\partial x} - i\omega_1 \right)^2 \epsilon_{1M} E_1 + \omega_{1b}^2 E_1 = 0 \quad (9)$$

where, $\epsilon_{1M} = 1 - \frac{\omega_p^2}{\omega_1 \tilde{\omega}_1}$ is the plasma dielectric, $\omega_{1b}^2 = \frac{\omega_b^2}{\alpha\gamma}$,

ω_s is the Langmuir frequency $\omega_s^2 = \frac{4\pi e^2 n_{0s}}{m}$. Intro-

ducing $E_1(x) = \frac{F_1(x)}{\epsilon_{1M}(x)} e^{\frac{i\omega_1}{v_b^{(0)}}x}$

into (9), we get

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\chi_{1M}^2}{\epsilon_{1M}} F_1 = 0, \quad \chi_{1M}^2 = \frac{\omega_{1b}^2}{v_b^{(0)2}} \quad (10)$$

In the empty regions $x \leq 0, x \geq a$, (10) have solutions:

$$F_1 = c_1 e^{i\chi_{1M}x}; \quad x \leq 0 \quad (11)$$

$$F_1 = c_2 e^{-i\chi_{1M}(x-a)}; \quad x \geq a \quad (12)$$

Inside the plasma layer $0 \leq x \leq a$, solution of (10) reads:

$$F_1 = c_3, \quad 0 \leq x \leq a \quad (13)$$

where, c_1, c_2, c_3 are constants. To obtain (13), we assumed much small plasma layer width compared to wavelengths, i.e., $\left| \frac{\chi_{1M}}{\sqrt{\epsilon_{1M}}} a \right| \ll 1$.

2.2. Warm Plasma - Cold Beam

Let us now introduce warmness to plasma. Accordingly, relations (7) and (8) read:

$$v_p^{(1)} = \frac{-ie}{m\tilde{\omega}_1^2} E_1 - \frac{iV_T^2}{\tilde{\omega}_1 n_p^{(0)}} \frac{\partial n_p^{(0)}}{\partial x} \quad (14)$$

$$n_p^{(1)} = \frac{-e}{m\omega_1 \tilde{\omega}_1} \frac{\partial}{\partial x} (n_p^{(0)} E_1) - \frac{V_T^2}{\omega_1 \tilde{\omega}_1} \frac{\partial^2 n_p^{(0)}}{\partial x^2} \quad (15)$$

Then, (10) converts to:

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\chi_{1MT}^2}{\epsilon_{1MT}} F_1 = R_{1T} \quad (16)$$

where, $\chi_{1MT}^2 = \chi_{1M}^2 = \frac{\omega_{pb}^2}{v_b^{(0)2}}$, $\epsilon_{1MT} = \epsilon_{1M} = 1 - \frac{\omega_p^2}{\omega_1 \tilde{\omega}_1}$,

$V_T = \sqrt{\frac{T_e}{m}}$ (thermal velocity), and

$$R_{1T}(x) = \frac{4\pi e}{v_b^{(0)2}} e^{\frac{-i\omega_1}{v_b^{(0)}} x} \frac{V_T^2}{\omega_1 \tilde{\omega}_1} \frac{\partial}{\partial x} \left(v_b^{(0)} \frac{\partial}{\partial x} - i\omega_1 \right)^2 n_p^{(0)}$$

It is clear that (16) differs from (10) because of the existence of the source term R_{1T} due to plasma warmness, which causes an inhomogeneity in the differential equation.

Solutions of (16) in the empty regions, are still given by (10) and (11), while in the inhomogeneous plasma layer, and under the condition $\left| \frac{\chi_{1MT}}{\sqrt{\epsilon_{1MT}}} a \right| \ll 1$, the solu-

tion is given by:

$$E_1 \cong \frac{e^{\frac{i\omega_1}{v_b^{(0)}} x}}{\epsilon_{1MT}} \left[\int_0^x dx' \int_0^{x'} R_{1T} e^{\frac{-i\omega_1}{v_b^{(0)}} x} dx + c_4 \right], \quad 0 \leq x \leq a \quad (17)$$

It is clear that we can easily the case for unmagnetized plasma by setting $\omega_c = 0$ into (1), i.e., $\tilde{\omega}_1 \rightarrow \omega_1$.

