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An Equilibrium Asset Pricing Model under the Dual Theory of the Smooth Ambiguity Model

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Abstract

This paper considers an equilibrium asset pricing model in a static pure exchange economy under ambiguity. Ambiguity preference is represented by the dual theory of the smooth ambiguity model [6]. We show the existence and the uniqueness of the equilibrium in the economy and derive the state price density (SPD). The equilibrium excess return, which can be seen as an extension of the capital asset pricing model (CAPM) under risk to ambiguity, is derived from the SPD. We also determine the effects of ambiguity preference on the excess returns of ambiguous securities through comparative statics of the SPD.

Keywords

The Dual Theory of the Smooth Ambiguity Model, Equilibrium, Asset Pricing, CAPM

1. Introduction

The state price density (SPD) is a central concept in modern asset pricing theory.[1] Given the SPD and its probability distribution, we can price every asset. Thus, it is essential to derive the SPD for asset pricing. A lot of studies have attempted to derive the SPD from both a theoretical and an empirical viewpoint.

Most asset pricing models suppose that an agent can assign a unique probability distribution over a state space. However, it is commonly observed that such a unique probability is not available in the economy. It is known that expected utility, which is a dominant tool in asset pricing theory, cannot describe choice under ambiguity.[2] Many ambiguity models, such as those of [5]

[1]There are various terminologies for representing the same concept, such as stochastic discount factor, pricing kernel and others (see, e.g., [3] [4]).

[2]Risk is defined as a situation in which the probabilities over the state space are uniquely assigned. Ambiguity is defined as a situation in which the probabilities over the state space are either not uniquely assigned or are unknown.

[17], have been proposed to capture these choices. Following [10] [18], ambiguity is represented by a second-order probability over the set of first-order probabilities over the state space in this paper.

This paper unites the above two lines of research. The purpose of this paper is to characterize the SPD in the presence of ambiguity. More precisely, we examine equilibrium in a single-period pure exchange economy under ambiguity and derive the SPD by using the dual theory of the smooth ambiguity model [6]. We also derive the equilibrium excess return using the SPD, which can be viewed as an extension of the classical capital asset pricing model (CAPM) [11] [19].

The original theory of the smooth ambiguity model by [10] takes the "double" expected utility form as the second-order expectation of the transformed first-order expected utility. The tractability of this model is a distinct advantage compared with other existing ambiguity models. The ambiguity preference is reflected by the shape of the transformation. Unlike the original theory, the dual theory of the smooth ambiguity model by [6] captures ambiguity preference by a distortion of the second-order probability distribution. As shown in [6], an equivalent representation of the dual theory is the "single" expected utility with respect to a mixture of the first-order probability distributions with the distorted second-order probability distribution. This form reinforces the advantage of the original theory that the existing results in the expected utility are applicable to decision problems under ambiguity, while maintaining descriptive validity for ambiguity. This advantage makes it easy to characterize equilibria under ambiguity compared with the original theory as shown later in this paper. Thus the dual theory might be a powerful tool for these kinds of analyses.

A series of studies on equilibrium analysis in securities markets precede this paper. This paper derives the SPD in an economy with a representative agent. For the construction of the representative agent, we use the idea of assigning proper weights to each agent, which goes back to [14]. [12] gave the standard approach for an optimal/production model of a representative agent leading to equilibrium. The SPD can be viewed as a generalization of Bühlmann's economic premium principle [2] in an economy under ambiguity. This paper also shows the existence and the uniqueness of the equilibrium in an economy. [15] pioneered establishing the existence of equilibria in a complete market. We adapt the dual method of [7] [8] to show the existence and the uniqueness of the equilibrium in an economy.

Using the SPD, we extend the classical CAPM to an economy under ambiguity. Thus, this paper is related to the recent literature on the CAPM under ambiguity. In particular, [13] and [16] considered the CAPM under ambiguity using the smooth ambiguity model. Even though these papers derive a similar form of the excess return in the CAPM, it should be noted that we do not require any approximations to derive the CAPM, whereas [13] and [16] used a quadratic approximation. Furthermore, the optimal portfolio for each agent can be shown to consist of his specific portfolio in addition to a safe security and the market portfolio. As a special case, the optimal portfolio degenerates to the classical separation by [20], that is, an agent's specific portfolio disappears.

Because we adopt the dual theory of the smooth ambiguity model by [6], ambiguity preference is embedded in the expected utility representation with a distorted probability. In other words, ambiguity preference does not explicitly appear for most of the main analysis. While this tractability is a distinct advantage compared with existing models, ambiguity preferences nonetheless have an effect on equilibria in securities markets. We clarify these effects through comparative statics of the SPD. Using the results, we also determine the effects of ambiguity preference on the excess returns of ambiguous securities based on the CAPM derived in the paper.

The organization of this paper is as follows. The next section introduces some settings and notions about the economy and agents, and provides some preliminary analysis. In Section 3, we derive the SPD and show its existence and uniqueness in the economy. In Section 4, we apply the SPD to obtain an equilibrium asset return which can be rewritten as the CAPM in the presence of ambiguity. Furthermore, we show how an agent's portfolio is decomposed and obtain the classical two-fund separation from [20] as a special case. In Section 5, we perform some comparative statics to examine the effect of ambiguity on the SPD and the equilibrium excess return. The final section contains concluding remarks. All proofs of the propositions are collected in the Appendix.

2. The Model

We consider a single-period pure exchange economy. All of the uncertainty is described by a finite discrete state space $\Omega = \{\omega_1, \cdots, \omega_S\}$. Agents in the economy can trade a safe security and S ambiguous securities. The rate of return of the safe security is a constant r_f. The rate of return of the i-th ambiguous security is given by a random variable $\tilde{r}_i$. The rate of return $\tilde{r}_i$, $i = 1, \cdots, S$, can be decomposed into a constant term μ_i and a random term $\tilde{z}_i$ such that

$$\tilde{r}_i = \mu_i + \tilde{z}_i,$$

where $\tilde{z}_i = z_{is}$ if $\omega_s \in \Omega$ occurs for $s = 1, \cdots, S$.

We define the matrix $\boldsymbol{R}$ by

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & \cdots & r_{S1} \\ \vdots & \ddots & \vdots \\ r_{1S} & \cdots & r_{SS} \end{pmatrix},$$

where $r_{is} = \mu_i + z_{is}$, $i, s = 1, \cdots, S$. We assume that the rank of $\boldsymbol{R}$ is equal to S, that is, the security market is assumed to be complete.[3] Ambiguity is represented by a finite set of probability distributions on Ω:

$$\boldsymbol{p}^{(\ell)} = \left(p_1^{(\ell)}, \cdots, p_S^{(\ell)}\right), \quad \ell \in \{1, \cdots, \mathfrak{L}\}.$$

[3]While we assume a complete market for simplicity, we can relax this assumption by using, for example, the embedded market approach in [9], *i.e.*, an augmented complete market with constraints on sales and purchases for some securities.

Given a unique probability distribution $\boldsymbol{\rho}=\left(\rho^{(1)},\cdots,\rho^{(\mathfrak{L})}\right)$ over the index set $\{1,\cdots,\mathfrak{L}\}$, a compound probability distribution $\boldsymbol{P}$ on Ω is defined by

$$\boldsymbol{P}=\left(P_1,\cdots,P_S\right)=\left(\sum_{\ell=1}^{\mathfrak{L}}p_1^{(\ell)}\rho^{(\ell)},\cdots,\sum_{\ell=1}^{\mathfrak{L}}p_S^{(\ell)}\rho^{(\ell)}\right).$$

Each agent is assumed to consider this to be the reference probability distribution in the economy.[4] Without loss of generality, we assume that $\mathbb{E}^P\left[\tilde{z}_i\right]=\sum_{s=1}^S P_s z_{is}=0$, $i=1,\cdots,S$.

