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Optimal Portfolio Strategy with Discounted Stochastic Cash Inflows When the Stock Price Is a Semimartingale

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Abstract

This paper discusses optimal portfolio with discounted stochastic cash inflows (SCI). The cash inflows are invested into a market that is characterized by a stock and a cash account. It is assumed that the stock and the cash inflows are stochastic and the stock is modeled by a semi-martingale. The Inflation linked bond and the cash inflows are Geometric. The cash account is deterministic. We do some scientific analyses to see how the discounted stochastic cash inflow is affected by some of the parameters. Under this setting, we develop an optimal portfolio formula and later give some numerical results.

Keywords

Stochastic Cash Inflows, Portfolio, Inflation-Linked Bond, Semimartingale

1. Introduction

For example in financial mathematics, the classical model for a stock price is that of a geometric Brownian motion. However, it is argued that this model fails to capture properly the jumps in price changes. A more realistic model should take jumps into account. In the Jump diffusion model, the underlying asset price has jumps superimposed upon a geometric Brownian motion. The model therefore consists of a noise component generated by the Wiener process, and a jump component. It involves modelling option prices and finding the replicating portfolio. Researchers have increasingly been studying models from economics and from the natural sciences where the underlying randomness contains jumps. According to Nkeki [1], the wars, decisions of the Federal Reserve, other central banks, and other news can cause the stock price to make a sudden shift. To model this, one would like to represent the stock price by a process that has jumps (Bass [2]). Liu *et al.* (2003) [3] solved for the optimal portfolio in a model with stochastic volatility and jumps when the investor can trade the stock and a risk-free asset only. They also found that Liu and Pan (2003) [4] extended this paper to the case of a complete market. Arai [5] considered an incomplete financial market composed of d risky assets and one riskless asset. Branger and Larsen [6] solved the portfolio planning problem of an ambiguity averse investor. They considered both an incomplete market where the investor can trade the stock and the bond only, and a complete market, where he also has access to derivatives. In Guo and Xu (2004) [7], researchers applied the mean-variance analysis approach to model the portfolio selection problem. They considered a financial market containing $d+1$ assets: d risky stocks and one bond. The security returns are assumed to follow a jump-diffusion process. Uncertainty is introduced by Brown motion processes and Poisson processes The general method to solve mean-variance model is the dynamic programming. Dynamic programming technique was firstly introduced by Richard Bellman in the 1950s to deal with calculus of variations and optimal control prob-lems (Weber *et al.* [8]). Further developments have been obtained since then by a number of scholars including Florentin (1961, 1962) and Kushner (2006), among others. In Jin and Zhang [9], researchers solved the optimal dynamic portfolio choice problem in a jump-diffusion model with some realistic constraints on portfolio weights, such as the no-short-selling constraint and the no-borrowing constraint. Beginning with work of Nkeki [1] which involves optimization of the portfolio strategy using discounted stochastic cash inflows, this work explores optimal portfolio strategy using jump diffussion model.

In Nkeki [1], the stock price is modelled by continuous process which is geometric and but in this work we assume that the stock price process is driven by a semimartingale; defined in Shiryaev *et al.* [10]. The jump diffusion model combines the usual geometric Brownian motion for the diffusion and the general jump process such that the jump amplitudes are normally distributed.

Semimartingales as a tool of modelling stock prices processes has a number of advantages. For example this class contains discrete-time processes, diffusion processes, diffusion processes with jumps, point processes with independent increments and many other processes (Shiryaev [11]). The class of semimartingales is stable with respect to many transformations: absolutely continuous changes of measure, time changes, localization, changes of filtration and so on as stated in (Sharyaev [11]). Stochastic integration with respect to semimartingales describes the growth of capital in self-financing strategies. In this research, a sufficient maximum principle for the optimal control of jump diffusions is used showing dynamic programming and going applications to financial optimization problem in a market described by such process. For jump diffusions with jumps, a necessary maximum principle was given by Tang and Li, see also Kabanov and Kohlmann (∅ksendal and Sulem [12]). If stochastic control satisfies the maximum principle conditions, then the control is indeed optimal for the stochastic control problem. It is believed that such results involves a useful complicated integro-differential equation (the Hamilton-Jacobi-Bellmann equation) in the jump diffusion case. The investor's stochastic Cash inflows (CSI) into the cash account, on inflation-linked bond and stock were considered. Most calculations and methods used were influenced by the works of Nkeki [1], Nkeki [13] ∅ksendal [14], ∅ksendal and Sulem [12], Klebaner [15] and Cont and Tankov [16].

2. Model Formulation

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space where $\mathcal{F}=(\mathcal{F}_t)_{t\geq 0}$ denotes the "flow of information" as discussed in the definition. Mathematically the latter means that $\mathcal{F}$ consists of σ-algebras, *i.e.* for all $s\leq t,\mathcal{F}_s\subset\mathcal{F}_t\subset\mathcal{F}$. The Brownian motions $W(t)=(W^I(t),W^S(t))$ is a 2-dimensional process on a given filtered probability space $(\Omega,\mathcal{F},F(\mathcal{F}),\mathbb{P}),t\in[0,T]$, where $\mathbb{P}$ is the real world probability measure, t is the time period, T is the terminal time period, $W^I(t)$ is the Brownian motion with respect to the "noise" arising from the inflation and $W^I(t)$ is the Brownian motion with respect to the "noise" arising from the stock market.

The dynamics of the cash account with the price $Q(t)$ is given by:

$$\mathrm{d}Q(t,I(t))=Q(t)r\mathrm{d}t \tag{1}$$

$$Q(0)=1$$

where r is the short term interest as defined in Nkeki [1].

The price of the inflation-linked bond $B(t,I(t))$ is given by the dynamics:

$$\mathrm{d}B(t,I(t))=B(t,I(t))\left[(r+\sigma_I\phi_I)\mathrm{d}t+\sigma_B\cdot\mathrm{d}W(t)\right] \tag{2}$$

$$B(0,I(0))=b$$

where $\sigma_B=(\sigma_I,0)$ is the volatility of inflation-linked bond, ϕ_I is the market price of inflation risk, I_t is the inflation index at time t and has the dynamics:

$$\mathrm{d}I(t)=I(t)\left[q\mathrm{d}t+\sigma_I\cdot\mathrm{d}W(t)\right]$$

where q is the expected rate of inflation, which is the difference between nominal interest rate, r real interest $\bar{r}$ and σ_I is the volatility of inflation index.

Suppose the financial process (stock return) $S=(S_t)_{t\geq 0}$ is given on a filtered probability space. Assume that $S(t)$ is of "exponential form".

$$S_t=S_0\mathrm{e}^{H_t},\ H_0=0,t\geq 0 \tag{3}$$

where $H=(H_t)_{t\geq 0}$ is a semi-martingale with respect to $\mathbb{F}$ and $\mathbb{P}$.

Using Itô formula for semimartingales (see **Appendix**) and then differentiating the process we have

$$\mathrm{d}S_t=S_{t-}\mathrm{d}\hat{H}_t \tag{4}$$

where

$$\hat{H}_t=H_t+\frac{1}{2}\langle H^c\rangle_s+\sum_{0<s\leq t}\left(\mathrm{e}^{\Delta H_s}-1-\Delta H_s\right) \tag{5}$$

Using random measure if jumps (see [11])

$$\hat{H}_t = H_t + \frac{1}{2}\langle H^c \rangle_s + \left(e^x - 1 - x\right) * \mu \tag{6}$$

hence

$$d\hat{H}_t = d\mathcal{B}(\varphi)_t + dH_t^c + \frac{1}{2}d\langle H^c \rangle_s + d\int_0^t \int_{|z|\le 1}\left(e^z - 1\right)d(\mu - \nu) + d\int_0^t \int_{|z|\le 1}\left(e^z - 1 - z\right)d\nu \tag{7}$$

Substituting on Equation (7) into Equation (50) we have

$$dS_t = S_{t-}d\left(\mathcal{B}(\varphi)_t + H_t^c + \frac{1}{2}\langle H^c \rangle_s + \int_0^t \int_{|z|\le 1}\left(e^z - 1\right)d(\mu - \nu) + \int_0^t \int_{|z|\le 1}\left(e^z - 1 - z\right)d\nu\right) \tag{8}$$

We know that differential of our stock price can written as

$$dS(t) = S(t-)\left(\alpha dt + \sigma_1 \cdot dW(t) + \text{jump}\right) \tag{9}$$

where $\sigma_1 = \left(\rho\sigma_S, \sqrt{1-\rho^2}\sigma_S\right)$ and $W(t)$ defined as before.

Now comparing Equation (8) with Equation (9), we can now see that when we equate the predictable parts we have

$$S_{t-}\left(d\mathcal{B}(\varphi)_t + \frac{1}{2}d\langle H^c \rangle_t\right) = S_{t-}\alpha dt,$$

$$d\left(\mathcal{B}(\varphi)_t + \frac{1}{2}\langle H^c \rangle_t\right) = \alpha dt, \ \mathcal{B}(\varphi)_t + \frac{1}{2}\langle H^c \rangle_t = \alpha t \tag{10}$$

Equating the continuous parts we get

$$dH_t^c = \sigma_1 dW_t, \quad H_t^c = \sigma_1 W_t \tag{11}$$

and the jump parts give

$$d\left(\int_0^t \int_{|z|\le 1}\left(e^z - 1\right)d(\mu - \nu) + \int_0^t \int_{|z|\le 1}\left(e^z - 1 - z\right)d\nu\right) = \text{jump}$$

and hence we let

$$\int_{|z|\le 1}\left(e^z - 1\right)d(\mu - \nu) + \int_{|z|\le 1}\left(e^z - 1 - z\right)d\nu = J \tag{12}$$

From (11) it follows that $\langle H^c \rangle_t = \sigma_1^2 t$ and hence it follows that

$$\mathcal{B}(\varphi)_t = \alpha t - \frac{1}{2}\langle H^c \rangle_t = \left(\alpha - \frac{1}{2}\sigma_1^2\right)t$$

Substituting Equation (12) into Equation (9) we have

$$dS(t) = S(t-)\left(\alpha dt + \sigma_1 \cdot dW(t) + \int_{|z|\le 1}\left(e^z - 1\right)d(\mu - \nu) + \int_{|z|\le 1}\left(e^z - 1 - z\right)d\nu\right) \tag{13}$$

and further simply it to

$$dS(t) = S(t-)\left(\alpha dt + \sigma_1 \cdot dW(t) + \int_{|z|\le 1}\varphi_1(z)d(\mu - \nu) + \int_{|z|\le 1}\varphi_2(z)d\nu\right) \tag{14}$$

where $\varphi_1(z) = \varphi_2(z) + z = e^z - 1$

Using Itó's formula for jump diffusion

$$S_t = S_0 \exp\left\{\left(\alpha - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t + \int_0^t \int_{|z|\le 0}\left(\ln\left(1-\varphi_1\right)+\varphi_1\right)\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right.$$
$$\left. + \int_0^t \int_{|z|\le 0}\left(\ln\left(1-\varphi_2\right)+\varphi_2\right)\nu\left(\mathrm{d}z,\mathrm{d}s\right) + \int_0^t \int_{\mathbb{R}}\left(\ln\left(1-\varphi_2\right)\right)\left(\mu-\nu\right)\left(\mathrm{d}z,\mathrm{d}s\right)\right\}$$