2.3. Warm Plasma-Warm Beam

By taking into consideration the pressure gradient $\bar{\nabla} P_b$ of the electron beam, it is easy to check that, in the linear regime, the results obtained for cold beam (relations (14-17)) are still valid.

Besides, if plasma is immersed in oscillating magnetic field

$$\vec{H}_{ext} = \vec{e}_z H_0 e^{-i\omega_M t}, \quad \omega_M = \omega_1$$

The systems, in the linear theory, will behaves as in case of un-magnetized plasma.

3. Nonlinear Theory

In this part we consider the nonlinear interaction and wave generation at second and third harmonic generation by REB.

3.1. Second Harmonic Wave Generation in Static Magnetic Field $\vec{H}_{ext} = \vec{e}_z H_0$

In presence of static magnetic field, and for warm beam - warm plasma interaction, the second perturbed quantities (with $(\omega_2 = 2\omega_1)$ for the plasma and beam reads:

$$v_p^{(2)} = \frac{-ie}{m\tilde{\omega}_2} E_2 + \frac{ie^2}{m^2 \tilde{\omega}_1^2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x} - \frac{iV_T^2}{n_p^{(0)} \tilde{\omega}_2} \frac{\partial n_p^{(0)}}{\partial x} \quad (18)$$

$$n_p^{(2)} = \frac{\partial}{\partial x} \left\{ \frac{-en_p^{(0)}}{m\omega_2 \tilde{\omega}_2} E_2 + \frac{e^2 n_p^{(0)} (\omega_1 + \tilde{\omega}_2)}{m^2 \omega_1 \tilde{\omega}_1^2 \omega_2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x} + \left(\frac{e^2 E_1^2}{m^2 \omega_1 \tilde{\omega}_1^2 \omega_2} - \frac{V_T^2}{\omega_2 \tilde{\omega}_2} \right) \frac{\partial n_p^{(0)}}{\partial x} \right\} \quad (19)$$

$$v_b^{(2)} = \frac{-1}{v_b^{(0)}} e^{\frac{i\omega_2}{v_b^{(0)}} x} \int e^{\frac{-i\omega_2}{v_b^{(0)}} x} \left[\frac{e}{m\gamma} E_2 + \alpha_2 v_b^{(1)} \frac{\partial v_b^{(1)}}{\partial x} + \frac{V_T^2}{n_b^{(0)} \gamma} \frac{\partial n_b^{(1)}}{\partial x} \right] dx \quad (20)$$

$$n_b^{(2)} = \frac{-1}{v_b^{(0)}} e^{\frac{i\omega_2}{v_b^{(0)}} x} \int e^{\frac{-i\omega_2}{v_b^{(0)}} x} \left\{ \frac{-n_b^{(0)}}{v_b^{(0)}} \frac{i\omega_2}{v_b^{(0)}} \int e^{\frac{i\omega_2}{v_b^{(0)}} x} E_2 dx - \frac{n_b^{(0)}}{v_b^{(0)}} E_2 + \frac{\partial}{\partial x} (n_b^{(1)} v_b^{(1)}) \right\} \quad (21)$$

Using (19) and (20) into Poisson's Equation (3), we obtain the following differential equation for the second harmonic electric field E_2 :

$$\frac{\partial^2 F_2}{\partial x^2} + \frac{\chi_2^2}{\epsilon_{2M}} F_2 = R'_{2MT} \quad (22)$$

where, R'_{2MT} is a nonlinear source term includes the effects of warmness of both the beam and plasma elec-

trons, assuming that both have the same thermal velocity V_T .

