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On the Solution of the Multi-Asset Black-Scholes Model: Correlations, Eigenvalues and Geometry

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Abstract

In this paper, the multi-asset Black-Scholes model is studied in terms of the importance that the correlation parameter space (equivalent to an N dimensional hypercube) has in the solution of the pricing problem. It is shown that inside of this hypercube there is a surface, called the Kummer surface Σ_K, where the determinant of the correlation matrix ρ is zero, so the usual formula for the propagator of the N asset Black-Scholes equation is no longer valid. Worse than that, in some regions outside this surface, the determinant of ρ becomes negative, so the usual propagator becomes complex and divergent. Thus the option pricing model is not well defined for these regions outside Σ_K. On the Kummer surface instead, the rank of the ρ matrix is a variable number. By using the Wei-Norman theorem, the propagator over the variable rank surface Σ_K for the general N asset case is computed. Finally, the three assets case and its implied geometry along the Kummer surface is also studied in detail.

Keywords

Multi-Asset Black-Scholes Equation, Wei-Norman Theorem, Correlation Matrix Eigenvalues, Kummer Surface, Propagators

1. Introduction

Since the seminal work of Black, Scholes and Merton on option pricing, see [1] [2], an important research agenda has been developed on the subject. This research has mainly centered in extending the basic Black and Scholes model to well known empirical regularities, with the hope of improving the predicting power for the famous formula, see for example [3]-[6]. An interesting extension has been the modeling of many

underlying assets, which has been called the multi-asset Black-Scholes model [3] [7]. In this case, the option price satisfies a diffusion equation considering many related assets. The first work addressing this problem in the literature was Margrabe (1978), see [8]. The Margrabe formula considered an exchange option, which gives its owner the right, but not the obligation, to exchange *b* units of one asset into a unit of another asset at a specific point in time. Specifically, Margrabe derived a closed-form expression for the option by taking one of the underlying assets as a numeraire and then applying the Black and Scholes standard formulation. Later Stulz [9] found analytical formulae for European put and call options on the minimum or the maximum of two risky assets. In this particular case, the solution is expressed in terms of bivariate cumulative standard normal distributions, and when the strike price of the option is zero the value reduces to the Margrabe pricing. Other interesting papers that follow in this literature are [10]-[15]. The numerical implementation of the solution of the multi-asset Black-Scholes model is increasingly difficult for models with more that three assets, see for instance [16]-[18]. One important point, that has been missed in the literature, is that in all of the multi-asset Black-Scholes models mentioned above, the relationship between assets is modeled by their correlations, and hence it is implicitly assumed that a well behaved multivariate Gaussian distribution must exist in order to have a valid solution.

In this paper, the multi-asset Black-Scholes model is studied in terms of the importance that the correlation parameter space (which is equivalent to an N dimensional hypercube) has in the solution of the option pricing problem. It is shown that inside of this hypercube there is a surface, called the Kummer surface Σ_K [19]-[22], where the determinant of the correlation matrix ρ is zero, so over Σ_K the usual formula for the propagator of the N asset Black-Scholes equation is no longer valid. Worse than that, outside this surface, there are points where the determinant of ρ becomes negative, so the usual propagator becomes complex and divergent. Thus the option pricing model is not well defined for some regions outside Σ_K. On Σ_K the rank of ρ matrix is a variable number, depending on which sector of the Kummer surface the correlation parameters are lying. By using the Wei-Norman theorem [23]-[26], the propagator along the Kummer surface Σ_K, for the N assets case is found. This expression is valid whatever the value of the ρ matrix ranks over Σ_K.

This paper is organized as follows. Section 2 describes the traditional multi-asset Black-Scholes model. In Section 3, the problem is formulated as a N dimensional diffusion equation. In Section 4, the implied geometry of the correlation matrix space is analyzed, specially when its determinant is zero, which coincides with a Kummer surface in algebraic geometry. The Kummer surface and its geometry are reviewed for the particular case of three assets in Section 4.1. In Section 5, by using the Wei-Norman theorem the propagator over the variable rank surface Σ_K for a general N asset case is computed. Finally, some conclusions and future research are presented in Section 6.