We define the SPD as a non-negative random variable $H=\left(H_1,\cdots,H_S\right)$ on Ω satisfying

$$\mathbb{E}^P\left[\left(1+r_f\right)H\right]=\left(1+r_f\right)\sum_{s=1}^S P_s H_s=1, \tag{1}$$

$$\mathbb{E}^P\left[\left(1+\tilde{r}_i\right)H\right]=\sum_{s=1}^S P_s\left(1+r_{is}\right)H_s=1,\quad i=1,\cdots,S. \tag{2}$$

There are K agents in the economy. Let $\epsilon_{ks}>0$, $k=1,\cdots,K$, denote agent k's terminal income if the state ω_s, $i=1,\cdots,S$, occurs, let $w_k>0$ denote agent k's initial wealth, and let π_{ki} denote the amount that agent k invests in the i-th ambiguous security. For a given state ω_s, the terminal wealth W_{ks}, $i=1,\cdots,S$, of agent k is given by

$$\begin{aligned}W_k=\begin{pmatrix}W_{k1}\\ \vdots\\ W_{kS}\end{pmatrix}&=\left(1+r_f\right)w_k\mathbf{1}_S+\left(R-r_f\mathbf{1}_{S\times S}\right)\boldsymbol{\pi}_k+\tilde{\epsilon}_k\\ &=\left(1+r_f\right)w_k\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}+\sum_{i=1}^S\pi_{ki}\begin{pmatrix}r_{i1}-r_f\\ \vdots\\ r_{iS}-r_f\end{pmatrix}+\begin{pmatrix}\epsilon_{k1}\\ \vdots\\ \epsilon_{kS}\end{pmatrix},\end{aligned} \tag{3}$$

where $\boldsymbol{\pi}_k=\left(\pi_{k1},\cdots,\pi_{kS}\right)^{\mathrm{T}}$ and $\tilde{\epsilon}_k=\left(\epsilon_{k1},\cdots,\epsilon_{kS}\right)^{\mathrm{T}}$, and where $\mathbf{1}_S=\left(1,\cdots,1\right)^{\mathrm{T}}$ is an S-dimensional vector of 1s and $\mathbf{1}_{S\times S}$ is an $S\times S$-matrix of 1s.[5] The portfolio $\boldsymbol{\pi}_k$ is called *admissible* for initial wealth w_k if

$$W_k\geq 0.$$ [6]

We assume that the axioms of the dual theory of the smooth ambiguity model [6] hold in the economy. Under this setting, agent k is assumed to evaluate their terminal wealth using

$$\mathbb{E}^{Q_k}\left[u_k\left(W_k\right)\right]=\sum_{s=1}^S Q_{ks}u_k\left(W_{ks}\right). \tag{4}$$

Here, $Q_k=\left(Q_{k1},\cdots,Q_{kS}\right)$ is a probability distribution on Ω defined by

$$Q_{ks}=\sum_{\ell=1}^{\mathfrak{L}}p_s^{(\ell)}\rho_k^{(\ell)},$$

where the probability distribution $\left(\rho_k^{(1)},\cdots,\rho_k^{(\mathfrak{L})}\right)$ is a distortion of

[4]Ambiguity neutral agents, who are equivalent to expected utility maximizers, evaluate ambiguity using this probability distribution.

[5] T denotes the transpose.

[6]See [9].

$\boldsymbol{\rho}=\left(\rho^{(1)},\cdots,\rho^{(\mathfrak{L})}\right)$ that reflects agent k's attitude towards ambiguity (see Corollary 1 of [6]). To avoid technical difficulties, we set the utility function u_k to be a map from $(0,\infty)$ to $\mathbb{R}$ that is strictly increasing, strictly concave and continuously differentiable, with $u_k'(\infty)=\lim_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}u_k(x)=0$ and $u_k'(0+)=\lim_{x\downarrow 0}\frac{\mathrm{d}}{\mathrm{d}x}u_k(x)=\infty$.

For a given initial wealth w_k, agent k chooses an admissible portfolio so as to maximize his welfare represented by (4) over the class of portfolios

$$\mathcal{A}(w_k)=\left\{\boldsymbol{\pi}_k:\mathbb{E}^P\left[H\left(W_k-\tilde{\epsilon}_k\right)\right]\le w_k,\ \mathbb{E}^{Q_k}\left[u_k^-\left(W_k\right)\right]<\infty\right\}.^7 \tag{5}$$

In other words, each agent computes the value function

$$V_k(w_k)=\sup_{\pi_k\in\mathcal{A}(w_k)}\mathbb{E}_k^Q\left[u_k\left(W_k\right)\right].$$

To solve this problem, we define a function $\mathcal{X}_k$ by

$$\mathcal{X}_k(\lambda)=\mathbb{E}^{Q_k}\left[HL_k^{-1}\left(I_k\left(\lambda L_k^{-1}H\right)-\tilde{\epsilon}_k\right)\right],\quad \lambda\in(0,\infty),$$

where L_k is the likelihood ratio defined by $L_k=\left(L_{k1},\cdots,L_{kS}\right)=\left(\frac{Q_{k1}}{P_1},\cdots,\frac{Q_{kS}}{P_S}\right)$, and I_k is the inverse function of the marginal utility u_k'. We note that I_k is a map from $(0,\infty)$ onto itself with $I_k(0+)=u_k'(0+)=\infty$ and $I_k(\infty)=u_k'(\infty)=0$.

Under the settings above, the agent's optimal wealth and the optimal portfolio are given by the following proposition.

Proposition 1. Suppose that

$$\mathcal{X}_k(\lambda)<\infty,\quad \lambda\in(0,\infty)$$

and that

$$V_k(w_k)<\infty,\quad w_k\in(0,\infty).$$

Then agent k's optimal wealth $W_k^*=\left(W_{k1}^*,\cdots,W_{kS}^*\right)^{\mathrm{T}}$ and optimal portfolio π_k^* are given by

$$W_{ks}^*=I_k\left(\lambda_k L_{ks}^{-1}H_s\right),\ s=1,\cdots,S, \tag{6}$$

$$\boldsymbol{\pi}_k^*=\left(R-r_f\mathbf{1}_{S\times S}\right)^{-1}\begin{pmatrix}W_{k1}^*-\left(1+r_f\right)w_k-\epsilon_{k1}\\ \vdots\\ W_{kS}^*-\left(1+r_f\right)w_k-\epsilon_{kS}\end{pmatrix}, \tag{7}$$

where λ_k is a solution to the equation of the budget constraint:

$$\mathcal{X}_k\left(\lambda_k\right)=\sum_{s=1}^{S}P_sH_s\left(I_k\left(\lambda_k L_{ks}^{-1}H_s\right)-\epsilon_{ks}\right)=w_k, \tag{8}$$

and $\left(R-r_f\mathbf{1}_{S\times S}\right)^{-1}$ is the inverse of $\left(R-r_f\mathbf{1}_{S\times S}\right)$.[8]

[7] $u_k^-(W_k)$ denotes the negative part of the utility, that is, $u_k^-(W_k)=\max\left\{0,-u_k\left(W_{ks}\right)\right\}$.

[8] Because the market is assumed to be complete, this inverse matrix exists.