$$R'_{2MT}(x) = \frac{1}{v_b^{(0)2}} e^{\frac{-i\omega_2}{v_b^{(0)}}x} [R_{pMT} + R_{bT}], \quad E_2 = \frac{F_2}{\varepsilon_{2M}} e^{\frac{i\omega_2}{v_b^{(0)}}x}, \quad \chi_2^2 = \frac{\omega_b^2}{\gamma\alpha_2 v_b^{(0)2}},$$

$$R_{bT}(x) = -4\pi e \left[\left(i\omega_2 - v_b^{(0)} \frac{\partial}{\partial x} \right) n_b^{(1)} v_b^{(1)} + n_b^{(0)} \left(\alpha_2 v_b^{(1)} \frac{\partial v_b^{(1)}}{\partial x} + \frac{V_T^2}{n_b^{(0)} \gamma} \frac{\partial n_b^{(1)}}{\partial x} \right) \right]$$

$$R_{pMT}(x) = - \left(v_b^{(0)} \frac{\partial}{\partial x} - i\omega_2 \right)^2 \left\{ \frac{e\omega_p^2 (\omega_1 + \tilde{\omega}_2)}{m\omega_2 \tilde{\omega}_2 \omega_1 \tilde{\omega}_1^2} E_1 \frac{\partial E_1}{\partial x} + \left(\frac{eE_1^2}{m\tilde{\omega}_1^2 \omega_1 \omega_2} - \frac{mV_T^2}{e\omega_1 \tilde{\omega}_2} \right) \frac{\partial \omega_p^2}{\partial x} \right\}$$

$$\tilde{\omega}_2 = \omega_2 - i\omega_c, \quad \varepsilon_{2M} = 1 - \frac{\omega_p^2}{\omega_2 \tilde{\omega}_2}, \quad \alpha_2 = 1 + 2\gamma^2 \left(\frac{v_b^{(0)}}{c} \right)^2$$

Solutions of (22) in the empty regions and in the inhomogeneous plasma layer, are given by:

$$F_2 = \frac{e^{i\chi_2 x}}{2i\chi_2} \int R_{bT} e^{-i\chi_2 x} dx - \frac{e^{-i\chi_2 x}}{2i\chi_2} \int R_{pMT} e^{i\chi_2 x} dx, \quad x \leq 0 \quad (23)$$

$$F_2 = \frac{e^{i\chi_2(x-a)}}{2i\chi_2} \int R_{bT} e^{-i\chi_2(x-a)} dx - \frac{e^{-i\chi_2(x-a)}}{2i\chi_2} \int R_{pMT} e^{i\chi_2(x-a)} dx, \quad x \geq a \quad (24)$$

$$F_2 \cong v_b^{(0)2} \int_0^x dx' \int_0^{x'} R'_{2MT}(x) e^{\frac{-i\omega_2}{v_b^{(0)}}x} dx'' + c, \quad 0 \leq x \leq a \quad (25)$$

It is clear that source terms, *i.e.*, R_{bT} , R_{pMT} are represented by products of fundamentals, and the electric field E_2 , as per solutions (23-25), are proportional to waves of second harmonics. This leads to a nonlinear amplification of waves in the inhomogeneous plasma layer and in turn additional plasma heating.

3.1.1. Second Harmonic Wave Generation in an

Oscillating Magnetic Field $\vec{H}_{ext} = \vec{e}_z H_0 e^{-i\omega_M t}$

Let us consider here the effect of external magnetic field

$$n_p^{(2)} = \frac{\partial}{\partial x} \left\{ \frac{-en_p^{(0)}}{m\omega_2^2} E_2 + \frac{e^2 n_p^{(0)} (\omega_1 + \omega_2)}{m^2 \omega_1^3 \omega_2^2} E_1 \frac{\partial E_1}{\partial x} + \left(\frac{e^2 E_1^2}{m^2 \omega_1^3 \omega_2} - \frac{V_T^2}{\omega_2^2} \right) \frac{\partial n_p^{(0)}}{\partial x} + \frac{ie n_p^{(0)} \omega_c(t)}{m c \omega_1 \omega_2^2} E_1 \right\} \quad (27)$$