2. The Multi-Asset Black-Scholes Model

Consider a portfolio consisting of one option and N underlying assets. Let S_i be the

price processes for the assets; $i = 1, \cdots, N$ where each asset satisfies the usual dynamic

$$dS_i = \alpha_i S_i d\tau + \sigma_i S_i dW_i \tag{1}$$

$i = 1, \cdots, N$ and the N Wiener processes W_i are correlated according to

$$dW_i dW_j = \rho_{ij} d\tau \tag{2}$$

where ρ is the symmetric matrix

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rho_{1N} & \rho_{2N} & \rho_{3N} & \rho_{4N} & \cdots & 1 \end{pmatrix} \tag{3}$$

so

$$dS_i dS_j = \sigma_i \sigma_j S_i S_j \rho_{ij} d\tau \tag{4}$$

If the price process for the option is $\Pi = \Pi(S_1, S_2, \cdots, S_n, \tau)$, the value V of the portfolio is given by

$$V = \Pi - \sum_i \Delta_i S_i \tag{5}$$

where Δ_i are the shares of each asset in the portfolio. The self-financing portfolio condition ensures that

$$dV = d\Pi - \sum_i \Delta_i dS_i \tag{6}$$

and applying It Lemma for Π one gets

$$dV = \left(\frac{\partial \Pi}{\partial \tau} d\tau + \sum_i \frac{\partial \Pi}{\partial S_i} dS_i + \sum_{i,j} \frac{1}{2} \frac{\partial^2 \Pi}{\partial S_i \partial S_j} dS_i dS_j \right) - \sum_i \Delta dS_i \tag{7}$$

According to [4], for a free arbitrage set of N assets, the return of the portfolio is

$$dV = rV d\tau \tag{8}$$

and from Equations (7) and (8) one has

$$\frac{\partial \Pi}{\partial \tau} d\tau + \sum_i \frac{\partial \Pi}{\partial S_i} \left(\alpha_i S_i d\tau + \sigma_i S_i dW_i \right) + \sum_{i,j} \frac{1}{2} \frac{\partial^2 \Pi}{\partial S_i \partial S_j} \sigma_i \sigma_j S_i S_j \rho_{ij} d\tau$$
$$- \sum_i \Delta_i \left(\alpha_i S_i d\tau + \sigma_i S_i dW_i \right) = r \left(\Pi - \sum_i \Delta_i S_i \right) d\tau \tag{9}$$

Collecting $d\tau$ and dW_i terms in the above equation one gets:

$$\frac{\partial \Pi}{\partial \tau} + \sum_i \frac{\partial \Pi}{\partial S_i} \alpha_i S_i + \sum_{i,j} \frac{1}{2} \frac{\partial^2 \Pi}{\partial S_i \partial S_j} \sigma_i \sigma_j S_i S_j \rho_{ij} - \sum_i \Delta_i \alpha_i S_i - r \left(\Pi - \sum_j \Delta_j S_j \right) = 0 \tag{10}$$

and

$$\sum_i \left[\frac{\partial \Pi}{\partial S_i} \sigma_i S_i - \Delta_i \sigma_i S_i \right] dW_i = 0 \tag{11}$$

From Equation (11), and given the independence of the W_i for $i = 1, \cdots, N$

$$\frac{\partial \Pi}{\partial S_i}\sigma_i S_i - \Delta_i \sigma_i S_i = 0 \tag{12}$$

or equivalently

$$\Delta_i = \frac{\partial \Pi}{\partial S_i} \tag{13}$$

so one arrives at the multi-asset Black-Scholes equation

$$\frac{\partial \Pi}{\partial \tau} + \sum_{i,j}\frac{1}{2}\frac{\partial^2 \Pi}{\partial S_i \partial S_j}\sigma_i \sigma_j S_i S_j \rho_{ij} + r\left(\sum_j S_j \frac{\partial \Pi}{\partial S_j} - \Pi\right) = 0 \tag{14}$$

which must be integrated with the final condition

$$\Pi(\boldsymbol{S}, T) = \Phi(\boldsymbol{S})$$

for constant r, α_i, σ_i and a simple contingent claim Φ.