3. Equilibrium

In this section, we show the existence and the uniqueness of the equilibrium and derive the SPD. Before proceeding with the analysis, let us start by defining equilibria in the economy.

Definition 1. An equilibrium is defined as a set of pairs $\left(\boldsymbol{\pi}_k^*, W_k^*\right)$, $k=1,\cdots,K$, of an optimal portfolio and an optimal terminal wealth satisfying the following equations:

$$\sum_{k=1}^{K} \boldsymbol{\pi}_k^* = \mathbf{0}, \tag{9}$$

$$\sum_{k=1}^{K} W_k^* = \left(1+r_f\right) w_0 \mathbf{1}_S + \tilde{\varepsilon}, \tag{10}$$

where w_0 and $\tilde{\varepsilon} = \left(\varepsilon_1,\cdots,\varepsilon_S\right)^{\mathrm{T}}$ are the aggregate initial wealth and the aggregate terminal income defined by $w_0 = \sum_{k=1}^{K} w_k$ and $\varepsilon_s = \sum_{k=1}^{K} \varepsilon_{ks}$, $s=1,\cdots,S$, respectively.

From (6), it follows that (10) is equivalent to

$$\sum_{k=1}^{K} I_k\left(\lambda_k L_{ks}^{-1} H_s\right) = w_0\left(1+r_f\right) + \varepsilon_s, \quad s=1,\cdots,S. \tag{11}$$

For arbitrary $\boldsymbol{\Gamma} = \left(\gamma_1,\cdots,\gamma_K\right) \in (0,\infty)^K$ and for each $s=1,\cdots,S$, let the function $\mathcal{I}_s\left(\cdot;\boldsymbol{\Gamma}\right)$ be defined by

$$\mathcal{I}_s\left(x;\boldsymbol{\Gamma}\right) := \sum_{k=1}^{K} I_k\left(\frac{x}{\gamma_k L_{ks}}\right).$$

Then (11) is equivalent to

$$\mathcal{I}_s\left(H_s;\boldsymbol{\Gamma}\right) = w_0\left(1+r_f\right) + \varepsilon_s, \quad s=1,\cdots,S,$$

with $\gamma_k = \dfrac{1}{\lambda_k}$. If the inverse function $\mathcal{H}_s\left(\cdot;\boldsymbol{\Gamma}\right)$ of $\mathcal{I}_s\left(\cdot;\boldsymbol{\Gamma}\right)$ is defined by

$$\mathcal{I}_s\left(\mathcal{H}_s\left(x;\boldsymbol{\Gamma}\right);\boldsymbol{\Gamma}\right) = x, \quad s=1,\cdots,S, \tag{12}$$

then the SPD in equilibrium is given by

$$H_s = \mathcal{H}_s\left(w_0\left(1+r_f\right) + \varepsilon_s;\boldsymbol{\Gamma}\right).$$

The budget constraint (8) in equilibrium can be rewritten as

$$\sum_{s=1}^{S} P_s \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\left[I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)}{\gamma_k L_{ks}}\right) - \epsilon_{ks}\right] = w_k. \tag{13}$$

This means that the equilibrium can be characterized by $\boldsymbol{\Gamma}$ to satisfy (13).

For each $s=1,\cdots,S$, and $\left(\gamma_1,\cdots,\gamma_K\right) \in (0,\infty)^K$, we define the utility function of the representative agent by

$$U\left(x;\mathcal{L}_s\right) = \max_{x=\sum_{k=1}^{K} x_k,\, x_k \ge 0} \sum_{k=1}^{K} \gamma_k L_{ks} u_k\left(x_k\right),$$

where $\mathcal{L}_s$ denotes $\mathcal{L}_s = \left(\gamma_1 L_{1s},\cdots,\gamma_K L_{Ks}\right) \in (0,\infty)^K$, $s=1,\cdots,S$.

The SPD in equilibrium is characterized in the following proposition.

Proposition 2.

$$U'\left(w_0\left(1+r_f\right)+\varepsilon_s,\mathcal{L}_s\right)=\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right).$$

We note that the utility function of the representative agent has positive homogeneity with respect to $\mathcal{L}_s$ by the definition, that is, for any positive constant c, $U\left(x;c\mathcal{L}_s\right)=cU\left(x;\mathcal{L}_s\right)$. Hence, this proposition implies that $\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)$ also has this property with respect to $\boldsymbol{\Gamma}$. That is, for each $s=1,\cdots,S$, and each positive constant *c*, the following equation holds:

$$\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;c\boldsymbol{\Gamma}\right)=c\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\quad\forall\boldsymbol{\Gamma}\in\left(0,\infty\right)^K. \tag{14}$$

From this fact, we can also confirm that the budget constraint (13) does not change for any positive homogeneity with respect to $\boldsymbol{\Gamma}$. That is, we can write

$$\begin{aligned}0&=\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\left[I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)}{\gamma_k L_{ks}}\right)-\epsilon_{ks}\right]-w_k\\&=\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\left[I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)}{\gamma_k L_{ks}}\right)-\epsilon_{ks}-w_k\left(1+r_f\right)\right]\\&=\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;c\boldsymbol{\Gamma}\right)\left[I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;c\boldsymbol{\Gamma}\right)}{c\gamma_k L_{ks}}\right)-\epsilon_{ks}-w_k\left(1+r_f\right)\right]\\&=\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;c\boldsymbol{\Gamma}\right)\left[I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;c\boldsymbol{\Gamma}\right)}{c\gamma_k L_{ks}}\right)-\epsilon_{ks}\right]-w_k\end{aligned} \tag{15}$$

for every positive constant c.

Let κ be the index of absolute risk aversion for the representative agent, defined by

$$\kappa\left(x;\mathcal{L}_s\right)=-\frac{U''\left(x;\mathcal{L}_s\right)}{U'\left(x;\mathcal{L}_s\right)},\quad s=1,\cdots,S. \tag{16}$$

We immediately obtain the following corollary to the above proposition.

Corollary 1.

$$\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)=\frac{\exp\left(-\int_0^{w_0\left(1+r_f\right)+\varepsilon_s}\kappa\left(x;\mathcal{L}_s\right)\mathrm{d}x\right)}{\left(1+r_f\right)\mathbb{E}^P\left[\exp\left(-\int_0^{w_0\left(1+r_f\right)+\tilde{\varepsilon}}\kappa\left(x;\mathcal{L}\right)\mathrm{d}x\right)\right]}.$$

We note that Corollary 1 is an extension of the general economic premium principle of Bühlmann under risk [2] to that under ambiguity.

The following proposition states the existence and the uniqueness of the equilibrium.

Proposition 3. There exists a $\Gamma\in\left(0,\infty\right)^K$ satisfying (13). Furthermore, suppose that the marginal utilities u_k', $k=1,\cdots,K$, satisfy the condition

$$xu_k'\left(x\right)\text{ is increasing with respect to }x\in\left(0,\infty\right). \tag{17}$$

Then Γ is unique up to positive constant multiples.

4. Asset Pricing

The following proposition characterizes the excess return in equilibrium.