(see **Appendix**). Hence we define the following

$$\Delta = \begin{pmatrix}\sigma_I \phi_I \\ \alpha - r\end{pmatrix},\ \Sigma = \begin{pmatrix}\sigma_B \\ \sigma_I\end{pmatrix},$$
$$\theta = \begin{pmatrix}\theta_I \\ \theta_S\end{pmatrix},\ \lambda = \begin{pmatrix}\lambda \\ 0\end{pmatrix} \tag{15}$$

The market price of the market risk is given by

$$\phi = \Sigma^{-1}\Delta = \begin{pmatrix}\phi_I \\ \phi_S\end{pmatrix} = \begin{pmatrix}\phi_I \\ \dfrac{\alpha - r - \rho\sigma_1\phi_I}{\sigma_1\sqrt{1-\rho^2}}\end{pmatrix} \tag{16}$$

where, ϕ_s is the market price of stock market risk. We assume the process $P(t)$ which is geometric and with the no arbitrage conditions applied to it obtain the following stochastic differential equation,

$$\mathrm{d}P(t) = P(t-)\left(-\phi_t \mathrm{d}W_t + \int_{|z|\le 1}\psi\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right) \tag{17}$$

Using Itó's formula for jump diffusion equation on 17 we have

$$P(t) = P(0)\exp\left(-\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t \int_{|z|\le 1}\psi_1(t,z)\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right) \tag{18}$$

where $\psi_1(t,z) = \ln\left(1+\psi(t,z)\right) - \psi(t,z)$ (see **Appendix**).

$P(t)$ is a martingale that is always positive and satisfies $\mathbb{E}\left[Z(T)\right] = 1$.

Now we have the price density given by

$$\Lambda(t) = \frac{P_t}{Q_t} \tag{19}$$

where

$$\Lambda = \frac{P(t)}{Q(t)} = P(0)\exp\left(-\phi W_t - \left(r + \frac{1}{2}\phi^2\right)t + \int_0^t \int_{|z|\le 1}\psi_1(t,z)\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right) \tag{20}$$

3. The Dynamics of Stochastic Cash Inflows

The dynamics of the stochastic cash inflows with process, $D(t)$ is given by

$$\mathrm{d}D(t) = D(t)\left(k\mathrm{d}t + \sigma_D' \mathrm{d}W(t)\right),\ D(0) = D_0 \tag{21}$$

where $\sigma_D = \left(\sigma_1^D, \sigma_2^D\right)'$ is the volatility of the cash inflows and k is the expected growth rate of the cash inflows. σ_1^D is the volatility arising from inflation and σ_2^D is the volatility arising from the stock market.

Solving for $D(t)$ we use Itò's formula for continuous processes. Let $f(t) = \ln D_t$ and

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial D}\mathrm{d}D(t) + \frac{1}{2}\frac{\partial^2 f}{\partial D^2}\left[\mathrm{d}D(t)\right]^2 \\ &= \frac{1}{D(t)}\left(D(t)\left(k\mathrm{d}t + \sigma_D'\mathrm{d}W(t)\right)\right) - \frac{1}{2}\frac{1}{D(t)^2}\left[D(t)\right]^2\sigma_D^2\mathrm{d}t \\ &= \left(k - \frac{1}{2}\sigma_D^2\right)\mathrm{d}t + \sigma_D'\mathrm{d}W(t)\end{aligned}$$

$$\mathrm{d}\ln D(t) = \left(k - \frac{1}{2}\sigma_D^2\right)\mathrm{d}t + \sigma_D'\mathrm{d}W(t)$$

$$\ln D(t) - \ln D(0) = \int_0^t \left(k - \frac{1}{2}\sigma_D^2\right)\mathrm{d}s + \int_0^t \sigma_D'\mathrm{d}W(s)$$

$$\ln\frac{D(t)}{D(0)} = \left(k - \frac{1}{2}\sigma_D^2\right)t + \sigma_D'W(t)$$

$$D(t) = D(0)\exp\left(\left(k - \frac{1}{2}\sigma_D^2\right)t + \sigma_D'W(t)\right) \tag{22}$$

4. The Dynamics of the Wealth Process

If $X(t)$ is the wealth process and $\theta(t) = (\theta_0, \theta_I, \theta_S)$ is the admissible portfolio where θ_0 is number of units in the cash account, θ_I is the number of units in the inflation bond and θ_S is the number of units in the stock. In an incomplete market with no arbitrage we have $\theta_0 = 1 - \theta_I - \theta_S$. The dynamics of the wealth process is given by

$$\begin{aligned}\mathrm{d}X(t) &= \left(X_t(r + \theta\Delta) + D(t)\right)\mathrm{d}t + X_t(\Sigma\theta)'\,\mathrm{d}W(t) \\ &\quad + X_t\theta\lambda\left(\int_{|z|\le 1}\varphi_1(\mu - \nu)(\mathrm{d}z, \mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z, \mathrm{d}s)\right)\end{aligned} \tag{23}$$

where

$$\lambda = \exp\left(\alpha\delta t + \sigma_1\delta W_t + \int_0^{\delta t}\int_{|z|\le 1}\varphi_1(\mu - \nu)(\mathrm{d}z, \mathrm{d}s) + \int_0^{\delta t}\int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z, \mathrm{d}s)\right) \tag{24}$$

(see **Appendix**). For $\mu = \nu = 0$ we have the dynamics of the wealth process as

$$\mathrm{d}X(t) = \left(X_t(r + \theta\Delta) + D(t)\right)\mathrm{d}t + X_t(\Sigma\theta)'\,\mathrm{d}W(t) \tag{25}$$

For the Poisson jump measure we have the dynamics of the wealth process as

$$\begin{aligned}\mathrm{d}X(t) &= \left(X_t(r + \theta\Delta) + D(t)\right)\mathrm{d}t + X_t(\Sigma\theta)'\,\mathrm{d}W(t) \\ &\quad + X_t\theta\lambda\left(\int_{|z|\le 1}\varphi_1(\bar{N} - \bar{\nu})(\mathrm{d}z, \mathrm{d}s) + \int_{|z|\le 1}\varphi_2\bar{\nu}(\mathrm{d}z, \mathrm{d}s)\right)\end{aligned} \tag{26}$$

where $\bar{N}$ is the Poisson measure and $\bar{\nu}$ is the compensator on the Poisson measure $\bar{N}$.

5. The Discounted Value of SCI

In this Section, we introduce

Definition 1. *The discounted value of the expected future SCI is defined as*

$$\Psi(t)=\mathbb{E}_t\left(\int_t^T \frac{\Lambda(u)}{\Lambda(t)}D(u)\mathrm{d}u\right) \tag{27}$$

where $\mathbb{E}_t=\mathbb{E}(\cdot\,|\,\mathrm{F}_t)$ is the conditional expectation with respect to the Brownian Filtration $\{\mathrm{F}_t\}_{t\geq 0}$ and $\Lambda(t)=P(t)\exp(-rt)$ is the stochastic discount factor which adjust for nominal interest rate and market price of risks for stock and inflation-linked bond (Nkeki [1]).

Proposition 1. *If* $\Psi(t)$ *is the discounted value of the expected future SCI, then*

$$\Psi(t)=\frac{D(t)\left(\exp\left\{(k-r-\sigma_D\phi)(T-t)+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+k-r-\sigma_D\phi} \tag{28}$$

Proof. By definition 1, we have that

$$\Psi(t)=D(t)\mathbb{E}_t\left(\int_t^T \frac{\Lambda(u)D(u)}{\Lambda(t)D(t)}\mathrm{d}u\right) \tag{29}$$

Applying change of variable on 30, we have

$$\Psi(t)=D(t)\mathbb{E}_t\left(\int_0^{T-t} \frac{\Lambda(\tau)D(\tau)}{\Lambda(0)D(0)}\mathrm{d}\tau\right) \tag{30}$$

starting with

$$\begin{aligned}\frac{\Lambda(\tau)D(\tau)}{\Lambda(0)D(0)}&=\exp\left\{-\phi W_\tau-\left(r+\frac{1}{2}\|\phi\|^2\right)\tau+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}\\&\quad\times\exp\left\{\left(k-\frac{1}{2}\|\sigma_D\|^2\right)\tau+\sigma_D W_\tau\right\}\\&=\exp\left\{(\sigma_D-\phi)W_\tau+\left(k+r-\left(\frac{1}{2}\|\sigma_D\|^2+\frac{1}{2}\|\phi\|^2\right)\right)\tau+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}\\&=\exp\left\{(\sigma_D-\phi)W_\tau+(k+r-\sigma_D'\phi)\tau+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}\end{aligned}$$

we have

$$\mathbb{E}\left[\frac{\Lambda(\tau)D(\tau)}{\Lambda(0)D(0)}\right]=\exp\left\{(k+r-\sigma_D'\phi)\tau+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}$$

and lastly

$$\Psi(t)=D(t)\int_0^{T-t}\exp\left\{(k+r-\sigma_D'\phi)\mathrm{d}\tau+\int_0^{T-t}\int_{|z|\leq 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}$$

We further take note that for $\nu=0$ we have the discounted value of the SCI as

$$\Psi(t)=\frac{D(t)\left(\exp\{(k-r-\sigma_D\phi)(T-t)-1\}\right)}{\eta+k-r-\sigma_D\phi} \tag{31}$$

The differential form of $\Psi(t)$ is given by

$$\mathrm{d}\Psi(t)=\Psi(t)\left((r+\sigma_D\phi)\mathrm{d}t+\sigma_D'\mathrm{d}W(t)+\int_{|z|\leq 1}\psi_1(\mathrm{dz})\mathrm{d}t\right)-D(t)\mathrm{d}t \tag{32}$$

Equation (32) is obtained by differentiating $\Psi(t)$ as shown in the proof below

$$\Psi(t)=\frac{D(t)\left(\exp\left\{(k-r-\sigma_D'\phi)(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+k-r-\sigma_D'\phi}$$

differentiating both sides,

$$\begin{aligned}
\mathrm{d}\Psi(t)&=\frac{D(t)\mathrm{d}\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+\beta}\\
&\quad+\frac{\mathrm{d}D(t)\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+\beta}\\
&=\frac{-(\eta+\beta)D(t)\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}\right)}{\eta+\beta}\mathrm{d}t\\
&\quad+\frac{D(t)\left(k\mathrm{d}t+\sigma_D'\mathrm{d}W(t)\right)\times\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+\beta}\\
&=\frac{-(\eta+\beta)D(t)\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})-1\right\}\right)}{\eta+\beta}\mathrm{d}t-\frac{\beta+\eta}{\beta+\eta}D(t)\mathrm{d}t\\
&\quad+\frac{D(t)\left(k\mathrm{d}t+\sigma_D'\mathrm{d}W(t)\right)\times\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})\right\}-1\right)}{\eta+\beta}\\
&=\frac{D(t)\left(\exp\left\{\beta(T-t)+\int_0^{T-t}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})-1\right\}\right)}{\eta+\beta}\\
&\quad\times\left(-(\eta+\beta)\mathrm{d}t+k\mathrm{d}t+\sigma_D'\mathrm{d}W(t)\right)-D(t)\mathrm{d}t\\
&=\Psi(t)\left((k-\beta-\eta)\mathrm{d}t+\sigma_D'\mathrm{d}W(t)\right)-D(t)\mathrm{d}t\\
&=\Psi(t)\left((r+\sigma_D\phi)\mathrm{d}t+\sigma_D'\mathrm{d}W(t)+\int_{|z|\le 1}\psi_1\nu(\mathrm{dz})\mathrm{d}t\right)-D(t)\mathrm{d}t
\end{aligned}$$