When we look for harmonic generation with $\omega_2 = 2\omega_1$, the perturbed cold electron beam perturbations reads:

$$v_b^{(2)} = \frac{-1}{v_b^{(0)}} e^{\frac{i\omega_2}{v_b^{(0)}}x} \int e^{\frac{-i\omega_2}{v_b^{(0)}}x} \left[\frac{e}{m\gamma} E_2 + \alpha_2 v_b^{(1)} \frac{\partial v_b^{(1)}}{\partial x} \right] dx \quad (28)$$

$$n_b^{(2)} = \frac{-1}{v_b^{(0)}} e^{\frac{i\omega_2}{v_b^{(0)}}x} \int e^{\frac{-i\omega_2}{v_b^{(0)}}x} \left\{ \frac{-n_b^{(0)}}{v_b^{(0)}} \frac{i\omega_2}{v_b^{(0)}} e^{\frac{i\omega_2}{v_b^{(0)}}x} \int e^{\frac{-i\omega_2}{v_b^{(0)}}x} \left[\frac{e}{m\gamma} E_2 + \alpha v_b^{(1)} \frac{\partial v_b^{(1)}}{\partial x} \right] dx \right. \\ \left. - \frac{n_b^{(0)}}{v_b^{(0)}} \left[\frac{e}{m\gamma} E_2 + \alpha_2 v_b^{(1)} \frac{\partial v_b^{(1)}}{\partial x} \right] + \frac{\partial}{\partial x} (n_b^{(1)} v_b^{(1)}) \right\} dx \quad (29)$$

Accordingly, the differential equation governing the electric field is given by:

$$\frac{\partial^2 F_2}{\partial x^2} + \frac{\chi_2^2}{\epsilon_2} F_2 = R_{2MOT} \quad (30)$$

where, $R_{2MOT} = \frac{1}{\nu_b^{(0)2}} e^{\frac{-i\omega_2 x}{\nu_b^{(0)}}} [R_{pMOT} + R_b]$

$$R_{pMOT} = -\left(\nu_b^{(0)} \frac{\partial}{\partial x} - i\omega_2\right)^2 \left[\frac{e\omega_p^2 (\omega_1 + \omega_2)}{m\omega_1^3 \omega_2^2} E_1 \frac{\partial E_1}{\partial x} + \left(\frac{eE_1^2}{m\omega_1^3 \omega_2} - \frac{mV_T^2}{e\omega_1 \omega_2} \right) \frac{\partial \omega_p^2}{\partial x} + \frac{i\omega_p^2 \omega_c(t)}{\omega_1 \omega_2^2} E_1^2 \right]$$

with solutions:

$$F_2 = \frac{e^{i\chi_2 x}}{2i\chi_2} \int R_b e^{-i\chi_2 x} dx - \frac{e^{-i\chi_2 x}}{2i\chi_2} \int_0^x R_{pMOT} e^{i\chi_2 x} dx, \quad x \leq 0 \quad (31)$$

$$F_2 = \frac{e^{i\chi_2(x-a)}}{2i\chi_2} \int R_b e^{-i\chi_2(x-a)} dx - \frac{e^{-i\chi_2(x-a)}}{2i\chi_2} \int_0^x R_{pMOT} e^{i\chi_2(x-a)} dx, \quad x \geq a \quad (32)$$

$$F_2 \cong \int_0^x dx' \int_0^{x'} (R_b + R_{pMOT}) e^{\frac{-i\omega_2 x}{\nu_b^{(0)}}} dx'' + c, \quad 0 \leq x \leq a \quad (33)$$