3. The Multi-Asset Black-Scholes Equation as a *N* Dimensional Diffusion Equation

Here, some transformations are developed, which maps the multi-asset option pricing equation in a more simpler diffusion equation. If one makes the change of variables

$$x_i = \ln(S_i) - \left(r - \frac{1}{2}\sigma_i^2\right)\tau \tag{15}$$

in (14), one can map this equation to

$$\frac{\partial \Pi}{\partial \tau} + \frac{1}{2}\sum_{i,j}\sigma_i \sigma_j \rho_{ij}\frac{\partial^2 \Pi}{\partial x_i \partial x_j} - r\Pi = 0$$

At least if one defines Ψ as

$$\Pi(\boldsymbol{x}, \tau) = \mathrm{e}^{-r(T-\tau)}\Psi(\boldsymbol{x}, \tau) \tag{16}$$

then Ψ satisfies the equation

$$\frac{\partial \Psi}{\partial \tau} + \frac{1}{2}\sum_{i,j}\sigma_i \sigma_j \rho_{ij}\frac{\partial^2 \Psi}{\partial x_i \partial x_j} = 0$$

Now, by defining the variables

$$\chi_i = \frac{x_i}{\sigma_i} \tag{17}$$

the above equation can be written as

$$\frac{\partial \Psi}{\partial \tau} + \frac{1}{2}\sum_{i,j}\rho_{ij}\frac{\partial^2 \Psi}{\partial \chi_i \partial \chi_j} = 0$$

And finally, by defining the forward time coordinate

$$t = T - \tau \tag{18}$$

one arrives at

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\sum_{i,j}\rho_{ij}\frac{\partial^2 \Psi}{\partial \chi_i \partial \chi_j} \tag{19}$$

Now performing the transformation

$$\boldsymbol{\zeta} = U^{-1}\boldsymbol{\chi} \tag{20}$$

one can change the χ_k variables to the ζ_k coordinates that diagonalizes the ρ matrix

$$D = U^{-1}\rho U \tag{21}$$

where

$$D = diag\left(\lambda_1, \lambda_2, \cdots, \lambda_N\right) \tag{22}$$

and U is the change basis matrix, with $U^{-1} = U^t$, $\det(U) = 1$. In this diagonal coordinate system, the diffusion equation read finally

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\sum_{i=1}^{N}\lambda_i \frac{\partial^2 \Psi}{\partial \zeta_i^2} \tag{23}$$

Now this equation is studied in terms of the behavior of the eigenvalues λ_i.

4. The Geometry of the ρ Matrix

The ρ matrix in (3) can be characterized completely for the $M = \dfrac{N(N-1)}{2}$ dimensional vector

$$\boldsymbol{r} = \left(\rho_{12}, \rho_{13}, \rho_{14}, \cdots, \rho_{(N-1)N}\right), \quad -1 \le \rho_{ij} \le 1 \tag{24}$$

which lies inside of an M dimensional hypercube centering in the origin and of length 2. Thus, the ρ matrix is a function of $\boldsymbol{r}$: $\rho = \rho(\boldsymbol{r})$. Note that, for some point $\boldsymbol{r}$ inside of the hypercube, the determinant of the ρ matrix vanishes. For example, for the vertex

$$\boldsymbol{r} = (1,1,1,\cdots,1) \Rightarrow \det(\rho) = 0 \tag{25}$$

In fact, exists a whole surface inside the hypercube, where the determinant of ρ vanishes. This surface, called Kummer surface Σ_K in algebraic geometry [19]-[22], is defined by the equation

$$\boldsymbol{r} \in \Sigma_K \Leftrightarrow \det\begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rho_{1N} & \rho_{2N} & \rho_{3N} & \rho_{4N} & \cdots & 1 \end{pmatrix} = \det\rho(\boldsymbol{r}) = 0 \tag{26}$$

In fact, one can think of the hypercube as the disjoint union of the subset of point or surfaces Σ_C of constant C determinant value:

$$\boldsymbol{r} \in \Sigma_C \Leftrightarrow \det\begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rho_{1N} & \rho_{2N} & \rho_{3N} & \rho_{4N} & \cdots & 1 \end{pmatrix} = \det\rho(\boldsymbol{r}) = C \tag{27}$$

Let r an arbitrary vector in $\mathbb{R}^M$ and let $\phi(r)$ the determinant of ρ in each point, that is $\phi(r) = \det(\rho(r))$. Note that $\phi(r)$ is a polynomial function in terms of the r coordinates.