Proposition 4.

$$\mu_i - r_f = -\frac{\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\tilde{z}_i\right]}{\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right]}. \tag{18}$$

Let $\boldsymbol{\pi}_M = \left(\pi_{M1},\cdots,\pi_{MS}\right)^{\mathrm{T}}$ be a portfolio satisfying

$$\begin{pmatrix} U'\left(w_0\left(1+r_f\right)+\varepsilon_1,\mathcal{L}_1\right) \\ \vdots \\ U'\left(w_0\left(1+r_f\right)+\varepsilon_S,\mathcal{L}_S\right) \end{pmatrix} = \left(R - r_f \mathbf{1}_{S\times S}\right)\boldsymbol{\pi}_M. \tag{19}$$

We note that there exists a unique $\boldsymbol{\pi}_M$ because the market is complete. We refer to $\boldsymbol{\pi}_M$ as *the market portfolio*. The terminology of the market portfolio is used as the common portfolio that every agent in the economy holds. At the end of this section, we show that the market portfolio degenerates to the classical market portfolio under certain conditions. The following corollary is a natural extension of CAPM [11] [19] under ambiguity.

Corollary 2. Let $\tilde{r}_M = \frac{\sum_{i=1}^{S}\pi_{Mi}\tilde{r}_i}{\sum_{i=1}^{S}\pi_{Mi}}$ be the rate of return of the market portfolio and let $\mu_M = \frac{\sum_{i=1}^{S}\pi_{Mi}\mu_i}{\sum_{i=1}^{S}\pi_{Mi}}$ be its expected return.

Then

$$\mu_i - r_f = \beta_i\left(\mu_M - r_f\right),$$

where

$$\beta_i = \frac{\mathrm{Cov}\left(\tilde{r}_i,\tilde{r}_M\right)}{\mathrm{Var}\left(\tilde{r}_M\right)},$$

and Cov and Var denote the covariance and the variance under P, respectively.

Next, we show that the classical two-fund separation theorem [20] holds in a special case of my model. By this result, our analysis can be seen as a natural extension of CAPM under risk to that under ambiguity. Before proceeding, we note that each agent's optimal portfolio can be decomposed as a sum of the market portfolio and his specific portfolio. From (7) and (19), the following proposition holds.

Proposition 5. For agent k, the optimal portfolio $\boldsymbol{\pi}_k^*$ can be decomposed as a sum of the market portfolio $\boldsymbol{\pi}_M$ and his specific portfolio $\boldsymbol{\pi}_k^e$ as follows:

$$\boldsymbol{\pi}_k^* = \boldsymbol{\pi}_M + \boldsymbol{\pi}_k^e, \tag{20}$$

where

$$\boldsymbol{\pi}_k^e = \left(R - r_f \mathbf{1}_{S\times S}\right)^{-1} \begin{pmatrix} W_{k1}^* - \left(1+r_f\right)w_k - \epsilon_{k1} - U'\left(w_0\left(1+r_f\right)+\varepsilon_1;\mathcal{L}_1\right) \\ \vdots \\ W_{kS}^* - \left(1+r_f\right)w_k - \epsilon_{kS} - U'\left(w_0\left(1+r_f\right)+\varepsilon_S;\mathcal{L}_S\right) \end{pmatrix}.$$

The following is the two-fund separation theorem under ambiguity.

Proposition 6. Assume that all agents have quadratic utility functions, that they are all ambiguity neutral, and that all their terminal incomes are proportional to the aggregate income $\tilde{\varepsilon}$. Then the optimal portfolio for each agent consists of the market portfolio and the safe security.

5. Some Comparative Statics

We examine how ambiguity preference influences the SPD in equilibrium. We also determine the effects of ambiguity preference on equilibrium excess returns based on the CAPM derived in the previous section.

To keep the analysis simple, we consider a specific case with $\mathfrak{L}=2$, $S=2$, and $K=2$.[9] We refer to this economy as a *two-state economy*. In the two-state economy, the probability distributions over the index set $\{1,2\}$ are given as follows:

$$\rho^{(1)}=\rho,\ \rho^{(2)}=1-\rho,$$

$$\rho_k^{(1)}=\rho_k,\ \rho_k^{(2)}=1-\rho_k,\ \rho,\rho_k\in(0,1),\ k=1,2.$$

Then the reference probability distribution (P_1,P_2), and agent k's probability distribution (Q_{k1},Q_{k2}), $k=1,2$, are given by

$$P_1=P=p^{(1)}\rho+p^{(2)}(1-\rho),$$

$$P_2=1-P=\left(1-p^{(1)}\right)\rho+\left(1-p^{(2)}\right)(1-\rho),$$

$$Q_{k1}=Q_k=p^{(1)}\rho_k+p^{(2)}(1-\rho_k),$$

$$Q_{k2}=1-Q_k=\left(1-p^{(1)}\right)\rho_k+\left(1-p^{(2)}\right)(1-\rho_k),\ k=1,2.$$

By Definition 1, the SPDs H_s, $s=1,2$, in equilibrium are given by a solution of the following simultaneous equations:

$$\begin{cases} PH_1\left(I_1\left(\dfrac{H_1}{\gamma_1}\dfrac{P}{Q_1}\right)-\epsilon_{11}\right)+(1-P)H_2\left(I_1\left(\dfrac{H_2}{\gamma_1}\dfrac{1-P}{1-Q_1}\right)-\epsilon_{12}\right)=w_1, \\ PH_1\left(I_2\left(\dfrac{H_1}{\gamma_2}\dfrac{P}{Q_2}\right)-\epsilon_{21}\right)+(1-P)H_2\left(I_2\left(\dfrac{H_2}{\gamma_2}\dfrac{1-P}{1-Q_2}\right)-\epsilon_{22}\right)=w_2, \\ I_1\left(\dfrac{H_1}{\gamma_1}\dfrac{P}{Q_1}\right)+I_2\left(\dfrac{H_1}{\gamma_2}\dfrac{P}{Q_2}\right)=(w_1+w_2)\left(1+r_f\right)+(\epsilon_{11}+\epsilon_{21}), \\ I_1\left(\dfrac{H_2}{\gamma_1}\dfrac{1-P}{1-Q_1}\right)+I_2\left(\dfrac{H_2}{\gamma_2}\dfrac{1-P}{1-Q_2}\right)=(w_1+w_2)\left(1+r_f\right)+(\epsilon_{12}+\epsilon_{22}). \end{cases} \tag{21}$$

We impose the following assumption on the economy to get explicit results.

[9]Although a similar result is expected to obtain in the general case, we leave this for future research.

Assumption 1. a) Each agent gains strictly higher expected utility under the first-order probabilities with $\ell = 1$ than with $\ell = 2$; that is, for $k = 1,2$,

$$p^{(1)}u_k(W_{k1}) + \left(1 - p^{(1)}\right)u_k(W_{k2}) < p^{(2)}u_k(W_{k1}) + \left(1 - p^{(2)}\right)u_k(W_{k2}).$$

b) Each agent has log utility; that is, $u_k(x) = \log(x)$, $k = 1,2$.

Remark 1. We note that if Assumption 1 (a) holds and agent k, $k = 1,2$, is strictly ambiguity averse (loving) in the two-state economy, then $\rho_k > (<)\rho$ holds from Corollary 1 and Proposition 1 of [6].

First, we consider the effects of ambiguity aversion and loving on the SPD in equilibrium.

Proposition 7. In the two-state economy, suppose that

$$p^{(1)} < (>)\, p^{(2)}$$

is satisfied under Assumption 1. Then the following statements hold.

1) If each agent is strictly ambiguity averse, then the SPD H_1 with ambiguity is strictly lower (higher) than that without ambiguity, and the SPD H_2 with ambiguity is strictly higher (lower) than that without ambiguity.

2) If each agent is strictly ambiguity loving, then the SPD H_1 with ambiguity is strictly higher (lower) than that without ambiguity, and the SPD H_2 with ambiguity is strictly lower (higher) than that without ambiguity.