The current discounted cash inflows can be obtained by putting $t=0$ into Equation (28),

$$\Psi(0)=\frac{D(0)\left(\exp\left\{(k-r-\sigma_D\phi)(T)+\int_0^{T}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{dt})\right\}-1\right)}{\eta+k-r-\sigma_D\phi} \tag{33}$$

If $r+\sigma_D\phi>k$ and $\int_0^{\infty}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{ds})<\infty$ we can change the horizon by allowing our T to go up to ∞ *i.e.*

$$\begin{aligned}
\lim_{T\to\infty}\Psi(0)&=\lim_{T\to\infty}\frac{D(0)\left(\exp\left\{(k-r-\sigma_D\phi)(T)+\int_0^{T}\int_{|z|\le 1}\psi_1\nu(\mathrm{dz},\mathrm{dt})\right\}-1\right)}{\eta+k-r-\sigma_D\phi}\\
&=\frac{D(0)}{\eta+k-r-\sigma_D\phi}
\end{aligned}$$

In case of deterministic case, we have $\phi=0$ and $\psi=0$, so

$$\Psi(0)=\frac{D(0)\left(\exp\{(k-r)(T)\}-1\right)}{k-r} \tag{34}$$

and for $r>k$, and $T\to\infty$ we have

$$\lim_{T\to\infty}\Psi(0)=\lim_{T\to\infty}\frac{D(0)\left(\exp\{(k-r)(T)\}-1\right)}{k-r}=\frac{D(0)}{r-k}$$

Since $\Psi(0)=\Psi_0(T,D_0,k,r,\sigma_D)$ is a constant, if we are interested to see how it behaves with respect to T,D_0,k,r and σ_D we need to take $\Psi(0)$ as a function of T,D_0,k,r and σ_D. Then we can look at the sensitivity analysis of $\Psi(0)$,

Finding partial derivatives of Ψ_0 we obtain the followng

Differentiating Ψ_0 with respect to *T*, we have

$$\frac{\partial}{\partial T}\Psi_0=D_0\exp\left(\beta T+\int_{|z|\le 1}\psi_1\nu(\mathrm{d}\xi)\right) \tag{35}$$

Differentiating Ψ_0 with respect to D_0, we have

$$\frac{\partial}{\partial D_0}\Psi_0=\frac{1}{D_0}\Psi_0 \tag{36}$$

Differentiating Ψ_0 with respect to *k*, we have

$$\frac{\partial}{\partial k}\Psi_0=\frac{1}{(\eta+\beta)^2}\left(\left((\eta+\beta)T-1\right)\left(\Psi_0+\frac{D_0}{\eta+\beta}\right)+D_0\right) \tag{37}$$

Differentiating Ψ_0 with respect to *r*, we have

$$\frac{\partial}{\partial r}\Psi_0=\frac{1}{(\eta+\beta)^2}\left(\left(1-(\eta+\beta)T\right)\left(\Psi_0+\frac{D_0}{\eta+\beta}\right)-D_0\right) \tag{38}$$

Differentiating Ψ_0 with respect to σ_D, we have

$$\frac{\partial}{\partial \sigma_D}\Psi_0=\frac{1}{(\eta+\beta)^2}\left(\left(1-(\eta+\beta)T\right)\left(\Psi_0+\frac{D_0}{\eta+\beta}\right)-D_0\right) \tag{39}$$

where $\beta=k-r-\sigma_D\phi$ and $\eta=\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z)$ and $\psi_1=\ln(1+\psi)-\psi$

The following calculations shows how we differentiated $\Psi(0)$ with respect to T

$$\begin{aligned}\Psi_0&=\frac{D(0)}{\beta+\eta}\left(\exp\left(\beta T+\int_0^T\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z,\mathrm{d}t)\right)-1\right)\\&=\frac{D(0)}{\beta+\eta}\exp\left(\beta T+\int_0^T\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z,\mathrm{d}t)\right)-\frac{D_0}{\beta+\eta}\end{aligned}$$

differentiating with respect to T

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}\Psi_0&=\frac{D_0}{\beta+\eta}\exp\left(\beta T+\int_0^T\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z,\mathrm{d}t)\right)\times(\beta+\eta)\\&=D_0\exp\left(\beta T+\int_0^T\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z,\mathrm{d}t)\right)\end{aligned}$$

We repeated the following procedure for all other variables.

When we have a deterministic case, differentiating Ψ_0 partially we have the following

Differentiating Ψ_0 with respect to T, we have

$$\frac{\partial}{\partial T}\Psi_0 = D_0 \exp(\beta T) \tag{40}$$

Differentiating Ψ_0 with respect to D_0, we have

$$\frac{\partial}{\partial D_0}\Psi_0 = \frac{1}{\beta}\left(\frac{1}{D_0}\frac{\partial \Psi_0}{\partial T} - 1\right) \tag{41}$$

Differentiating Ψ_0 with respect to r, we have

$$\frac{\partial}{\partial r}\Psi_0 = \frac{1}{\beta^2}\left((1-\beta T)\frac{\partial \Psi_0}{\partial T} - D_0\right) \tag{42}$$

Differentiating Ψ_0 with respect to σ_D, we have

$$\frac{\partial}{\partial \sigma_D}\Psi_0 = \frac{\phi}{\beta^2}\left((1-\beta T)\frac{\partial \Psi_0}{\partial T} - D_0\right) \tag{43}$$

Differentiating Ψ_0 with respect to k, we have

$$\frac{\partial}{\partial k}(\Psi_0) = \frac{1}{\beta^2}\left((\beta T - 1)\frac{\partial \Psi_0}{\partial T} + D_0\right) \tag{44}$$

Table 1 shows the sensitivity of variables. Sensitivity analysis can be incorporated into discounted cash inflows analysis by examining how the discounted cash inflows of each project changes with changes in the inputs used. These could include changes in revenue assumptions, cost assumptions, tax rate assumptions, and discount rates. It also enables management to have contingency plans in place if assumptions are not met. It also shows that the return on the project is sensitive to changes in the projected revenues and costs. Looking at Table 1, one can see that changing a variable can make

Table 1. Simulation of the sensitivity analysis.

T	$\frac{\partial \Psi_0}{\partial D_0}$	$\frac{\partial \Psi_0}{\partial \sigma_1^D}$	$\frac{\partial \Psi_0}{\partial \sigma_2^D}$	$\frac{\partial \Psi_0}{\partial T}$	$\frac{\partial \Psi_0}{\partial r}$	$\frac{\partial \Psi_0}{\partial \theta_I}$	$\frac{\partial \Psi_0}{\partial \theta_S}$	$\frac{\partial \Psi_0}{\partial k}$
1	1.00	−4.01	−4.83	100.4359	−50.15	−12.54	−18.05	50.15
2	2.01	−16.09	−19.36	100.87	−201.16	−50.29	−72.42	201.16
3	3.02	−36.31	−43.69	101.31	−453.93	−113.42	−163.42	453.93
4	4.04	−64.753	−77.90	101.76	−809.34	−202.34	−291.36	809.34
5	5.05	−101.46	−122.07	102.20	−1268.27	−317.07	−456.58	1268.27
6	6.08	−146.53	−176.29	102.64	−1831.63	−457.91	−659.39	1831.63
7	7.11	−200.02	−240.65	103.09	−2500.31	−625.08	−900.11	2500.31
8	8.14	−262.02	−315.24	103.54	−3275.22	−818.80	−1179.08	3275.22
9	9.18	−332.58	−400.14	103.99	−4157.27	−1039.32	−1496.62	4157.27
10	10.22	−411.79	−495.44	104.45	−5147.39	−1286.85	−1853.06	5147.39

an impact on the SCI. An investor must do the sensitivity analysis in order to know changes can be made on the market to improve the results of an investment.

6. The Dynamics of the Value Process

Proposition 2. *If* $V(t)$ *is the value process and* $V(t)=\Psi(t)+X(t)$ *where* Ψ *is the discounted value of the expected future SCI then the differential form of* $V(t)$ *is given by*

$$\begin{aligned} \mathrm{d}V(t) &= \left(r\left(X_t+\Psi_t\right)+\Delta\theta X_t+\sigma_D\phi\Psi_t\right)\mathrm{d}t+\left(\Sigma\theta X_t+\sigma_D\Psi_t\right)\mathrm{d}W(t) \\ &\quad +\Psi(t-)\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z)\mathrm{d}t+\lambda\theta X(t-)\int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z)\mathrm{d}t \\ &\quad +\lambda\theta X(t-)\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z)\mathrm{d}t \end{aligned} \tag{45}$$

Proof. Differentiating $V(t)$ and substituting Equations (32) and (26) on the differential obtained we have

$$\begin{aligned} \mathrm{d}V(t) &= \mathrm{d}\Psi(t)+\mathrm{d}X(t) \\ &= \Psi(t)\left(\left(r+\sigma_D\phi\right)\mathrm{d}t+\sigma_D'\mathrm{d}W(t)\right)+\Psi(t-)\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z)\mathrm{d}t-D(t)\mathrm{d}t \\ &\quad +X(t)\Big(\left(r+\Sigma\theta\right)\mathrm{d}t+\left(\Sigma\theta\right)'\mathrm{d}W(t)\Big)\mathrm{d}t+\lambda\theta X(t-)\int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z)\mathrm{d}t \\ &\quad +\lambda\theta X(t-)\int_{|z|\le 1}\psi_2(\mu-\nu)(\mathrm{d}z)\mathrm{d}t+D(t)\mathrm{d}t \\ &= \left(r\left(X(t)+\Psi(t)\right)+\left(\sigma_D\phi\Psi(t)+\Delta\theta X(t)\right)\right)\mathrm{d}t+\left(\Psi(t)\sigma_D+X(t)\Sigma\theta\right)'\mathrm{d}W(t) \\ &\quad +\Psi(t-)\int_{|z|\le 1}\psi_1\nu(\mathrm{d}z)\mathrm{d}t+\lambda\theta X(t-)\int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z)\mathrm{d}t \\ &\quad +\lambda\theta X(t-)\int_{|z|\le 1}\psi_2(\mu-\nu)(\mathrm{d}z)\mathrm{d}t \end{aligned}$$

For $\mu=\nu=0$, the jump part becomes zero and we obtain

$$\mathrm{d}V(t)=\left(r\left(X_t+\Psi_t\right)+\Delta\theta X_t+\sigma_D\phi\Psi_t\right)\mathrm{d}t+\left(\Sigma\theta X_t+\sigma_D\Psi_t\right)\mathrm{d}W(t)$$