In absence of magnetic field, and for cold beam - cold plasma interaction, the third perturbed quantities with ($\omega_3 = 3\omega_1$) for the plasma and beam reads:

$$\nu_b^{(3)} = \frac{-1}{\nu_b^{(0)}} e^{\frac{i\omega_3 x}{\nu_b^{(0)}}} \int e^{\frac{-i\omega_3 x}{\nu_b^{(0)}}} \left[\frac{e}{m\gamma} E_3 + \alpha \nu_b^{(2)} \frac{\partial \nu_b^{(1)}}{\partial x} + \nu_b^{(1)} \frac{\partial \nu_b^{(2)}}{\partial x} \right] dx \quad (34)$$

3.1.2. Third Harmonic Wave Generation in Static Magnetic Field $\vec{H}_{ext} = \vec{e}_z H_0$

$$n_b^{(3)} = \frac{-1}{\nu_b^{(0)}} e^{\frac{i\omega_3 x}{\nu_b^{(0)}}} \int e^{\frac{-i\omega_3 x}{\nu_b^{(0)}}} \left\{ \frac{-n_b^{(0)}}{\nu_b^{(0)}} \frac{i\omega_3}{\nu_b^{(0)}} e^{\frac{i\omega_3 x}{\nu_b^{(0)}}} \int e^{\frac{-i\omega_3 x}{\nu_b^{(0)}}} \left[\frac{e}{m\gamma} E_3 + \alpha_2 \nu_b^{(2)} \frac{\partial \nu_b^{(1)}}{\partial x} + \nu_b^{(1)} \frac{\partial \nu_b^{(2)}}{\partial x} \right] dx \right. \\ \left. - \frac{n_b^{(0)}}{\nu_b^{(0)}} \left[\frac{e}{m\gamma} E_3 + \alpha_2 \nu_b^{(2)} \frac{\partial \nu_b^{(1)}}{\partial x} + \nu_b^{(1)} \frac{\partial \nu_b^{(2)}}{\partial x} \right] + \frac{\partial}{\partial x} (n_b^{(1)} \nu_b^{(2)} + n_b^{(2)} \nu_b^{(1)}) \right\} dx \quad (35)$$

$$\nu_p^{(3)} = \frac{-ie}{m\omega_3} E_3 - \frac{eE_1}{m\omega_1 \omega_3} \frac{\partial}{\partial x} \left[\frac{-ie}{m\omega_2} E_2 + \frac{ie^2}{m^2 \omega_1^2 \omega_2} E_1 \frac{\partial E_1}{\partial x} \right] - \frac{e}{m\omega_1 \omega_3} \left[\frac{-ie}{m\omega_2} E_2 + \frac{ie^2}{m^2 \omega_1^2 \omega_2} E_1 \frac{\partial E_1}{\partial x} \right] \frac{\partial E_1}{\partial x} \quad (36)$$

$$n_p^{(3)} = \frac{\partial}{\partial x} \left\{ \frac{-en_p^{(0)}}{m\omega_3^2} E_3 + \frac{ien_p^{(0)} E_1}{m\omega_1 \omega_3^2} \frac{\partial}{\partial x} L + \frac{ien_p^{(0)} (\omega_1 + \omega_3)}{m\omega_1^2 \omega_3^2} L \frac{\partial E_1}{\partial x} + \frac{ieE_1}{m\omega_1^2 \omega_3} L \frac{\partial n_p^{(0)}}{\partial x} \right. \\ \left. - \frac{e}{m\omega_1 \omega_3} \frac{\partial}{\partial x} \left[\frac{-en_p^{(0)}}{m\omega_2^2} E_2 + \frac{e^2 n_p^{(0)} (\omega_1 + \omega_2)}{m^2 \omega_1^3 \omega_2^2} E_1 \frac{\partial E_1}{\partial x} + \frac{e^2 E_1^2}{m^2 \omega_1^3 \omega_2} \frac{\partial n_p^{(0)}}{\partial x} \right] \right\} \quad (37)$$

where, $L = \frac{-ie}{m\omega_2} E_2 + \frac{ie^2}{m^2 \omega_1^2 \omega_2} E_1 \frac{\partial E_1}{\partial x}$,

$$\frac{\partial^2 F_3}{\partial x^2} + \frac{\chi_3^2}{\epsilon_3} F_3 = R_3(x) \quad (38)$$