As stated in Remark 1, if all agents are strictly ambiguity averse, then they uniformly increase the weights ρ_k, $k = 1,2$, of the index $\ell = 1$ as $\rho_k > \rho$ under Assumption 1 (a). This leads to $Q_k < (>) P$ for both $k = 1,2$ under the condition $p^{(1)} < (>)\, p^{(2)}$. As a result, ambiguity aversion increases (decreases) H_1 and decreases (increases) H_2. The same reasoning can be applied to the case of ambiguity loving.

Next, we consider the effect of more ambiguity aversion on the excess returns of the ambiguous securities in equilibrium. We compare two economies consisting of the same two-state economy except for the ambiguity preferences of each agent. To distinguish between the two economies, we call them Economy A and Economy B. We assume that all of the agents in Economy A are more ambiguity averse than those in Economy B in the sense of [6]. Under this assumption, we obtain comparative static predictions for how more ambiguity aversion influences the equilibrium excess returns as a corollary of Proposition 7.

Corollary 3. Assume that the conditions of Proposition 7 hold. If the random terms of the rate of return for the i-th ambiguous security are arranged in the order

$$z_{i1} < z_{i2}, \tag{22}$$

then the excess return $\mu_i - r_f$ in equilibrium in Economy A is lower (higher) than that in Economy B.

If the order of the random terms is reversed, that is, $z_{i1} > z_{i2}$, then the excess return $\mu_i - r_f$ in equilibrium in Economy A is higher (lower) than that in Economy B.

We note that, from Theorem 2 of [6] and Proposition 7, the SPD H_1 in Economy A is lower (higher) than that in Economy B if $p^{(1)} < (>) p^{(2)}$. Thus, the excess return $\mu_i - r_f$ in Economy A is lower (higher) than that in Economy B in the case where (22) holds. The last statement of the corollary also holds by the same reasoning.

6. Concluding Remarks

This paper studies an equilibrium asset pricing model for a static pure exchange economy with ambiguity. The preference of an agent in the economy is represented by the dual theory of the smooth ambiguity model from [6]. An equilibrium is fully characterized by the SPD, so we derive the SPD and show its existence and uniqueness in the economy. Applying the SPD, the equilibrium excess return is derived. Equilibrium excess returns can be rewritten in an extended version of CAPM under ambiguity. The optimal portfolio consists of the agent's specific portfolio in addition to the safe asset and the market portfolio. The classical separation theorem is obtained as a special case. We also conduct comparative statics analysis on a specific case of the two-state economy and show how ambiguity preferences influence the SPD and returns of ambiguous securities in the equilibrium.

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Appendix A. Proofs

A.1. Proof of Proposition 1

$\boldsymbol{\pi}_k^*$ is trivially an admissible portfolio because $W_{ks}^* \ge 0$ from (6). We first show that $\boldsymbol{\pi}_k^* \in \mathcal{A}(w_k)$. It is obvious that $\mathbb{E}^P\left[H\left(W_k^* - \tilde{\epsilon}_k\right)\right] \le w_k$ holds by (8).

Following [7] [8] and [9], the convex dual $\hat{u}_k$ of u_k is defined by

$$\hat{u}_k(x) = \max_{y\in(0,\infty)}\left[u_k(y) - xy\right], \quad x \in (0,\infty),$$

and is a decreasing, convex and continuously differentiable function on $(0,\infty)$, satisfying

$$\hat{u}_k(x) = u_k\left(I_k(x)\right) - xI_k(x),$$
$$\hat{u}_k'(x) = -I_k(x), \quad x \in (0,\infty).$$

Using the convex dual, we have

$$u_k\left(I_k\left(\lambda_k L_{ks}^{-1} H_s\right)\right) - \lambda_k L_{ks}^{-1} H_s I_k\left(\lambda_k L_{ks}^{-1} H_s\right) \ge u_k(1) - \lambda_k L_{ks}^{-1} H_s$$
$$\Leftrightarrow u_k\left(W_{ks}^*\right) \ge u_k(1) + \lambda_k L_{ks}^{-1} H_s\left(W_k^* - 1\right), \quad s = 1,\cdots,S.$$

Because $W_{ks}^* \ge 0$, we have $\lambda_k L_{ks}^{-1} H_s W_{ks}^* \ge 0$, and so

$$u_k\left(W_{ks}^*\right) \ge u_k(1) - \lambda_k L_{ks}^{-1} H_s, \quad s = 1,\cdots,S. \tag{23}$$

From (23), we have

$$\mathbb{E}^{Q_k}\left[u_k^-\left(W_{ks}^*\right)\right] \le \left|u_k(1)\right| + \lambda_k \mathbb{E}^{Q_k}\left[L_k^{-1} H\right]$$
$$= \left|u_k(1)\right| + \lambda_k \mathbb{E}^P[H] = \left|u_k(1)\right| + \frac{\lambda_k}{1 + r_f} < \infty.$$

Thus $\boldsymbol{\pi}_k^* \in \mathcal{A}(w_k)$.

Next, we show that $\boldsymbol{\pi}_k^*$ is optimal. For all $\lambda > 0$ and $\boldsymbol{\pi}_k \in \mathcal{A}(w_k)$,

$$\begin{aligned}
&\mathbb{E}^{Q_k}\left[u_k\left(W_k\right)\right] \\
&\le \mathbb{E}^{Q_k}\left[u_k\left(W_k\right)\right] + \lambda\left\{w_k - \mathbb{E}^Q\left[L_k^{-1} H\left(W_k - \tilde{\epsilon}_k\right)\right]\right\} \\
&\le \mathbb{E}^{Q_k}\left[\hat{u}_k\left(\lambda L_k^{-1} H\right)\right] + \lambda\left(w_k + \mathbb{E}^Q\left[L_k^{-1} H \tilde{\epsilon}_k\right]\right) \\
&= \mathbb{E}^{Q_k}\left[u_k\left(I_k\left(\lambda L_k^{-1} H\right)\right)\right] + \lambda\left\{w_k - \mathbb{E}^Q\left[L_k^{-1} H\left(I_k\left(\lambda L_k^{-1} H\right) - \tilde{\epsilon}_k\right)\right]\right\}.
\end{aligned} \tag{24}$$

The first inequality is due to the budget constraint, $\mathbb{E}^Q\left[L_k H\left(W_k - \tilde{\epsilon}_k\right)\right] \le w_k$. The second inequality and the equality are due to the definition of $\hat{u}_k$. From (6) and (8), the expression (24) holds with equality if and only if $W_k = W_k^*$ and $\lambda = \lambda_k$. This means that $\boldsymbol{\pi}_k = \boldsymbol{\pi}_k^*$ is optimal from (7).

A.2. Proof of Proposition 2

Let $W_{ks}^\lozenge$ be Defined by

$$W_{ks}^\lozenge = I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1 + r_f\right) + \varepsilon_s; \boldsymbol{\Gamma}\right)}{\gamma_k L_{ks}}\right). \tag{25}$$

Then

$$\sum_{k=1}^{K} W_{ks}^{\Diamond} = \sum_{k=1}^{K} I_k \left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}\right)}{\gamma_k L_{ks}} \right)$$

$$= \mathcal{I}_s \mathcal{H}_s\left(\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}\right);\mathbf{\Gamma}\right) = w_0\left(1+r_f\right)+\varepsilon_s$$

by the definitions of $\mathcal{I}$ and $\mathcal{H}$. This means that $W_{ks}^{\Diamond}$ satisfies (10).