7. Finding Optimal Portfolio

Theorem 3. *Let* $X(t)$ *be the worth process whose dynamics is defined by* Equation (23), $\Psi(t)$ *the discounted value of expected future stochastic cash inflow as defined in proportion* (1), $V(t)$ *the value process as defined in proportion* (2) *and* $U(v)=\frac{v^{1-\gamma}}{1-\gamma}$ *the utility function and if we assume that* $\nu=0$, *the optimal portfolio is given by* $\theta(t)^*=\left(\theta_0^*,\theta_I^*,\theta_S^*\right)$ *where*

$$\begin{pmatrix}\theta_I^*\\ \theta_S^*\end{pmatrix}=\begin{pmatrix}\dfrac{\phi_I V^*(t)}{\sigma_I\gamma X^*(t)}-\dfrac{\sigma_1^D\Psi(t)}{\sigma_I X^*(t)}\\ \dfrac{\left(\sigma_I(\alpha-r)+\sigma_s\phi_I\rho\left(\sigma_s\sqrt{1-\rho^2}-\sigma_I\right)\right)V^*(t)}{\sigma_s^2\sigma_I\left(1-\rho^2\right)\gamma X^*(t)}-\dfrac{\left(\sigma_I\sigma_2^D-\sigma_s\sigma_1^D\right)\Psi(t)}{\sigma_s\sigma_I\sqrt{1-\rho^2}X^*(t)}\end{pmatrix}$$

and

$$\theta_0^* = 1 - \theta_I^* - \theta_s^*$$

$$= 1 - \frac{\phi_I V^*(t)}{\sigma_I \gamma X^*(t)} + \frac{\sigma_1^D \Psi(t)}{\sigma_I X^*(t)} - \frac{\left(\sigma_I(\alpha - r) + \sigma_s \phi_I \rho \left(\sigma_s \sqrt{1-\rho^2} - \sigma_I\right)\right) V^*(t)}{\sigma_s^2 \sigma_I \left(1-\rho^2\right) \gamma X^*(t)} \tag{46}$$

$$+ \frac{\left(\sigma_I \sigma_2^D - \sigma_s \sigma_1^D\right) \Psi(t)}{\sigma_s \sigma_I \sqrt{1-\rho^2} X^*(t)}$$

The proof is given in **Appendix**.

From Equation (71), $\frac{\left(\sigma_I(\alpha - r) + \sigma_s \phi_I \rho \left(\sigma_s \sqrt{1-\rho^2} - \sigma_I\right)\right) V^*(t)}{\sigma_s^2 \sigma_I \left(1-\rho^2\right) \gamma X^*(t)}$ represents the classical portfolio strategy at time t and $\frac{\left(\sigma_I \sigma_2^D - \sigma_s \sigma_1^D\right) \Psi(t)}{\sigma_s \sigma_I \sqrt{1-\rho^2} X^*(t)}$ represents the inter-temporal hedging term that offset shock from the SCI at time t.

Some Numerical Values

Figure 1 was obtained by setting $T = 10$, $\phi_I = 0.08$, $r = 0.04$, $k = 0.099$, $\alpha = 0.09$, $\sigma_d = [0.25; 0.36]$, $\sigma_s = 0.4$, $\sigma_I = 0.4$, $D = 100$, $\rho = 0.6$ and $\gamma = 0.5$ in Equation (70). This figure shows that when $t = 0$, the portfolio value is 0.151 which is equivalent to 15.1% when the value of the wealth is 40,000 and the portfolio value is 0.159 which is equivalent to 15.9% when the value of the wealth is 1,000,000. When $t = 10$, the portfolio value is 0.16 which is equivalent to 16% when the value of the wealth is 40,000

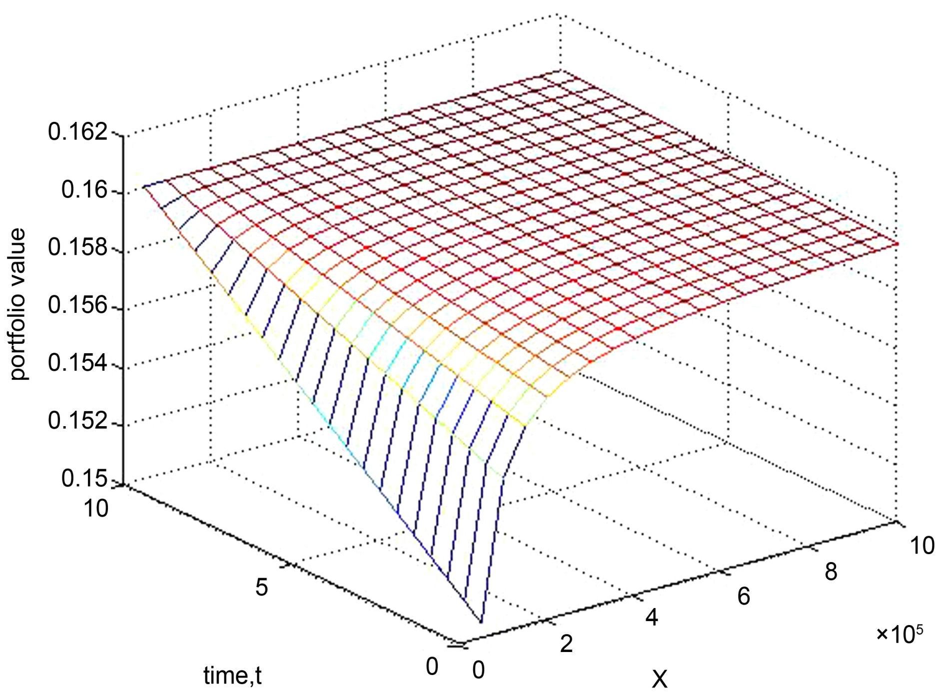

Figure 1. Portfolio value in inflation-linked bond.

and the portfolio value is 0.1604 which is equivalent to 16.04% when the value of the wealth is 1,000,000. This shows that there is a huge increase on the portfolio value from $t = 0$ to $t = 10$ when the value of the wealth is small and there in less change when the value of the wealth is large.

Figure 2 was obtained by setting $T = 10$, $\phi_I = 0.08$, $r = 0.04$, $k = 0.099$, $\alpha = 0.09$, $\sigma_d = [0.25; 0.36]$, $\sigma_s = 0.4$, $\sigma_I = 0.4$, $D = 100$, $\rho = 0.6$ and $\gamma = 0.5$ in Equation (71). This figure shows that when $t = 0$, the portfolio value is 0.907 which is equivalent to 90.7% when the value of the wealth is 40,000 and the portfolio value is 0.9019 which is equivalent to 90.19% when the value of the wealth is 1,000,000. When $t = 10$, the portfolio value is 0.9017 which is equivalent to 90.17% when the value of the wealth is 40,000 and the portfolio value is 0.9017 which is equivalent to 90.17% when the value of the wealth is 1,000,000. This shows that there is a huge decrease on the portfolio value from $t = 0$ to $t = 10$ when the value of the wealth is small and there in less change when the value of the wealth is large.

Figure 3 was obtained by setting $T = 10$, $\phi_I = 0.08$, $r = 0.04$, $k = 0.099$, $\alpha = 0.09$, $\sigma_d = [0.25; 0.36]$, $\sigma_s = 0.4$, $\sigma_I = 0.4$, $D = 100$, $\rho = 0.6$ and $\gamma = 0.5$ in Equation (72). This figure shows that when $t = 0$, the portfolio value is −0.057 which is equivalent to −5.7% when the value of the wealth is 40,000 and the portfolio value is −0.0613 which is equivalent to −6.13% when the value of the wealth is 1,000,000. When $t = 10$, the portfolio value is −0.0615 which is equivalent to −6.15% when the value of the wealth is 40,000 and the portfolio value is −0.0613 which is equivalent to 6.13% when the value of the wealth is 1,000,000. This shows that there is a huge decrease on the portfolio value from $t = 0$ to $t = 10$ when the value of the wealth is small and

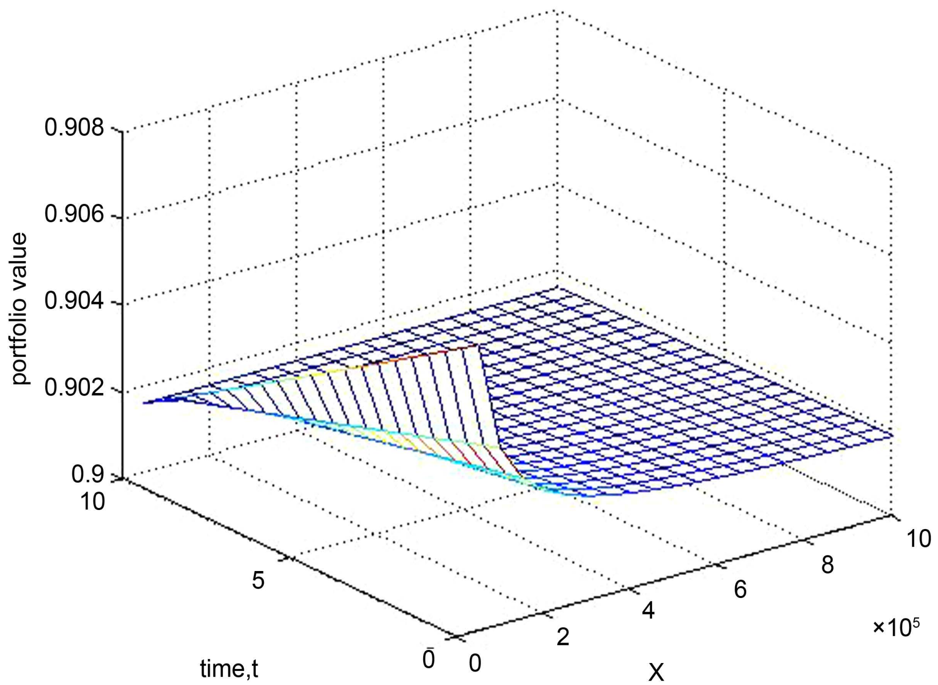

Figure 2. Portfolio value in stock.