The vector η given by the M dimensional gradient $\eta = \nabla_r \phi(r)$ is perpendicular to the level surfaces Σ_C and gives the direction for greater growth of the function $\phi(r)$. Note also that the components of this vector are also polynomial functions of the r coordinates, so $\eta = \eta(r)$ is a continuous vector function.

Consider now a point $r_0 \in \Sigma_K$, that is, $\phi(r_0) = 0$. As ϕ and η are continuous, there is a neighbor of r_0 on Σ_K, such that for $\epsilon > 0$ the vector $r_+ = r_0 + \epsilon \eta \in \Sigma_C$ with $C > 0$, whereas the vector $r_- = r_0 - \epsilon \eta \in \Sigma_C$ with $C < 0$, due to the ϕ function growths along the η direction. Thus, the Kummer surface Σ_K separates spacial regions with positive ρ determinant from that with negative ρ determinant.

In its diagonal form, Equation (26) is

$$r \in \Sigma_K \Leftrightarrow \det \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_N \end{pmatrix} = 0 \tag{28}$$

where the $\lambda_i = \lambda_i(r)$, that is

$$r \in \Sigma_K \Leftrightarrow \phi(r) = \lambda_1(r)\lambda_2(r)\cdots\lambda_n(r) = 0 \tag{29}$$

Note that Equation (29) implies that there is at least one eigenvalue that is zero over all the Kummer surface. But on Σ_K other eigenvalues can also become null. Thus, the Kummer surface is a variable ρ rank surface.

As $\phi(r)$ is equal to $\lambda_1(r)\lambda_2(r)\cdots\lambda_n(r)$, the vector η can be written as

$$\begin{aligned} \eta(r) = &\left[\nabla_r \lambda_1(r)\right]\lambda_2(r)\cdots\lambda_n(r) + \lambda_1(r)\left[\nabla_r \lambda_2(r)\right]\cdots\lambda_n(r) \\ &+ \lambda_1(r)\lambda_2(r)\cdots\left[\nabla_r \lambda_n(r)\right] \end{aligned} \tag{30}$$

Let say that λ_1 is the zero eigenvalue over all Kummer surface. Then over Σ_K, the vector η is given by

$$\eta(r) = \left[\nabla_r \lambda_1(r)\right]\lambda_2(r)\cdots\lambda_n(r) \tag{31}$$

If Σ_{K_n} is the subregion of Σ_K over which there are $n > 1$ null eigenvalues, then by (31)

$$\eta(r) = 0, \quad \forall r \in \Sigma_{K_n} \tag{32}$$

Thus higher order rank subregions Σ_{K_n} of the Kummer surface are characterized by the fact that the η vector vanishes on them.

Consider now, the origin $r_O = (0, 0, \cdots, 0)$ where $\phi(r_O) = 1$. It is easy to show that for points r near to the origin, the function ϕ goes as $\phi(r) \approx 1 - \|r\|^2$ by expanding ϕ in Taylor series around the origin and keeping the least order terms in the expansion. The η vector near the origin is then $\eta = -2r$ and its an inward radial vector. So near the origin, the constant determinant surfaces Σ_C are given approximately by M di- mensional spheres and ϕ growths inward to the origin.

Let Γ a curve that starts in the origin and that is normal to all Σ_C surfaces, that is, its tangent vector is parallel to the $-\boldsymbol{\eta}$ vector in each point. Because, near the origin the vector $-\boldsymbol{\eta}$ is radial, one can reach any point of the space starting from the origin using such a curve. Moving along Γ in the outer direction, the ϕ function always decreases from its initial value 1. Thus, at some point $\boldsymbol{r}_0$ in Γ, the ϕ function vanishes. Thus means that the Kummer surface Σ_K must contain a closed subsurface Σ_0 that enclosed the origin. Then inside of this closed subsurface Σ_0 the determinant of the ρ matrix must be positive and outside Σ_0 there are points where the determinant of the correlation matrix is necessarily negative. Note that Σ_0 can be contained totally inside the hypercube or can cut it in different regions with positive or negative determinant values respectively.