Since I_k is the inverse of u_k', (25) can be rewritten as

$$\gamma_k L_{ks} u_k'\left(W_{ks}^{\Diamond}\right) = \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}\right).$$

Since u_k is strictly concave,

$$\begin{aligned}
&\sum_{k=1}^{K} \gamma_k L_{ks} u_k\left(x_k\right) \\
&\le \sum_{k=1}^{K} \gamma_k L_{ks} \left\{u_k\left(W_{ks}^{\Diamond}\right)+\left(x_k-W_{ks}^{\Diamond}\right)u_k'\left(W_{ks}^{\Diamond}\right)\right\} \\
&= \sum_{k=1}^{K} \gamma_k L_{ks} u_k\left(W_{ks}^{\Diamond}\right)+\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}\right)\sum_{k=1}^{K}\left(x_k-W_{ks}^{\Diamond}\right)
\end{aligned}$$

for all $x_k \in (0,\infty)$. This holds with equality if and only if $x_k = W_{ks}^{\Diamond}$. Hence, we have

$$\begin{aligned}
&U\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathcal{L}_s\right) \\
&= \sum_{k=1}^{K} \gamma_k L_{ks} u_k\left(W_{ks}^{\Diamond}\right) = \sum_{k=1}^{K} \gamma_k L_{ks} u_k\left(I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}\right)}{\gamma_k L_{ks}}\right)\right).
\end{aligned}$$

Differentiating the above equation completes the proof.[10]

A.3. Proof of Corollary 1

From (16), there is a constant A for which

$$U'\left(x;\mathcal{L}_s\right) = A\exp\left(-\int_0^x \kappa\left(t;\mathcal{L}_s\right)\mathrm{d}t\right).$$

The result then follows from Proposition 2 and the fact that $\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right] = \mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\right] = \frac{1}{1+r_f}$.

A.4. Proof of Proposition 3

We first show the existence. Let $\mathbb{K} = \{1,\cdots,K\}$ be an index set of agents and let $\boldsymbol{e}_1,\cdots,\boldsymbol{e}_K \in (0,\infty)^K$ be the K-dimensional unit coordinate vectors. For any $\mathbb{B} \subset \mathbb{K}$, we denote the convex hull of $\{\boldsymbol{e}_k : k \in \mathbb{B}\}$ by

$$S_{\mathbb{B}} = \left\{\sum_{k\in\mathbb{B}} \gamma_k \boldsymbol{e}_k : \sum_{k\in\mathbb{B}} \gamma_k = 1, \gamma_k \ge 0, k \in \mathbb{B}\right\}.$$

Let $S_{\mathbb{B}}^{+} \subseteq S_{\mathbb{B}}$ be the set

[10]From (12), we use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=1}^{K} I_k\left(\frac{\mathcal{H}_s\left(x;\mathbf{\Gamma}\right)}{\gamma_k L_{ks}}\right) = 1.$$

$$S_{\mathbb{B}}^{+}=\left\{\sum_{k\in\mathbb{B}}\gamma_k\boldsymbol{e}_k:\sum_{k\in\mathbb{B}}\gamma_k=1,\gamma_k>0,k\in\mathbb{B}\right\}.$$

For each $k\in\mathbb{K}$, we define a function $C_k:S_{\mathbb{K}}\to\mathbb{R}$ by

$$C_k(\boldsymbol{\Gamma})=\begin{cases}\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\left(I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)}{\gamma_k L_{ks}}\right)-\epsilon_{ks}\right)-w_k & \text{if } \gamma_k>0,\\ -\sum_{s=1}^{S}P_s\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\boldsymbol{\Gamma}\right)\epsilon_{ks}-w_k & \text{if } \gamma_k=0.\end{cases}$$

To prove that the proposition holds, we have to show that there exists a $\Gamma\in S_{\mathbb{K}}$ satisfying $C_k(\boldsymbol{\Gamma})=0$ for each $k\in\mathbb{K}$.

Because the function C_k is continuous, the set

$$F_k=\left\{\boldsymbol{\Gamma}\in S_{\mathbb{K}};C_k(\boldsymbol{\Gamma})\geq 0\right\}$$

is closed. On the other hand, from (11) and (12),

$$\sum_{k\in\mathbb{K}}C_k(\boldsymbol{\Gamma})=0\quad\forall\boldsymbol{\Gamma}\in S_{\mathbb{K}}. \tag{26}$$

Now, suppose that there exists a $\hat{\boldsymbol{\Gamma}}\in S_{\mathbb{K}}$ such that $\hat{\boldsymbol{\Gamma}}\notin\bigcup_{k\in\mathbb{K}}F_k$. Then $C_k(\hat{\boldsymbol{\Gamma}})<0$ for all $k\in\mathbb{K}$, which contradicts (26). Therefore,

$$S_{\mathbb{K}}=\bigcup_{k\in\mathbb{K}}F_k.$$

Furthermore, suppose that there exists a $\hat{\boldsymbol{\Gamma}}\in S_{\mathbb{B}}$ such that $\hat{\boldsymbol{\Gamma}}\notin\bigcup_{k\in\mathbb{B}}F_k$. Then $C_k(\hat{\boldsymbol{\Gamma}})<0$ for all $k\in\mathbb{B}$. In this case, let $\hat{\gamma}_j=0$ for $j\in\mathbb{K}\setminus\mathbb{B}$. Then $\hat{\boldsymbol{\Gamma}}\in S_{\mathbb{K}}$ and $\sum_{k\in\mathbb{K}}C_k(\hat{\boldsymbol{\Gamma}})<0$, which again contradicts (26). Therefore,

$$S_{\mathbb{B}}\subset\bigcup_{k\in\mathbb{B}}F_k\quad\forall\mathbb{B}\subset\mathbb{K}. \tag{27}$$

From (27) and the Knaster-Kratowski-Mazurkiewicz Theorem (cf. p. 26 of [1]), the set $\cap_{k\in\mathbb{K}}F_k$ is nonempty. For any $\boldsymbol{\Gamma}^*\in\bigcap_{k\in\mathbb{K}}F_k$, we have

$$C_k(\boldsymbol{\Gamma}^*)=0\quad\forall k\in\mathbb{K}. \tag{28}$$

Otherwise we would have $\sum_{k\in\mathbb{K}}C_k(\boldsymbol{\Gamma}^*)>0$, which contradicts (26). We also have $\boldsymbol{\Gamma}^*\in S_{\mathbb{K}}^{+}$, because if there exists a $k\in\mathbb{K}$ such that $\gamma_k^*=0$, then $C_k(\boldsymbol{\Gamma}^*)<0$, which contradicts (28). Therefore, we can conclude that there exists a $\boldsymbol{\Gamma}^*\in\bigcap_{k\in\mathbb{K}}F_k$ that belongs to $(0,\infty)^K$.