Using (35) and (39) into Poisson's Equation (3), the differential equation governing the third harmonic electric field E_3 :

where,

$$\epsilon_3 = 1 - \frac{\omega_p^2}{\omega_3^2}, \quad \chi_3^2 = \frac{\omega_b^2}{\gamma \nu_b^{(0)2}}, \quad E_3 = \frac{F_3}{\epsilon_3} e^{\frac{i\omega_3 x}{\nu_b^{(0)}}}, \quad R_3(x) = \frac{1}{\nu_b^{(0)2}} e^{\frac{-i\omega_3 x}{\nu_b^{(0)}}} [R_{3p} + R_{3b}]$$

$$R_{3p}(x) = -\left(\nu_b^{(0)} \frac{\partial}{\partial x} - i\omega_3\right)^2 \left\{ \left[\frac{i\omega_p^2 E_1}{\omega_1 \omega_3^2} \frac{\partial}{\partial x} + \frac{i\omega_p^2 (\omega_1 + \omega_3)}{\omega_1^2 \omega_3^2} \frac{\partial E_1}{\partial x} + \frac{iE_1}{\omega_1^2 \omega_3} \frac{\partial \omega_p^2}{\partial x} \right] \left[\frac{-ie}{m\omega_2} E_2 + \frac{ie^2 E_1}{m^2 \omega_1^2 \omega_2} \frac{\partial E_1}{\partial x} \right] + \frac{E_1}{\omega_1 \omega_2 \omega_3} \frac{\partial}{\partial x} (\omega_p^2 E_2) - \frac{E_1}{\omega_1 \omega_3} \frac{\partial}{\partial x} \left[\frac{e\omega_p^2 (\omega_1 + \omega_2)}{m\omega_1^3 \omega_2^2} E_1 \frac{\partial E_1}{\partial x} + \frac{eE_1^2}{m\omega_1^3 \omega_2} \frac{\partial \omega_p^2}{\partial x} \right] \right\}$$

$$R_{3b}(x) = -4\pi e \left[\left(i\omega_3 - \nu_b^{(0)} \frac{\partial}{\partial x} \right) \left(n_b^{(1)} \nu_b^{(2)} + n_b^{(2)} \nu_b^{(1)} \right) + n_b^{(0)} \left(\alpha_2 \nu_b^{(2)} \frac{\partial \nu_b^{(1)}}{\partial x} + \nu_b^{(1)} \frac{\partial \nu_b^{(2)}}{\partial x} \right) \right]$$

with solutions

$$F_3 = \frac{e^{i\chi_3 x}}{3i\chi_3} \int R_{3b} e^{-i\chi_3 x} dx - \frac{e^{-i\chi_3 x}}{3i\chi_3} \int_0^x R_{3p} e^{i\chi_3 x} dx, \quad x \leq 0 \tag{39}$$

$$F_3 = \frac{e^{i\chi_3(x-a)}}{3i\chi_3} \int R_{3b} e^{-i\chi_3(x-a)} dx - \frac{e^{-i\chi_3(x-a)}}{3i\chi_3} \int_0^x R_{3p} e^{i\chi_3(x-a)} dx, \quad x \geq a \tag{40}$$

$$F_3 \cong \int_0^x dx' \int_0^{x'} \left(R_{3b} + R_{3p} \right) e^{\frac{-i\omega_3}{\nu_b^{(0)}} x} dx'' + c, \quad 0 \leq x \leq a \tag{41}$$