Thus, outside Σ_0 there are regions where the determinant

$$\lambda_1 \lambda_2 \cdots \lambda_N < 0 \tag{33}$$

so at least one of the eigenvalues must be negative outside Σ_0. Inside Σ_0 however

$$\lambda_1 \lambda_2 \cdots \lambda_N > 0 \tag{34}$$

This implies that pairs of eigenvalues can be negative. But inside Σ_0 the eigenvalue cannot be negative. To prove that, consider the origin $\boldsymbol{r}_O$ where all eigenvalues $\lambda_i = \lambda_i(\boldsymbol{r}_O)$ are equal to one. When $\boldsymbol{r}$ moves outward along a curve Γ that start at the origin, each eigenvalue $\lambda_i = \lambda_i(\boldsymbol{r})$ will change its value from its initial positive value 1, but cannot become negative. If $\lambda_i = \lambda_i(\boldsymbol{r}) < 0$ for some points $\boldsymbol{r}$ along Γ inside of Σ_0, then there is a point $\boldsymbol{r}_0$ where $\lambda_i = 0$. This implies that the vector $\boldsymbol{r}$ would cross the surface Σ_0, but it is impossible because $\boldsymbol{r}$ is inside of Σ_0 where $\det \rho > 0$. Then inside the surface Σ_0 all eigenvalues of the correlation matrix are positive.

In order to grasp the above ideas in detail the case of three assets is studied in the next sub section.

The Geometry of the *N* = 3 Assets Case

The ρ matrix, for the three assets case, is equal to

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \tag{35}$$

where the vector $\boldsymbol{r} = (\rho_{12}, \rho_{13}, \rho_{23})$ is written as $\boldsymbol{r} = (x, y, z)$. For this parameterization the determinant of the ρ matrix is

$$\det(\rho) = 2xyz - x^2 - y^2 - z^2 + 1$$

The constant determinant Σ_C surfaces $\det(\rho(\boldsymbol{r})) = C$ in the interior of the hypercube are shown in **Figure 1**, for some positive values between $0 < C < 1$. Instead, in **Figure 2**, some surfaces for negative *C* values are displayed with $-3 < C < 0$.