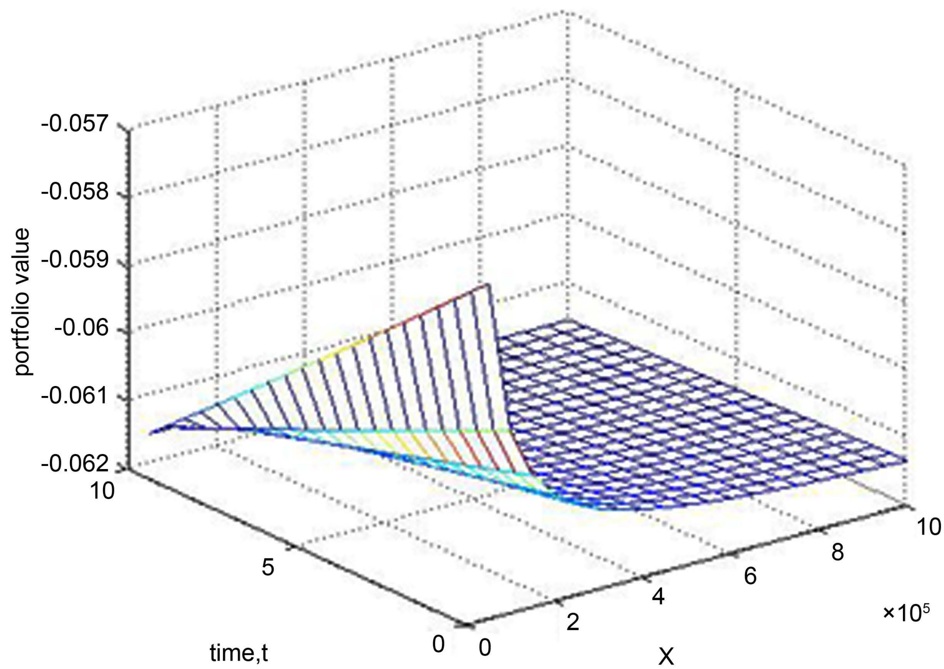

Figure 3. Portfolio value in cash account.

there in less change when the value of the wealth is large.

For $\nu \neq 0$, we have a problem because we cannot solve the equation explicitly. we need to come up with a computer program.

8. Conclusion

Semimartingales seems to model financial processes better since the cater for the jumps that occur in the system. The continuous processes may be convenient because one can easily produce results. For example, in our situation we managed to find the portfolio for continuous processes but we couldn't for the ones with jumps. This work can be extended designing a MATLAB program that will solve the equation for portfolio θ.

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References

[1] Nkeki, C.I. (2013) Optimal Portfolio Strategy with Discounted Stochastic Cash Inflows, *Journal of Mathematical Finance*, **3**, 130-137. http://dx.doi.org/10.4236/jmf.2013.31012

[2] Bass, B.F. (2004) Stochastic Differential Equations with Jumps. *Probability Surveys*, **1**, 1-19. http://dx.doi.org/10.1214/154957804100000015

[3] Liu, J., Pan, J. and Wang, T. (2001) An Equilibrium Model of Rare Event Premia. Working

Paper, UCLA, MIT Sloan, and UBC.

[4] Liu, J. and Pan, J. (2003) Dynamic Derivative Strategies. *Journal of Financial Economics*, **69**, 401-430. http://dx.doi.org/10.1016/S0304-405X(03)00118-1

[5] Arai, T. (2004) Minimal Martingale Measures for Jump Diffussion Processes. *Journal of Applied Probability*, **41**, 263-270. http://dx.doi.org/10.1017/S0021900200014194

[6] Branger, N. and Larsen, L.S. (2013) Robust Portfolio Choice with Uncertainty about Jump and Diffusion Risk. *Journal of Banking and Finance*, **37**, 5036-5047. http://dx.doi.org/10.1016/j.jbankfin.2013.08.023

[7] Guo, W. and Xu, C. (2004) Optimal Portfolio Selection When Stock Prices Follow an Jump-Diffusion Process. *Mathematical Methods of Operations Research*, **60**, 485-496. http://dx.doi.org/10.1007/s001860400365

[8] Azevedo, N., Pinheiro, D. and Weber, G.W. (2014) Dynamic Programming for a Markov-Switching Jump Diffusion. *Journal of Computational and Applied Mathematics*, **267**, 1-19. http://dx.doi.org/10.1016/j.cam.2014.01.021

[9] Jin, X. and Zhang, K. (2013) Dynamic Optimal Portfolio Choice in a Jump-Diffusion Model with Investment Constraints. *Journal of Banking and Finance*, **37**, 1733-1746. http://dx.doi.org/10.1016/j.jbankfin.2013.01.017

[10] Shiryaev, A.N., Buhlmann, H., Delbaen, F. and Embrechts, P. (1995) No-Arbitrage, Change of Measure and Conditional Esscher Transformations. https://people.math.ethz.ch/~delbaen/ftp/preprints/BDES-CWI.pdf

[11] Shiryaev, A. (2003) Essentials of Stochastic Finance: Facts, Models and Theory. World Scientific, Singapore, 3.

[12] Oksendal, B. and Sulem, A. (2009) Applied Stochastic Control of Jump Diffusions. 3rd Edition, Springer, Berlin.

[13] Nkeki, C.I. (2013) Optimal Investment under Inflation Protection and Optimal Portfolios with Discounted Cash Flows Strategy. *Journal of Mathematical Finance*, **3**, 130-137. http://dx.doi.org/10.4236/jmf.2013.31012

[14] Oksendal, B. (2010) Stochastic Differential Equations: An Introduction with Applications. 6th Edition, Springer, Berlin.

[15] Klebaner, F.C. (1998) Introduction to Stochastic Calculus with Applications. Imperial College Press, London. http://dx.doi.org/10.1142/p110

[16] Cont, R. and Tankov, P. (2004) Financial Modelling with Jump Processes. CRC Press, Boca Raton.

Appendix

Appendix A

Assume that $f = e^{H_t}$ and $f \in C^2$. Using Itô formula for sememartingales (see Jacod [?], Protter [?], Shiryaev [11], Shiryaev [10]) f, one obtains

$$f(H_t) = f(H_0) + \int_0^t f'(H_{s-})\,\mathrm{d}H_s + \frac{1}{2}\int_0^t f''(H_{s-})\,\mathrm{d}\langle H^c\rangle_s + \sum_{0<s\le t}\left[f(H_s) - f(H_{s-}) - f'(H_{s-})\Delta H_s\right] \tag{47}$$

to find our SDE, assume that $f(x) = e^x$ and substitute on Equation (47). Simplifying will give the following results

$$\begin{aligned} e_t^H &= e_0^H + \int_0^t \left(e^{H_{s-}}\right)' \mathrm{d}H_s + \frac{1}{2}\int_0^t \left(e^{H_{s-}}\right)'' \mathrm{d}\langle H^c\rangle_s + \sum_{0<s\le t}\left[e^{H_s} - e^{H_{s-}} - \left(e^{H_{s-}}\right)' \Delta H_s\right] \\ &= e_0^H + \int_0^t e^{H_{s-}}\mathrm{d}H_s + \frac{1}{2}\int_0^t e^{H_{s-}}\mathrm{d}\langle H^c\rangle_s + \sum_{0<s\le t}\left[e^{H_s} - e^{H_{s-}} - e^{H_{s-}}\Delta H_s\right] \end{aligned} \tag{48}$$

Differentiating will give;

$$\begin{aligned} \mathrm{d}e_t^H &= 0 + \mathrm{d}\int_0^t e^{H_{s-}}\mathrm{d}H_s + \frac{1}{2}\mathrm{d}\int_0^t e^{H_{s-}}\mathrm{d}\langle H^c\rangle_s + \mathrm{d}\sum_{0<s\le t}\left[e^{H_s} - e^{H_{s-}} - e^{H_{s-}}\Delta H_s\right] \\ &= e^{H_{s-}}\mathrm{d}H_s + \frac{1}{2}e^{H_{s-}}\mathrm{d}\langle H^c\rangle_s + \left[e^{H_s} - e^{H_{s-}} - e^{H_{s-}}\Delta H_s\right] \\ &= e^{H_{s-}}\mathrm{d}H_s + \frac{1}{2}e^{H_{s-}}\mathrm{d}\langle H^c\rangle_s + \left(e^{\Delta H_s + H_{s-}} - e^{H_{s-}} - e^{H_{s-}}\Delta H_s\right),\ \Delta H_s = H_s - H_{s-} \\ &= e^{H_{s-}}\mathrm{d}H_s + \frac{1}{2}e^{H_{s-}}\mathrm{d}\langle H^c\rangle_s + e_{s-}^H\left(e^{\Delta H_s} - 1 - \Delta H_s\right) \\ &= e^{H_{t-}}\left\{\mathrm{d}H_t + \frac{1}{2}\mathrm{d}\langle H^c\rangle_t + \left(e^{\Delta H_s} - 1 - \Delta H_s\right)\right\} \\ &= e^{H_{t-}}\left\{\mathrm{d}H_t + \frac{1}{2}\mathrm{d}\langle H^c\rangle_t + \mathrm{d}\sum_{0<s\le t}\left(e^{\Delta H_s} - 1 - \Delta H_s\right)\right\} \\ &= e^{H_{t-}}\mathrm{d}\left\{H_t + \frac{1}{2}\langle H^c\rangle_t + \sum_{0<s\le t}\left(e^{\Delta H_s} - 1 - \Delta H_s\right)\right\} \end{aligned}$$

$$\mathrm{d}e_t^H = e^{H_{t-}}\mathrm{d}\hat{H}_t \tag{49}$$

Now the differential of the stock process is given by

$$\mathrm{d}S_t = S_{t-}\mathrm{d}\hat{H}_t \tag{50}$$

where

$$\hat{H}_t = H_t + \frac{1}{2}\langle H^c\rangle_s + \sum_{0<s\le t}\left(e^{\Delta H_s} - 1 - \Delta H_s\right) \tag{51}$$

then, using Ito's formula for semimartingales (Protter [?]), we have

$$\begin{aligned} Y_t &= e^{X_0} + \int_0^t \left(e^{X_{s-}}\right)' \mathrm{d}X_s + \frac{1}{2}\int_0^t \left(e^{X_{s-}}\right)'' \mathrm{d}\langle X^c\rangle_s + \sum_{0\le s\le t}\left(e^{X_s} - e^{X_{s-}} - \left(e^{X_{s-}}\right)' \Delta X_s\right) \\ &= e^{X_0} + \int_0^t e^{X_{s-}}\mathrm{d}X_s + \frac{1}{2}\int_0^t e^{X_{s-}}\mathrm{d}\langle X^c\rangle_s + \sum_{0\le s\le t}\left(e^{X_s} - e^{X_{s-}} - e^{X_{s-}}\Delta X_s\right) \end{aligned} \tag{52}$$

and in differential form, this can be expressed as

$$\begin{aligned}
\mathrm{d}Y_t &= e^{X_{t-}}\mathrm{d}X_t + \frac{1}{2}e^{X_{t-}}\mathrm{d}\left\langle X^c\right\rangle_t + e^{X_t} - e^{X_{t-}} - e^{X_{t-}}\Delta X_t = e^{X_{t-}}\mathrm{d}X_t + \frac{1}{2}e^{X_{t-}}\mathrm{d}\left\langle X^c\right\rangle_t + e^{X_{t-}+\Delta X_t} - e^{X_{t-}} - e^{X_{t-}}\Delta X_t \\
&= e^{X_{t-}}\mathrm{d}X_t + \frac{1}{2}e^{X_{t-}}\mathrm{d}\left\langle X^c\right\rangle_t + e^{X_{t-}}\left(e^{\Delta X_t} - 1 - \Delta X_t\right) \\
&= e^{X_{t-}}\mathrm{d}\left[X_t + \frac{1}{2}\left\langle X^c\right\rangle_t + \sum_{0<s\le t}\left(e^{\Delta X_s} - 1 - \Delta X_s\right)\right] = Y_{t-}\mathrm{d}\hat{X}_t,
\end{aligned} \tag{53}$$