Next, we show the uniqueness. For any pair of vectors $\left(\boldsymbol{\Gamma}^{(i)},\boldsymbol{\Gamma}^{(j)}\right)\in(0,\infty)^K\times(0,\infty)^K$, we consider the usual order:

$$\boldsymbol{\Gamma}^{(i)}\leq\boldsymbol{\Gamma}^{(j)}\Leftrightarrow\gamma_k^{(i)}\leq\gamma_k^{(j)}\quad\forall k=1,\cdots,K,$$

$$\boldsymbol{\Gamma}^{(i)}<\boldsymbol{\Gamma}^{(j)}\Leftrightarrow\boldsymbol{\Gamma}^{(i)}\leq\boldsymbol{\Gamma}^{(j)},\boldsymbol{\Gamma}^{(i)}\neq\boldsymbol{\Gamma}^{(j)}.$$

Let $\boldsymbol{\Gamma}^{(1)}$ and $\hat{\boldsymbol{\Gamma}}$ be vectors in $(0,\infty)$, both of which satisfy (13). We define another vector $\boldsymbol{\Gamma}^{(2)}$ by $\boldsymbol{\Gamma}^{(2)}=c\hat{\boldsymbol{\Gamma}}$ for a positive constant $c=\max_{k\in\mathbb{K}}\gamma_k^{(1)}/\hat{\gamma}_k$. Then, from (15), $\boldsymbol{\Gamma}^{(2)}$ also satisfies (13), and $\boldsymbol{\Gamma}^{(1)}\leq\boldsymbol{\Gamma}^{(2)}$. If $\boldsymbol{\Gamma}^{(1)}=\boldsymbol{\Gamma}^{(2)}$, then $\boldsymbol{\Gamma}^{(1)}$ is a positive constant multiple of $\hat{\boldsymbol{\Gamma}}$. Hence, we have only to show that $\boldsymbol{\Gamma}^{(1)}<\boldsymbol{\Gamma}^{(2)}$ does not hold.

Suppose that $\mathbf{\Gamma}^{(1)} < \mathbf{\Gamma}^{(2)}$. Then, from the definition of $\mathcal{H}_s$,

$$\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right) < \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right)$$

holds. Hence, for $k \in \mathbb{K}$ such that $\gamma_k^{(1)} = c\hat{\gamma}_k = \gamma_k^{(2)}$, we have

$$\begin{aligned} &\frac{1}{\gamma_k^{(1)}}\sum_{s=1}^{S} P_s \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right)\epsilon_{ks} \\ &< \frac{1}{\gamma_k^{(2)}}\sum_{s=1}^{S} P_s \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right)\epsilon_{ks}. \end{aligned} \tag{29}$$

On the other hand, because I_k is the inverse of u_k, (17) is equivalent to $xI_k(x)$ decreasing with respect to $x \in (0,\infty)$. Hence, noting that $L_{ks} > 0$, $s = 1,\cdots,S$, $k \in \mathbb{K}$, we have

$$\begin{aligned} &\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right)}{\gamma_k^{(1)}} I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right)}{\gamma_k^{(1)} L_{ks}}\right) \\ &\geq \frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right)}{\gamma_k^{(2)}} I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right)}{\gamma_k^{(2)} L_{ks}}\right). \end{aligned}$$

This inequality leads to

$$\begin{aligned} &\frac{1}{\gamma_k^{(1)}}\sum_{s=1}^{S} P_s \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right) I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(1)}\right)}{\gamma_k^{(1)} L_{ks}}\right) \\ &\geq \frac{1}{\gamma_k^{(2)}}\sum_{s=1}^{S} P_s \mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right) I_k\left(\frac{\mathcal{H}_s\left(w_0\left(1+r_f\right)+\varepsilon_s;\mathbf{\Gamma}^{(2)}\right)}{\gamma_k^{(2)} L_{ks}}\right). \end{aligned} \tag{30}$$

Combining (29) and (30), we have

$$\frac{1}{\gamma_k^{(1)}} C_k\left(\mathbf{\Gamma}^{(1)}\right) > \frac{1}{\gamma_k^{(2)}} C_k\left(\mathbf{\Gamma}^{(2)}\right).$$

However, this contradicts (28). That is, $\mathbf{\Gamma}^{(1)} < \mathbf{\Gamma}^{(2)}$ never holds.

A.5. Proof of Proposition 4

From (2), we have

$$\begin{aligned} &\mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\left(1+\tilde{r}_i\right)\right] \\ &= \mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\left(1+r_f+\mu_i-r_f+\tilde{z}_i\right)\right] = 1 \\ &\Leftrightarrow \mu_i - r_f = -\frac{\mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\tilde{z}_i\right]}{\mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\right]}, \end{aligned}$$

where we use $\mathbb{E}^P\left[\mathcal{H}\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathbf{\Gamma}\right)\right] = 1/\left(1+r_f\right)$ by (1).

A.6. Proof of Corollary 2

From (18) and (19),

$$\mu_i - r_f = -\frac{\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\tilde{z}_i\right]}{\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right]} = -\frac{\mathbb{E}^P\left[\left(\sum_{j=1}^{S}\tilde{z}_j\pi_{Mj}\right)\tilde{z}_i\right]}{\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right]}, \tag{31}$$

where we have used the fact: $\mathbb{E}^P\left[\tilde{z}_j\right]=0$ in the second equality. From (31) and the definition of μ_M, we have

$$\mu_M - r_f = \sum_{i=1}^{S}\frac{\pi_{Mi}}{\sum_{j=1}^{S}\pi_{Mj}}\left(\mu_i - r_f\right) = -\frac{\mathbb{E}^P\left[\left(\sum_{j=1}^{S}\pi_{Mj}\tilde{z}_j\right)^2\right]}{\sum_{j=1}^{S}\pi_{Mj}\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right]}. \tag{32}$$

Canceling out $\mathbb{E}^P\left[U'\left(w_0\left(1+r_f\right)+\tilde{\varepsilon};\mathcal{L}\right)\right]$ from (31) and (32), we obtain

$$\mu_i - r_f = \sum_{j=1}^{S}\pi_{Mj}\frac{\mathbb{E}^P\left[\tilde{z}_i\sum_{j=1}^{S}\tilde{z}_j\pi_{Mj}\right]}{\mathbb{E}^P\left[\left(\sum_{j=1}^{S}\pi_{Mj}\tilde{z}_j\right)^2\right]}\left(\mu_M - r_f\right).$$

Here, recalling

$$\mathrm{Cov}\left(\tilde{r}_i,\tilde{r}_M\right) = \frac{1}{\sum_{j=1}^{S}\pi_{Mj}}\mathbb{E}^P\left[\tilde{z}_i\sum_{j=1}^{S}\pi_{Mj}\tilde{z}_j\right]$$

and

$$\mathrm{Var}\left(\tilde{r}_M\right) = \frac{1}{\left(\sum_{j=1}^{S}\pi_{Mj}\right)^2}\mathbb{E}^P\left[\left(\sum_{j=1}^{S}\pi_{Mj}\tilde{z}_j\right)^2\right],$$

we obtain the result.

A.7. Proof of Proposition 6

Because all agents have quadratic utility functions, we can assume that the marginal utility for each agent *k*, $k=1,\cdots,K$, is given by

$$u_k'(x) = -x + \alpha_k, \quad x < \alpha_k,$$

for some constant α_k. Noting that ambiguity neutrality implies that $L_{ks}=1$, $s=1,\cdots,S$, (see, [6]), it follows from (6) that

$$W_{ks}^* = \alpha_k - \lambda_k H_s. \tag{33}$$

Because the terminal income is proportional to the aggregate income $\tilde{\varepsilon}$, there exists a constant η_k satisfying $\tilde{\epsilon}_k = \eta_k\tilde{\varepsilon}$ and $\sum_{k=1}^{K}\eta_k = 1$. Hence, from (8),

$$\lambda_k = \frac{\frac{\alpha_k}{1+r_f} - w_k - \eta_k\mathbb{E}^P\left[H\tilde{\varepsilon}\right]}{\mathbb{E}^P\left[H^2\right]}. \tag{34}$$