The Kummer Σ_K surface is given by the condition $\det \rho(\boldsymbol{r}) = 0$, that is
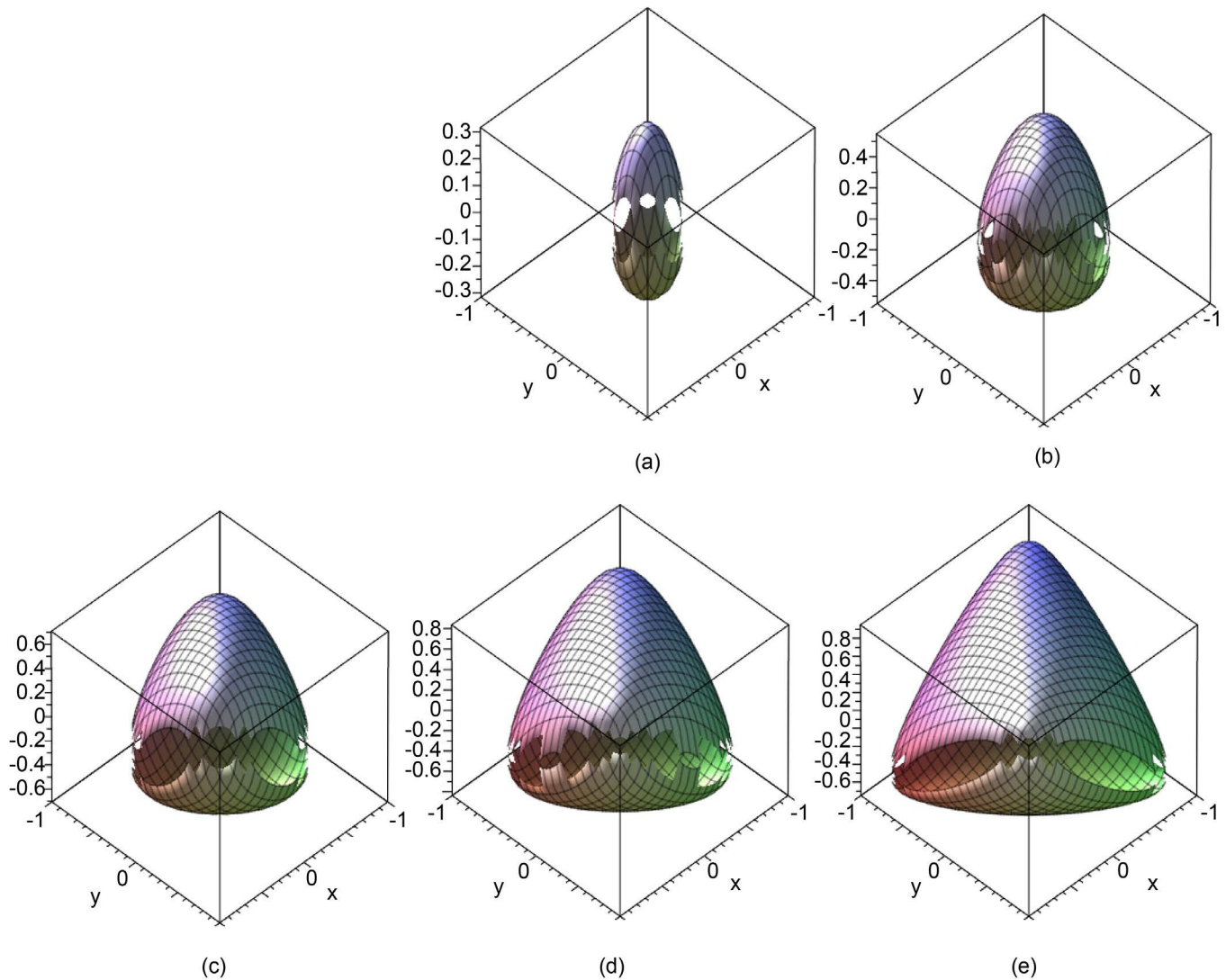
$$2xyz - x^2 - y^2 - z^2 + 1 = 0 \tag{36}$$

Figure 1. (a) $C = 0.9$, (b) $C = 0.7$, (c) $C = 0.5$, (d) $C = 0.3$, (e) $C = 0.1$.

From (36) one found that the Kummer Σ_0 subsurface inside the hypercube is given by the parametric equations

$$z = z^{\pm}(x, y) = xy \pm \sqrt{x^2 y^2 - x^2 - y^2 + 1} \tag{37}$$

Figure 3 shows the Kummer superior subsurface Σ_0^+ given by $z = z^+(x, y)$, the Kummer inferior subsurface Σ_0^- given by $z = z^-(x, y)$ and the complete Kummer subsurface Σ_0.

Because Σ_0 separates a region with $\det \rho > 0$ from that with $\det \rho < 0$ and due to the origin $\mathbf{r} = (0, 0, 0)$ the determinant is one, then inside of Σ_0 the determinant of the ρ matrix must be positive, which is consistent with **Figure 1**. The region situated between Σ_0 and the cube has negative determinant in this case.