Appendix B

Assuming $Y_t = \ln S_t$ and substituting it on the formula we get

$$\begin{aligned}
\mathrm{d}Y_t &= \frac{\partial Y_t}{\partial t}\mathrm{d}t + \frac{\partial Y_t}{\partial S_t}\left[\mathrm{d}S_t\right]^c + \frac{1}{2}\frac{\partial^2 Y_t}{\partial S_t^2}\left[\mathrm{d}S_t^c\right]^2 + \int_{|z|\le 0}\left(\ln\left(S_t - S_t\varphi_1\right) - \ln S_t + \frac{\partial Y_t}{\partial S_t}S_t\varphi_1\right)\nu(\mathrm{d}z) \\
&\quad + \int_{|z|\le 0}\left(\ln\left(S_t - S_t\varphi_2\right) - \ln S_t + \frac{\partial Y_t}{\partial S_t}S_t\varphi_2\right)\nu(\mathrm{d}z) + \int_{\mathbb{R}}\left(\ln\left(S_t - S_t\varphi_2\right) - \ln S_t\right)(\mu - \nu)(\mathrm{d}z) \\
&= \alpha\mathrm{d}t + \sigma_1\mathrm{d}W_t - \frac{1}{2}\sigma_1^2\mathrm{d}t + \int_{|z|\le 0}\left(\ln\left(1-\varphi_1\right) + \varphi_1\right)\nu(\mathrm{d}z) \\
&\quad + \int_{|z|\le 0}\left(\ln\left(1-\varphi_2\right) + \varphi_2\right)\nu(\mathrm{d}z) + \int_{\mathbb{R}}\left(\ln\left(1-\varphi_2\right)\right)(\mu - \nu)(\mathrm{d}z)
\end{aligned}$$

$$\begin{aligned}
\ln S_t - \ln S_0 &= \left(\alpha - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t + \int_0^t\int_{|z|\le 0}\left(\ln\left(1-\varphi_1\right) + \varphi_1\right)\nu(\mathrm{d}z,\mathrm{d}s) \\
&\quad + \int_0^t\int_{|z|\le 0}\left(\ln\left(1-\varphi_2\right) + \varphi_2\right)\nu(\mathrm{d}z,\mathrm{d}s) + \int_0^t\int_{\mathbb{R}}\left(\ln\left(1-\varphi_2\right)\right)(\mu - \nu)(\mathrm{d}z,\mathrm{d}s)
\end{aligned}$$

$$\begin{aligned}
S_t = S_0\exp\Bigg\{&\left(\alpha - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t + \int_0^t\int_{|z|\le 0}\left(\ln\left(1-\varphi_1\right) + \varphi_1\right)\nu(\mathrm{d}z,\mathrm{d}s) \\
&+ \int_0^t\int_{|z|\le 0}\left(\ln\left(1-\varphi_2\right) + \varphi_2\right)\nu(\mathrm{d}z,\mathrm{d}s) + \int_0^t\int_{\mathbb{R}}\left(\ln\left(1-\varphi_2\right)\right)(\mu - \nu)(\mathrm{d}z,\mathrm{d}s)\Bigg\}
\end{aligned} \tag{54}$$

Appendix C

Let $f\left(t, P_t\right) = Y_t = \ln P(t)$ and

$$\begin{aligned}
\mathrm{d}Y_t &= \frac{\partial Y}{\partial t}\mathrm{d}t + \frac{\partial Y}{\partial P}\mathrm{d}P^c + \frac{1}{2}\frac{\partial^2 Y}{\partial P^2}\left[\mathrm{d}P\right]^2 \\
&\quad + \int_{|z|\le 1}\left(Y\left(t, P(t-) - \xi(t,z)\right) - Y\left(t, P(t-)\right) - \frac{\partial Y}{\partial P}\xi(t,z)\right)\nu(\mathrm{d}t,\mathrm{d}z) \\
&= \frac{1}{P}\left(P\left(-\phi\mathrm{d}W_t\right)\right) - \frac{1}{2}\frac{1}{P^2}\left[P\left(\phi\mathrm{d}W_t\right)\right]^2 \\
&\quad + \int_{|x|\le 1}\left(\ln\left(P(t-) + P(t-)\psi(t,z)\right) - \ln P(t-) - \frac{1}{P(t-)}P(t-)\psi(t,z)\right)\nu(\mathrm{d}t,\mathrm{d}z) \\
&= -\phi\mathrm{d}W_t - \frac{1}{2}\phi^2\mathrm{d}t + \int_{|z|\le 1}\left(\ln\frac{P(t-)\left(1+\psi(t,z)\right)}{P(t-)} - \psi(t,z)\right)\nu(\mathrm{d}t,\mathrm{d}z) \\
&= -\phi\mathrm{d}W_t - \frac{1}{2}\phi^2\mathrm{d}t + \int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}t,\mathrm{d}z)
\end{aligned}$$

$$\mathrm{d}\ln P(t) = -\phi \mathrm{d}W_t - \frac{1}{2}\phi^2 \mathrm{d}t + \int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}t,\mathrm{d}z)$$

$$\ln P(t) - \ln P(0) = -\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t\int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}z,\mathrm{d}s)$$

$$\ln\frac{P(t)}{P(0)} = -\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t\int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}z,\mathrm{d}s)$$

$$\frac{P(t)}{P(0)} = \exp\left(-\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t\int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}z,\mathrm{d}s)\right)$$

$$\begin{aligned} P(t) &= P(0)\exp\left(-\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t\int_{|z|\le 1}\left(\ln\left(1+\psi(t,z)\right) - \psi(t,z)\right)\nu(\mathrm{d}z,\mathrm{d}s)\right) \\ &= P(0)\exp\left(-\phi W_t - \frac{1}{2}\phi^2 t + \int_0^t\int_{|z|\le 1}\psi_1(t,z)\nu(\mathrm{d}z,\mathrm{d}s)\right) \end{aligned} \tag{55}$$

Appendix D

$$\begin{aligned} \mathrm{d}X(t) &= X(t)\theta_S(t)\frac{\mathrm{d}S(t)}{S(t)} + X(t)\theta_I(t)\frac{\mathrm{d}B(t,I(t))}{B(t(I(t)))} + X(t)\theta_0(t)\frac{\mathrm{d}Q(t)}{Q(t)} + D(t)\mathrm{d}t \\ &= X(t)\theta_S(t)\frac{S_t}{S_t}\left(\alpha\mathrm{d}t + \sigma_1\cdot\mathrm{d}W(t)\right) + \frac{S_{t-}}{S_t}\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right) \\ &\quad + X(t)\theta_I(t)\frac{\mathrm{d}B(t,I(t))}{B(t(I(t)))} + X(t)\theta_0(t)\frac{\mathrm{d}Q(t)}{Q(t)} + D(t)\mathrm{d}t \\ &= X(t)\theta_S(t)\left(\alpha\mathrm{d}t + \sigma_1\cdot\mathrm{d}W(t)\right) + \frac{S_{t-}}{S_t}\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right) \\ &\quad + X(t)\theta_I(t)\left(\left(r+\sigma_I\phi_I\right)\mathrm{d}t + \sigma_B\mathrm{d}W(t)\right) + X(t)\left(1-\theta_s(t)-\theta_I(t)\right)(r\mathrm{d}t) + D(t)\mathrm{d}t \\ &= X(t)\theta_S(t)\left(\alpha\mathrm{d}t + \sigma_1\cdot\mathrm{d}W(t)\right) + \lambda\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right) \\ &\quad + X(t)\theta_I(t)\left(\left(r+\sigma_I\phi_I\right)\mathrm{d}t + \sigma_B\mathrm{d}W(t)\right) + X(t)\left(1-\theta_s(t)-\theta_I(t)\right)(r\mathrm{d}t) + D(t)\mathrm{d}t \\ &= X(t)\theta_S(t)\left(\alpha\mathrm{d}t + \sigma_1\cdot\mathrm{d}W(t)\right) + X(t)\theta_S(t)\lambda\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s)\right. \\ &\quad \left. + \int_{|z|\le 1}\varphi_2(\nu)(\mathrm{d}z,\mathrm{d}s)\right) + X(t)\theta_I(t)\left(\left(r+\sigma_I\phi_I\right)\mathrm{d}t + \sigma_B\mathrm{d}W(t)\right) \\ &\quad + X(t)\left(1-\theta_s(t)-\theta_I(t)\right)(r\mathrm{d}t) + D(t)\mathrm{d}t \\ &= X(t)\theta_S(\alpha-r)\mathrm{d}t + X(t)\theta_I\sigma_I\phi_I\mathrm{d}t + rX(t)\mathrm{d}t + X(t)\theta S\sigma_1\mathrm{d}W(t) \\ &\quad + X(t)\theta_I\sigma B\mathrm{d}W(t) + X(t)\theta_S\lambda\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right) + D(t)\mathrm{d}t \\ &= X(t)\left(\theta_S,\theta_I\right)'\begin{pmatrix}\alpha-r\\ \sigma_I\phi_I\end{pmatrix}\mathrm{d}t + rX(t)\mathrm{d}t + X(t)\left(\theta_S,\theta_I\right)'\begin{pmatrix}\sigma_1\\ \sigma_B\end{pmatrix}\mathrm{d}W(t) \\ &\quad + X(t)\left(\theta_S,\theta_I\right)'\begin{pmatrix}\lambda\\ 0\end{pmatrix}\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right) + D(t)\mathrm{d}t \\ &= X(t)\left\{(r+\theta\Delta)\mathrm{d}t + (\Sigma\theta)'\mathrm{d}W(t) + \theta\lambda\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{d}z,\mathrm{d}s) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{d}z,\mathrm{d}s)\right)\right\} + D(t)\mathrm{d}t \end{aligned}$$

then

$$\mathrm{d}X(t) = \left(X_t(r+\theta\Delta) + D(t)\right)\mathrm{d}t + X_t(\Sigma\theta)'\,\mathrm{d}W(t)$$
$$+ X_t\theta\lambda\left(\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_{|z|\le 1}\varphi_2\nu(\mathrm{dz},\mathrm{ds})\right) \tag{56}$$

where

$$\lambda = \exp\left(\alpha\delta t + \sigma_1\delta W_t + \int_0^{\delta t}\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_0^{\delta t}\int_{|z|\le 1}\varphi_2\nu(\mathrm{dz},\mathrm{ds})\right)$$

with $\delta t = t - t-$ and $\delta W_t = W_t - W_{t-}$

λ was found by simply dividing S_t by S_{t-} *i.e.*

$$\lambda = \frac{S_0\exp\left(\alpha t + \sigma_1 W_t + \int_0^t\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_0^t\int_{|z|\le 1}\varphi_2(\nu)(\mathrm{dz},\mathrm{ds})\right)}{S_0\exp\left(\alpha t - + \sigma_1 W_{t-} + \int_0^{t-}\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_0^{t-}\int_{|z|\le 1}\varphi_2\nu(\mathrm{dz},\mathrm{ds})\right)}$$

$$= \exp\left(\alpha(t-t-) + \sigma_1(W_t - W_{t-}) + \int_{t-}^t\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_{t-}^t\int_{|z|\le 1}\varphi_2\nu(\mathrm{dz},\mathrm{ds})\right)$$

$$= \exp\left(\alpha\delta t + \sigma_1\delta W_t + \int_0^{\delta t}\int_{|z|\le 1}\varphi_1(\mu-\nu)(\mathrm{dz},\mathrm{ds}) + \int_0^{\delta t}\int_{|z|\le 1}\varphi_2\nu(\mathrm{dz},\mathrm{ds})\right)$$