Substituting (34) into (33), we have from (3) that

$$\begin{aligned} W_{ks}^* &= \alpha_k - \frac{\dfrac{\alpha_k}{1+r_f} - w_k - \eta_k \mathbb{E}^P\left[H\tilde{\varepsilon}\right]}{\mathbb{E}^P\left[H^2\right]} H_s \\ &= \left(1+r_f\right) w_k + \left(r_{1s} - r_f, \cdots, r_{Ss} - r_f\right) \boldsymbol{\pi}_k^* + \eta_k \varepsilon_s \\ \Leftrightarrow & \left(\left(1+r_f\right) - \frac{H_s}{\mathbb{E}^P\left[H^2\right]}\right)\left(\frac{\alpha_k}{1+r_f} - w_k\right) + \eta_k \left(\frac{\mathbb{E}^P\left[H\tilde{\varepsilon}\right] H_s}{\mathbb{E}^P\left[H^2\right]} - \varepsilon_s\right) \\ &= \left(r_{1s} - r_f, \cdots, r_{Ss} - r_f\right) \boldsymbol{\pi}_k^*. \end{aligned} \tag{35}$$

Summing the above equations for $k = 1, \cdots, K$, and applying (9), we have

$$\frac{\mathbb{E}^P\left[H\tilde{\varepsilon}\right] H_s}{\mathbb{E}^P\left[H^2\right]} - \varepsilon_s = -\left(\left(1+r_f\right) - \frac{H_s}{\mathbb{E}^P\left[H^2\right]}\right)\zeta,$$

where $\zeta = \sum_{k=1}^{K}\left(\dfrac{\alpha_k}{1+r_f} - w_k\right)$. Substituting this into (35), we have

$$\begin{aligned} &\left(\left(1+r_f\right) - \frac{H_s}{\mathbb{E}^P\left[H^2\right]}\right)\left(\frac{\alpha_k}{1+r_f} - w_k - \zeta\eta_k\right) \\ &= \left(r_{1s} - r_f, \cdots, r_{Ss} - r_f\right) \boldsymbol{\pi}_k^*. \end{aligned}$$

Hence, from Proposition 2 and by the definition of the market portfolio $\boldsymbol{\pi}_M$, the optimal portfolio $\boldsymbol{\pi}_k^*$ satisfies

$$\begin{aligned} &\left(r_{1s} - r_f, \cdots, r_{Ss} - r_f\right) \boldsymbol{\pi}_k^* \\ &= \iota_k \left(\left(1+r_f\right) - \frac{1}{\mathbb{E}^P\left[H^2\right]}\left(r_{1s} - r_f, \cdots, r_{Ss} - r_f\right) \boldsymbol{\pi}_M\right), \end{aligned}$$

where $\iota_k = \dfrac{\alpha_k}{1+r_f} - w_k - \zeta\eta_k$. This means that the optimal portfolio consists of the safe security and the market portfolio.

A.8. Proof of Proposition 7

Under the condition: $p^{(1)} < (>)\, p^{(2)}$, if each agent is strictly ambiguity averse, from the definition of the probability distributions $\left(Q_{k1}, Q_{k2}\right)$, $k = 1, 2$, and Remark 1,

$$Q_k = p^{(1)}\rho_k + p^{(2)}\left(1-\rho_k\right) < (>)\, p^{(1)}\rho + p^{(2)}\left(1-\rho\right) = P.$$

Similarly, under the same condition, if each agent is strictly ambiguity loving, $Q_k > (<)\, P$. Hence, to prove the proposition it is sufficient to show that

$$\frac{\partial H_1}{\partial Q_k} > 0, \quad \frac{\partial H_2}{\partial Q_k} < 0, \quad k = 1, 2. \tag{36}$$

From Assumption 1 (b), (21) can be explicitly rewritten as

$$\begin{cases} \gamma_1 = w_1 + PH_1\epsilon_{11} + (1-P)H_2\epsilon_{12}, \\ \gamma_2 = w_2 + PH_1\epsilon_{21} + (1-P)H_2\epsilon_{22}, \\ \dfrac{\gamma_1 Q_1 + \gamma_2 Q_2}{H_1 P} = (w_1 + w_2)(1+r_f) + \epsilon_{11} + \epsilon_{21}, \\ \dfrac{\gamma_1(1-Q_1) + \gamma_2(1-Q_2)}{H_2(1-P)} = (w_1 + w_2)(1+r_f) + \epsilon_{12} + \epsilon_{22}. \end{cases} \tag{37}$$

Solving the above equations with respect to (H_1, H_2), we obtain

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \frac{1}{W_0 + a + b} \begin{pmatrix} \dfrac{1}{P}\left(Q_1 w_1 + Q_2 w_2 + \dfrac{b}{1+r_f}\right) \\ \dfrac{1}{1-P}\left((1-Q_1)w_1 + (1-Q_2)w_2 + \dfrac{a}{1+r_f}\right) \end{pmatrix},$$

where we put $W_0 = (w_1 + w_2)(1+r_f)$, $a = (1-Q_1)\epsilon_{11} + (1-Q_2)\epsilon_{21}$ and $b = Q_1\epsilon_{12} + Q_2\epsilon_{22}$. From this, we obtain for $k = 1,2$:

$$\frac{\partial H_1}{\partial Q_k} = \frac{w_k(W_0 + a + b) + \bar{\epsilon}_k^{(1)} w_1 + \bar{\epsilon}_k^{(2)} w_2 + \dfrac{\epsilon_{k1} b + \epsilon_{k2} a}{1+r_f}}{P(W_0 + a + b)^2},$$

$$\frac{\partial H_2}{\partial Q_k} = -\frac{w_k(W_0 + a + b) + \bar{\epsilon}_k^{(1)} w_1 + \bar{\epsilon}_k^{(2)} w_2 + \dfrac{\epsilon_{k1} b + \epsilon_{k2} a}{1+r_f}}{(1-P)(W_0 + a + b)^2},$$

where we put $\bar{\epsilon}_s^{(k)} = Q_k\epsilon_{s1} + (1-Q_k)\epsilon_{s2}$, $s,k = 1,2$. Hence we obtain (36).

A.9. Proof of Corollary 3

Let $\mathfrak{D} = (\mathfrak{D}_1, \mathfrak{D}_2)$ be a probability distribution over the states defined by

$$\mathfrak{D}_s = \frac{H_s}{\mathbb{E}^P[H]} P_s, \quad s = 1,2. \tag{38}$$

We first note that, from (18) and Proposition 2, the excess returns in equilibrium are given by

$$\mu_i - r_f = -\mathbb{E}^{\mathfrak{D}}[\tilde{z}_i] \tag{39}$$

where $\mathbb{E}^{\mathfrak{D}}$ denotes the expectation under $\mathfrak{D}$. For each state $s = 1,2$, let H_s^A and H_s^B denote the SPD in equilibrium for Economies A and B, respectively. Similarly, let $\mathfrak{D}^A$ and $\mathfrak{D}^B$ denote the probability distribution $\mathfrak{D}$ s for Economies A and B, respectively. From Theorem 2 of [6] and Proposition 7, the following implication holds:

$$p^{(1)} < (>)\, p^{(2)} \Rightarrow H_1^A < (>)\, H_1^B \text{ and } H_2^A > (<)\, H_2^B.$$

From (38), this implies that if $p^{(1)} < p^{(2)}$, then $\mathfrak{D}_1^A < \mathfrak{D}_1^B$ and $\mathfrak{D}_2^A = 1 - \mathfrak{D}_1^A > 1 - \mathfrak{D}_1^B = \mathfrak{D}_2^B$, while if $p^{(1)} > p^{(2)}$, then $\mathfrak{D}_1^A > \mathfrak{D}_1^B$ and $\mathfrak{D}_2^A < \mathfrak{D}_2^B$. Hence, from (39), we obtain the result.