In a static magnetic field, and warm plasma, beam perturbations (34) and (35) are still the same, while for plasma, (36) and (37) reads:

$$\nu_p^{(3)} = \frac{-ie}{m\tilde{\omega}_3} E_3 - \frac{iV_T^2}{n_p^{(0)} \tilde{\omega}_3} \frac{\partial n_p^{(0)}}{\partial x} - \frac{eE_1}{m\tilde{\omega}_1 \tilde{\omega}_3} \frac{\partial}{\partial x} \left[\frac{-ie}{m\tilde{\omega}_2} E_2 + \frac{ie^2}{m^2 \tilde{\omega}_1^2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x} \right] - \frac{e}{m\tilde{\omega}_1 \tilde{\omega}_3} \left[\frac{-ie}{m\tilde{\omega}_2} E_2 + \frac{ie^2}{m^2 \tilde{\omega}_1^2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x} \right] \frac{\partial E_1}{\partial x} \tag{42}$$

$$n_p^{(3)} = \frac{\partial}{\partial x} \left\{ \frac{-en_p^{(0)}}{m\tilde{\omega}_3 \omega_3} E_3 - \frac{V_T^2}{\tilde{\omega}_3 \omega_3} \frac{\partial n_p^{(0)}}{\partial x} + \frac{ien_p^{(0)} E_1}{m\tilde{\omega}_1 \tilde{\omega}_3 \omega_3} \frac{\partial}{\partial x} L_M + \frac{ien_p^{(0)} (\omega_1 + \tilde{\omega}_3)}{m\tilde{\omega}_1 \omega_1 \omega_3^2} L_M \frac{\partial E_1}{\partial x} + \frac{ieE_1}{m\tilde{\omega}_1 \omega_1 \omega_3} L_M \frac{\partial n_p^{(0)}}{\partial x} - \frac{e}{m\tilde{\omega}_1 \omega_3} \frac{\partial}{\partial x} \left[\frac{-en_p^{(0)}}{m\tilde{\omega}_2 \omega_2} E_2 + \frac{e^2 n_p^{(0)} (\omega_1 + \tilde{\omega}_2)}{m^2 \tilde{\omega}_1^2 \omega_1 \tilde{\omega}_2 \omega_2} E_1 \frac{\partial E_1}{\partial x} + \frac{e^2 E_1^2}{m^2 \tilde{\omega}_1^2 \omega_1 \omega_2} \frac{\partial n_p^{(0)}}{\partial x} \right] \right\} \tag{43}$$

where, $L_M = \frac{-ie}{m\tilde{\omega}_2} E_2 + \frac{ie^2}{m^2 \tilde{\omega}_1^2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x}$ $\frac{\partial^2 F_3}{\partial x^2} + \frac{\chi_3^2}{\epsilon_{3m}} F_3 = R_{3MT}(x)$ (44)

In this case the differential equation governing the third harmonic electric field E_3 reads

with source terms given by

$$R_{3MT}(x) = \frac{1}{\nu_b^{(0)2}} e^{\frac{-i\omega_3}{\nu_b^{(0)}} x} \left[R_{3pMT} + R_{3b} \right],$$

$$R_{3pMT}(x) = -\left(\nu_b^{(0)} \frac{\partial}{\partial x} - i\omega_3\right)^2 \left\{ \left[\frac{i\omega_p^2 E_1}{\tilde{\omega}_1 \omega_3 \tilde{\omega}_3} \frac{\partial}{\partial x} + \frac{i\omega_p^2 (\omega_1 + \tilde{\omega}_3)}{\omega_1 \tilde{\omega}_1 \omega_3} \frac{\partial E_1}{\partial x} + \frac{iE_1}{\omega_1 \tilde{\omega}_1 \omega_3} \frac{\partial \omega_p^2}{\partial x} \right] \left[\frac{-ie}{m\tilde{\omega}_2} E_2 + \frac{ie^2 E_1}{m^2 \tilde{\omega}_1^2 \tilde{\omega}_2} \frac{\partial E_1}{\partial x} \right] + \frac{E_1}{\tilde{\omega}_1 \omega_3} \frac{\partial}{\partial x} \left[\frac{-\omega_p^2}{\omega_2 \tilde{\omega}_2} E_2 + \frac{e\omega_p^2 (\omega_1 + \tilde{\omega}_2)}{m\omega_1 \tilde{\omega}_1^2 \omega_2 \tilde{\omega}_2} E_1 \frac{\partial E_1}{\partial x} + \frac{eE_1^2}{m\omega_1 \tilde{\omega}_1^2 \omega_2} \frac{\partial \omega_p^2}{\partial x} \right] - \frac{mV_T^2}{e\omega_3 \tilde{\omega}_3} \frac{\partial \omega_p^2}{\partial x} \right\}$$