Appendix E

Let $f = C^{1,2}$ and define $Y(t) = f(t, V(t)) = f(t, X(t), \Psi(t))$. Then $Y(t)$ is a stochastic process with jumps and

$$\begin{aligned}
\mathrm{d}Y(t) = {} & \frac{\partial f}{\partial t}(t, X(t), \Psi(t))\mathrm{d}t + \frac{\partial f}{\partial X(t)}(t, X(t), \Psi(t))\mathrm{d}X^c(t) \\
& + \frac{\partial f}{\partial \Psi(t)}(t, X(t), \Psi(t))\mathrm{d}\Psi^c(t) \\
& + \frac{1}{2}\frac{\partial^2 f}{\partial X^2(t)}(t, X(t), \Psi(t))\left[\mathrm{d}X^c(t)\right]^2 \\
& + \frac{1}{2}\frac{\partial^2 f}{\partial \Psi^2(t)}(t, X(t), \Psi(t))\left[\mathrm{d}\Psi^c(t)\right]^2 \\
& + \frac{\partial^2 f}{\partial \Psi(t)\partial X(t)}(t, X(t), \Psi(t))\left[\mathrm{d}\Psi^c(t)\mathrm{d}X^c(t)\right] \\
& + \int\left\{f(t, V(t-) + \kappa_1(t,z,\omega)) - f(t, V(t-)) - \frac{\partial f}{\partial X(t)}(t, V(t-))\kappa_1(t,z,\omega)\right\}\nu(\mathrm{dz}) \\
& + \int\left\{f(t, V(t-) + \kappa_2(t,z,\omega)) - f(t, V(t-)) - \frac{\partial f}{\partial X(t)}(t, V(t-))\kappa_2(t,z,\omega)\right\}\nu(\mathrm{dz}) \\
& + \int\left\{f(t, V(t-) + \kappa_2(t,z,\omega)) - f(t, V(t-))\right\}(\mu-\nu)(\mathrm{dz}) \\
& + \int\left\{f(t, V(t-) + \kappa_3(t,z,\omega)) - f(t, V(t-)) - \frac{\partial f}{\partial \Psi(t)}(t, V(t-))\kappa_3(t,z,\omega)\right\}\nu(\mathrm{dz})
\end{aligned}$$

take $Y = f\left(t,V\left(t\right)\right)$ and substituting on 58 to have

$$\begin{aligned}
\mathrm{d}Y\left(t\right) &= \frac{\partial f}{\partial t}\left(t,X\left(t\right),\Psi\left(t\right)\right)\mathrm{d}t + \frac{\partial f}{\partial X\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[X_t\left(r+\Delta\theta\right)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right] \\
&+ \frac{\partial f}{\partial \Psi\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[\Psi_t\left(r+\sigma_D\phi\right)\mathrm{d}t + \sigma_D\mathrm{d}W_t\right] \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial X^2\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[X_t\left(r+\Delta\theta\right)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right]^2 \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial \Psi^2\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[\left(r+\Delta\theta\right)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right]^2 \\
&+ \frac{\partial^2 f}{\partial \Psi\left(t\right)\partial X\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[X_t\left(r+\Delta\theta\right)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right]\left[\Psi_t\left(r+\sigma_D\phi\right)\mathrm{d}t + \sigma_D\mathrm{d}W_t\right] \\
&+ \int\left\{f\left(t,V\left(t-\right)+\kappa_1\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial X\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_1\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right) \\
&+ \int\left\{f\left(t,V\left(t-\right)+\kappa_2\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial X\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_2\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right) \\
&+ \int\left\{f\left(t,V\left(t-\right)+\kappa_2\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right)\right\}\left(\mu-\nu\right)\left(\mathrm{d}z\right) \\
&+ \int\left\{f\left(t,V\left(t-\right)+\kappa_3\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial \Psi\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_3\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}Y\left(t\right) &= \frac{\partial f}{\partial t}\left(t,X\left(t\right),\Psi\left(t\right)\right)\mathrm{d}t + \frac{\partial f}{\partial X\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[X_t\left(r+\Delta\theta\right)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right] \\
&+ \frac{\partial f}{\partial \Psi\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left[\Psi_t\left(r+\sigma_D\phi\right)\mathrm{d}t + \sigma_D\mathrm{d}W_t\right] \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial X^2\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left(X_t^2\Sigma^2\theta^2\mathrm{d}t\right) \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial \Psi^2\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left(\Psi_t^2\sigma_D'\sigma_D\mathrm{d}t\right) \\
&+ \frac{\partial^2 f}{\partial \Psi\left(t\right)\partial X\left(t\right)}\left(t,X\left(t\right),\Psi\left(t\right)\right)\left(X_t\Psi_t\sigma_D\Sigma\theta\mathrm{d}t\right) \\
&+ \int_{|z|\le 1}\left\{f\left(t,V\left(t-\right)+\kappa_1\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial X\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_1\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right) \\
&+ \int_{|z|\le 1}\left\{f\left(t,V\left(t-\right)+\kappa_2\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial X\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_2\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right) \\
&+ \int_{|z|\le 1}\left\{f\left(t,V\left(t-\right)+\kappa_2\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right)\right\}\left(\mu-\nu\right)\left(\mathrm{d}z\right) \\
&+ \int_{|z|\le 1}\left\{f\left(t,V\left(t-\right)+\kappa_3\left(t,z,\omega\right)\right) - f\left(t,V\left(t-\right)\right) - \frac{\partial f}{\partial \Psi\left(t\right)}\left(t,V\left(t-\right)\right)\kappa_3\left(t,z,\omega\right)\right\}\nu\left(\mathrm{d}z\right)
\end{aligned}$$

Choosing $f\left(t,V\left(t\right)\right) = J\left(t,V\left(t\right)\right)$ such that for a given portfolio strategy θ (not necessarily optimum, we introduce the associated utility

$$J(t,x,\Psi,\theta)=\mathbb{E}_{t,x,\Psi}\left[U\left(V^{\theta}(T)\right)\right] \tag{57}$$

Substituting $\kappa_1 = x\theta\lambda\varphi_1$, $\kappa_2 = x\theta\lambda\varphi_2$ and $\kappa_3 = \Psi\psi_1$ we now have

$$\begin{aligned}
\mathrm{d}J(t,V(t)) &= J_t\mathrm{d}t + J_x\left[x(r+\Delta\theta)\mathrm{d}t + \Sigma\theta\mathrm{d}W_t\right]\\
&\quad + J_\Psi\left[\Psi(r+\sigma_D\varphi)\mathrm{d}t + \sigma_D\mathrm{d}W_t\right] + \frac{1}{2}J_{xx}\left(x^2\Sigma^2\theta^2\mathrm{d}t\right)\\
&\quad + \frac{1}{2}J_{\Psi\Psi}\left(\Psi^2\sigma_D'\sigma_D\mathrm{d}t\right) + J_{x\Psi}\left(x\Psi\sigma_D\Sigma\theta\mathrm{d}t\right)\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_2) - J(v) - J_x x\theta\lambda\varphi_2\right\}\nu(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v) - J_x x\theta\lambda\varphi_1\right\}\nu(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v)\right\}(\mu-\nu)(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+\Psi\psi_1) - J(v) - J_\Psi\Psi\psi_1\right\}\nu(\mathrm{d}z)\\
&= \Big(J_t + x(r+\Delta\theta)J_x + \Psi(r+\sigma_D\varphi)J_\Psi + \frac{1}{2}x^2\Sigma^2\theta^2 J_{xx}\\
&\quad + \frac{1}{2}\Psi^2\sigma_D^2 J_{\Psi\Psi} + x\Psi\sigma_D\Sigma\theta J_{x\Psi}\Big)\mathrm{d}t + \left(\Sigma\theta J_x + \sigma_D J_\Psi\right)\mathrm{d}W_t\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_2) - J(v) - J_x x\theta\lambda\varphi_2\right\}\nu(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v) - J_x x\theta\lambda\varphi_1\right\}\nu(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v)\right\}(\mu-\nu)(\mathrm{d}z)\\
&\quad + \int_{|z|\le1}\left\{J(v+\Psi\psi_1) - J(v) - J_\Psi\Psi\psi_1\right\}\nu(\mathrm{d}z)
\end{aligned}$$

Integrating both sides we get

$$\begin{aligned}
J(T,V(T)) &= J(t,V(t)) + \int_t^T\Big(J_s + x(r+\Delta\theta)J_x + \Psi(r+\sigma_D\phi)J_\Psi\\
&\quad + \frac{1}{2}x^2\Sigma^2\theta^2 J_{xx} + \frac{1}{2}\Psi^2\sigma_D^2 J_{\Psi\Psi} + x\Psi\sigma_D\Sigma\theta J_{x\Psi}\Big)\mathrm{d}s\\
&\quad + \int_t^T\left(x\Sigma\theta J_x + \sigma_D J_\Psi\right)\mathrm{d}W_s\\
&\quad + \int_t^T\int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_2) - J(v) - J_x x\theta\lambda\varphi_2\right\}\nu(\mathrm{d}z,\mathrm{d}s)\\
&\quad + \int_t^T\int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v) - J_x x\theta\lambda\varphi_1\right\}\nu(\mathrm{d}z,\mathrm{d}s)\\
&\quad + \int_t^T\int_{|z|\le1}\left\{J(v+x\theta\lambda\varphi_1) - J(v)\right\}(\mu-\nu)(\mathrm{d}z,\mathrm{d}s)\\
&\quad + \int_t^T\int_{|z|\le1}\left\{J(v+\Psi\psi_1) - J(v) - J_\Psi\Psi\psi_1\right\}\nu(\mathrm{d}z,\mathrm{d}s)
\end{aligned}$$

Taking the expectations on both sides we have

$$\begin{aligned}
&\mathbb{E}_{t,x,\Psi}\left[J\left(T,V\left(T\right)\right)\right] \\
&= J\left(t,V\left(t\right)\right)+\mathbb{E}_{t,x,\Psi}\left[\int_t^T\left(J_s+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi\right.\right. \\
&\quad \left.+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}\right)\mathrm{d}s \\
&\quad +\int_t^T\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\left(\varphi_1+\varphi_2\right)\right)-2J\left(v\right)-J_x x\theta\lambda\left(\varphi_1+\varphi_2\right)\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right) \\
&\quad \left.+\int_t^T\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right]
\end{aligned}$$

For simplicity we have

$$\begin{aligned}
&\mathbb{E}_{t,x,\Psi}\left[J\left(T,V\left(T\right)\right)\right] \\
&= J\left(t,x,\Psi\right)+\mathbb{E}_{t,x,\Psi}\left[\int_t^T\left(J_s+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi\right.\right. \\
&\quad \left.+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}\right)\mathrm{d}s \\
&\quad +\int_t^T\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\psi_2\right)-2J\left(v\right)-J_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right) \\
&\quad \left.+\int_t^T\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right]
\end{aligned}$$