and

Solutions of (45) are:

$$F_3 = \frac{e^{i\chi_3 x}}{3i\chi_3} \int R_{3b} e^{-i\chi_3 x} dx - \frac{e^{-i\chi_3 x}}{3i\chi_3} \int_0^x R_{3pMT} e^{i\chi_3 x} dx, \quad x \leq 0 \tag{45}$$

$$F_3 = \frac{e^{i\chi_3(x-a)}}{3i\chi_3} \int R_{3b} e^{-i\chi_3(x-a)} dx - \frac{e^{-i\chi_3(x-a)}}{3i\chi_3} \int_0^x R_{3pMT} e^{i\chi_3(x-a)} dx, \quad x \geq a \tag{46}$$

$$F_3 \equiv \int_0^x dx' \int_0^{x'} \left(R_{3b} + R_{3pMT} \right) e^{\frac{-i\omega_3}{v_b^{(0)}} x} dx'' + c, \quad 0 \leq x \leq a \quad (47)$$

4. Conclusions

The nonlinear interaction of a relativistic electron beam (REB) with an inhomogeneous, magneto-active, relativistic warm plasma layer is investigated. The nonlinear effects associated with the generation of second and third harmonics, plays an important role in the process of energy transfer from the beam to the plasma as compared with linear stage. This is due to the fact that the electric field intensity at double and third harmonics is stronger than that of the basic frequency.

Fields are found to be inversely proportional to ε as:

$$E_i \propto \frac{1}{\varepsilon_{iM}},$$

where $\varepsilon_{iM} = 1 - \frac{\omega_p^2}{\omega_i \tilde{\omega}_i}$, $i = 1, 2, 3$. 1, 2 and 3 indicates fundamental, second and third harmonics respectively and $\tilde{\omega}_i = \omega_i - i\omega_c$.

In static strong magnetic field, at resonance $|\tilde{\omega}_i| \rightarrow 0$, the wave's amplitudes and the electric fields for fundamentals and higher harmonics are sharply increases.

As seen, the amplitudes of the exponentially growing oscillations at frequencies 2ω and 3ω are expressed through the amplitudes of the beam Langmuir oscillations in the region where is no plasma present. Thus, once there are Langmuir oscillations in the beam, even in the presence of external magnetic field, their frequency being equal to ω in the laboratory frame, and the oscillations with frequencies 2ω and 3ω are always generated at the inlet of the beam into the plasma when the beam in the plasma is unstable in relation to the excitation of oscillations with frequencies 2ω and 3ω . This lead to the conclusion that far enough from the plasma boundary inwards, the external magnetic field may increase the amplitude of grounded waves, but still the electric field of waves with double frequency would be stronger than that of basic frequency.

Also we can say that, the nonlinear source terms (R'_{2MT} , R_{2MOT} , R_3 , R_{3MT}) which is due to nonlinearity, static or oscillating magnetic field effects, and warmness of plasma and beam electrons, are in our case play a crucial role via wave amplification at second and third harmonic.

It is also shown that the nonlinear effects associated with the generation of second and third harmonics, play an important role in the process of energy transfer from the beam to the plasma as compared with linear stage. This due to the fact that the electric field intensity at

higher harmonics is stronger than that of basic frequency.

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