Where $\psi_2=\varphi_1+\varphi_2$ and $J\left(t,V\left(t\right)\right)=J\left(t,x,\Psi\right)$. Since we know that $J\left(T,V^\theta\left(T\right)\right)=U\left(V^\theta\left(T\right)\right)$, we now have

$$\begin{aligned}
&\mathbb{E}_{t,x,\Psi}\left[U\left(V^\theta\left(T\right)\right)\right] \\
&= J\left(t,x,\Psi\right)+\mathbb{E}_{t,x,\Psi}\left[\int_t^T\left(J_s+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi\right.\right. \\
&\quad \left.+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}\right)\mathrm{d}s \\
&\quad +\int_t^T\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\psi_2\right)-2J\left(v\right)-J_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right) \\
&\quad \left.+\int_t^T\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right]
\end{aligned}$$

Which gives us

$$\begin{aligned}
J\left(t,x,\Psi\right) &= \mathbb{E}_{t,x,\Psi}\left[U\left(V^\theta\left(T\right)\right)\right]-\mathbb{E}_{t,x,\Psi}\left[\int_t^T\left(J_s+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi\right.\right. \\
&\quad \left.+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}\right)\mathrm{d}s \\
&\quad +\int_t^T\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\psi_2\right)-2J\left(v\right)-J_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right) \\
&\quad \left.+\int_t^T\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{d}z,\mathrm{d}s\right)\right]
\end{aligned}$$

By Equation (57), we have the integral on the right hand side being equals to zero. That is

$$\mathbb{E}_{t,x,\Psi}\left[\int_t^T\left(J_s+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi\right.\right.$$
$$\left.+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}\right)\mathrm{ds}$$
$$+\int_t^T\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\psi_2\right)-2J\left(v\right)-J_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{dz},\mathrm{ds}\right)$$
$$\left.+\int_t^T\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{dz},\mathrm{ds}\right)\right]=0$$

Differentiating both sides we obtain following partial differential equation with jumps.

$$J_t+x\left(r+\Delta\theta\right)J_x+\Psi\left(r+\sigma_D\phi\right)J_\Psi$$
$$+\frac{1}{2}x^2\Sigma^2\theta^2J_{xx}+\frac{1}{2}\Psi^2\sigma_D^2J_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta J_{x\Psi}$$
$$+\int_{|z|\le 1}\left\{J\left(v+x\theta\lambda\psi_2\right)-2J\left(v\right)-J_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{dz}\right)$$
$$+\int_{|z|\le 1}\left\{J\left(v+\Psi\psi_1\right)-J\left(v\right)-J_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{dz}\right)=0$$

Consider the value function

$$V\left(t,x,\Psi\right)=\sup_\theta J\left(t,x,\Psi,\theta\right) \tag{58}$$

where J is as in Equation (57) Under technical conditions, the value function V satisfies

$$U_t+\sup_\theta\left[\frac{1}{2}x^2\Sigma^2\theta^2U_{xx}+\Delta\theta U_x+x\Psi\sigma_D\Sigma\theta U_{x\Psi}\right.$$
$$\left.+\int_{|z|\le 1}\left\{U\left(v+x\theta\lambda\psi_2\right)-U_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{dz}\right)\right]+\frac{1}{2}\Psi^2\sigma_D^2U_{\Psi\Psi} \tag{59}$$
$$+\Psi\left(r+\sigma_D\phi\right)U_\Psi+rxU_x\int_{|z|\le 1}\left\{U\left(v+\Psi\psi_1\right)-3J\left(v\right)-U_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{dz}\right)=0$$

This takes us to the HJB equation;

$$U_t\left(t,v\right)+\max_{\theta\in\pi}\mathcal{L}^\theta\Phi\left(t,x,\Psi\right)=0 \tag{60}$$

where $\mathcal{L}^\theta$ is the second linear operator for jump diffusion. Hence

$$\mathcal{L}^\theta U=x\left(r+\Delta\theta\right)U_x+\Psi\left(r+\sigma_D\phi\right)U_\Psi+\frac{1}{2}x^2\Sigma^2\theta^2U_{xx}$$
$$+\frac{1}{2}\Psi^2\sigma_D^2U_{\Psi\Psi}+x\Psi\sigma_D\Sigma\theta U_{x\Psi} \tag{61}$$
$$+\int_{|z|\le 1}\left\{U\left(v+x\theta\lambda\psi_2\right)-2U\left(v\right)-U_x x\theta\lambda\psi_2\right\}\nu\left(\mathrm{dz}\right)$$
$$+\int_{|z|\le 1}\left\{U\left(v+\Psi\psi_1\right)-U\left(v\right)-U_\Psi\Psi\psi_1\right\}\nu\left(\mathrm{dz}\right)$$

Taking our utility function as

$$U\left(v\right)=\frac{v^{1-\gamma}}{1-\gamma} \tag{62}$$

We consider the function of θ which is

$$\Gamma(\theta)=\frac{1}{2}x^2\Sigma^2\theta^2U_{xx}+\Delta\theta U_x+x\Psi\sigma_D\Sigma\theta U_{x\Psi}$$
$$+\int_{|z|\le 1}\left\{U\left(v+x\theta\lambda\psi_2\right)-U_x x\theta\lambda\psi_2\right\}\nu(\mathrm{d}z) \tag{63}$$

Differentiating $U(v)$ and substitute on (63), we get

$$\Gamma(\theta)=-\frac{1}{2}x^2\gamma\Sigma^2\theta^2v^{-\gamma-1}+\Delta\theta v^{-\gamma}-\gamma x\Psi\sigma_D\Sigma\theta v^{-\gamma-1}$$
$$+\int_{|z|\le 1}\left\{\frac{1}{1-\gamma}\left(v+x\theta\lambda\psi_2\right)^{1-\gamma}-v^{-\gamma}x\theta\lambda\psi_2\right\}\nu(\mathrm{d}z) \tag{64}$$

Since $\Gamma(\theta)$ is a concave function of θ, to find its maximum we differentiate (64) with respect to θ to obtain

$$\frac{\partial}{\partial\theta}\Gamma(\theta)=-x^2\gamma\Sigma^2\theta v^{-\gamma-1}+\Delta v^{-\gamma}-\gamma x\Psi\sigma_D\Sigma v^{-\gamma-1}$$
$$+\int_{|z|\le 1}\left\{x\lambda\psi_2\left(v+x\theta\lambda\psi_2\right)^{-\gamma}-v^{-\gamma}x\lambda\psi_2\right\}\nu(\mathrm{d}z) \tag{65}$$

For $\nu=0$ we can solve for θ because we have a linear equation below

$$x\Delta v^{-\gamma}-\gamma x^2\Sigma^2\theta v^{-\gamma-1}-\gamma x\Psi\sigma_D\Sigma v^{-\gamma-1}=0 \tag{66}$$

$$\theta=\frac{x\Delta v^{-\gamma}}{\gamma x^2\Sigma^2v^{-\gamma-1}}-\frac{\gamma x\Psi\sigma_D\Sigma v^{-\gamma-1}}{\gamma x^2\Sigma^2v^{-\gamma-1}}$$
$$=\frac{\Delta v}{\gamma x\Sigma^2}-\frac{\Psi\sigma_D}{x\Sigma}=\frac{\Sigma^{-2}\Delta v}{\gamma x}-\frac{\Psi\Sigma^{-1}\sigma_D}{x}$$

and θ^* will be given by

$$\theta^*=\frac{\Sigma^{-2}\Delta V^*(t)}{\gamma X^*(t)}-\frac{\Psi\Sigma^{-1}\sigma_D}{X^*(t)} \tag{67}$$

substituting Σ,Δ and σ^D as defined , we obtain the following

$$\theta^*=\begin{pmatrix}\dfrac{\phi_I V^*(t)}{\gamma\sigma_I^2X^*(t)}-\dfrac{\sigma_1^D\Psi(t)}{\sigma_I X^*(t)}\\ \dfrac{\sigma_I(\alpha-r)+\sigma_S\sigma_I\rho\left(\sigma_S\sqrt{1-\rho^2}-\sigma_I\right)V^*(t)}{\gamma\sigma_S^2\sigma_I\left(1-\rho^2\right)X^*(t)}-\dfrac{\left(\sigma_I\sigma_2^D-\rho\sigma_S\sigma_1^D\right)\Psi(t)}{\sigma_S\sigma_I\sqrt{1-\rho^2}X^*(t)}\end{pmatrix} \tag{68}$$

$$\begin{pmatrix}\theta_I^*\\ \theta_S^*\end{pmatrix}=\begin{pmatrix}\dfrac{\phi_I V^*(t)}{\sigma_I\gamma X^*(t)}-\dfrac{\sigma_1^D\Psi(t)}{\sigma_I X^*(t)}\\ \dfrac{\left(\sigma_I(\alpha-r)+\sigma_s\phi_I\rho\left(\sigma_s\sqrt{1-\rho^2}-\sigma_I\right)\right)V^*(t)}{\sigma_s^2\sigma_I\left(1-\rho^2\right)\gamma X^*(t)}-\dfrac{\left(\sigma_I\sigma_2^D-\rho\sigma_s\sigma_1^D\right)\Psi(t)}{\sigma_s\sigma_I\sqrt{1-\rho^2}X^*(t)}\end{pmatrix} \tag{69}$$

where

$$\theta_I^*=\frac{\phi_I V^*(t)}{\sigma_I\gamma X^*(t)}-\frac{\sigma_1^D\Psi(t)}{\sigma_I X^*(t)} \tag{70}$$

and

$$\theta_S^* = \frac{\left(\sigma_I(\alpha - r) + \sigma_s\phi_I\rho\left(\sigma_s\sqrt{1-\rho^2} - \sigma_I\right)\right)V^*(t)}{\sigma_s^2\sigma_I\left(1-\rho^2\right)\gamma X^*(t)} - \frac{\left(\sigma_I\sigma_2^D - \sigma_s\sigma_1^D\right)\Psi(t)}{\sigma_s\sigma_I\sqrt{1-\rho^2}X^*(t)} \tag{71}$$

We can now see that

$$\begin{aligned}\theta_0^* &= 1 - \theta_I^* - \theta_s^* \\ &= 1 - \frac{\phi_I V^*(t)}{\sigma_I\gamma X^*(t)} + \frac{\sigma_1^D\Psi(t)}{\sigma_I X^*(t)} - \frac{\left(\sigma_I(\alpha - r) + \sigma_s\phi_I\rho\left(\sigma_s\sqrt{1-\rho^2} - \sigma_I\right)\right)V^*(t)}{\sigma_s^2\sigma_I\left(1-\rho^2\right)\gamma X^*(t)} \\ &\quad + \frac{\left(\sigma_I\sigma_2^D - \sigma_s\sigma_1^D\right)\Psi(t)}{\sigma_s\sigma_I\sqrt{1-\rho^2}X^*(t)}\end{aligned} \tag{72}$$