In terms of its diagonal form, the ρ matrix inside or outside Σ_0 where $\det \rho \neq 0$, is
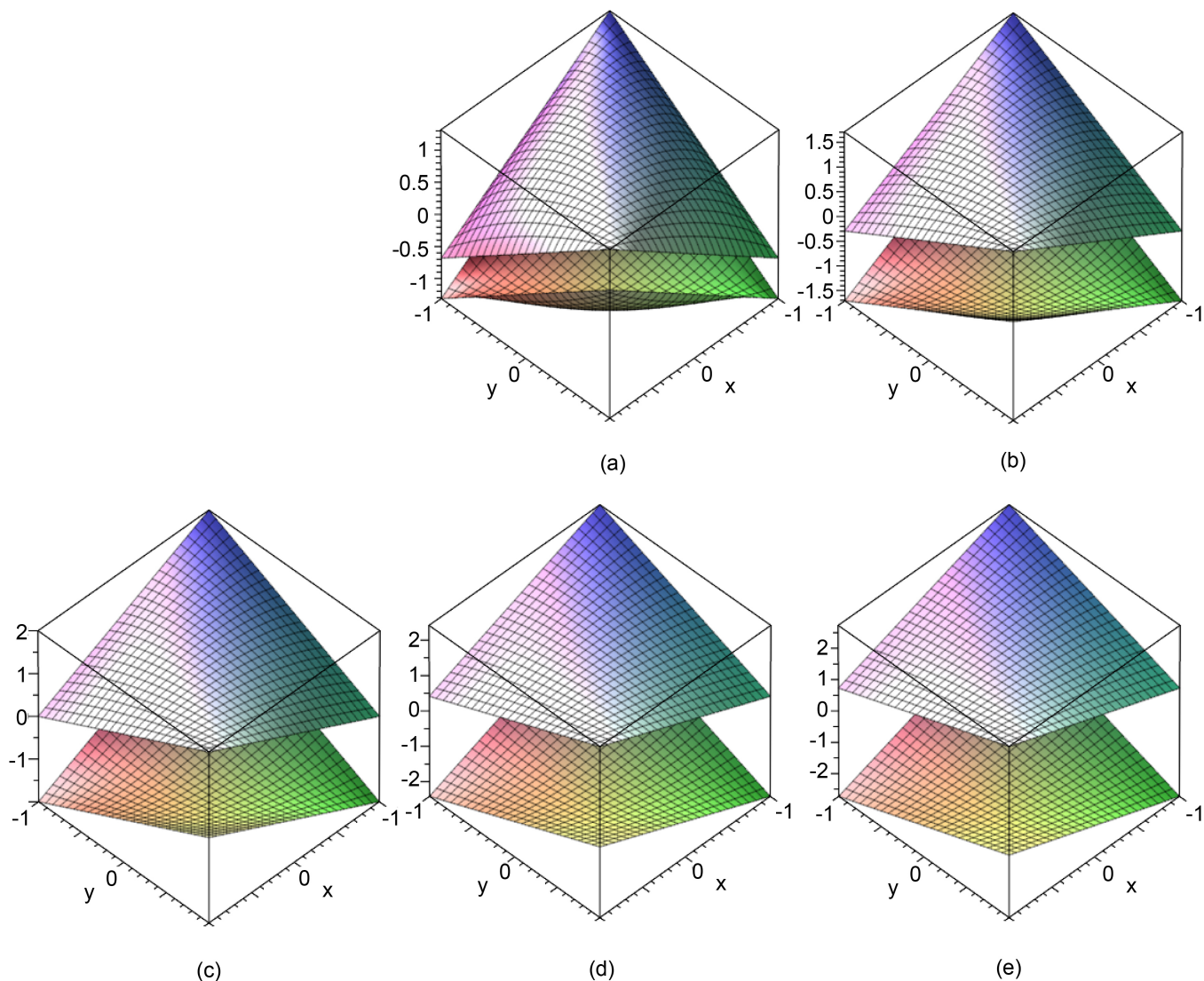
Figure 2. (a) $C = -0.1$, (b) $C = -0.5$, (c) $C = -1$, (d) $C = -2$, (e) $C = -3$.

$$\begin{pmatrix} \lambda_1(x, y, z) & 0 & 0 \\ 0 & \lambda_2(x, y, z) & 0 \\ 0 & 0 & \lambda_3(x, y, z) \end{pmatrix} \tag{38}$$

where the three eigenvalues $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$ when $\boldsymbol{r} = (x, y, z) \notin \Sigma_0$.

On the Kummer superior subsurface Σ_0^+, the diagonal form of the ρ matrix is

$$\begin{pmatrix} \lambda_1^+(x, y) & 0 & 0 \\ 0 & \lambda_2^+(x, y) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{39}$$

where

$$\lambda_1^+(x, y) = \frac{3}{2} + \frac{1}{2}\sqrt{1 + 8x^2 y^2 + 8xy\sqrt{x^2 y^2 - x^2 - y^2 + 1}} \tag{40}$$
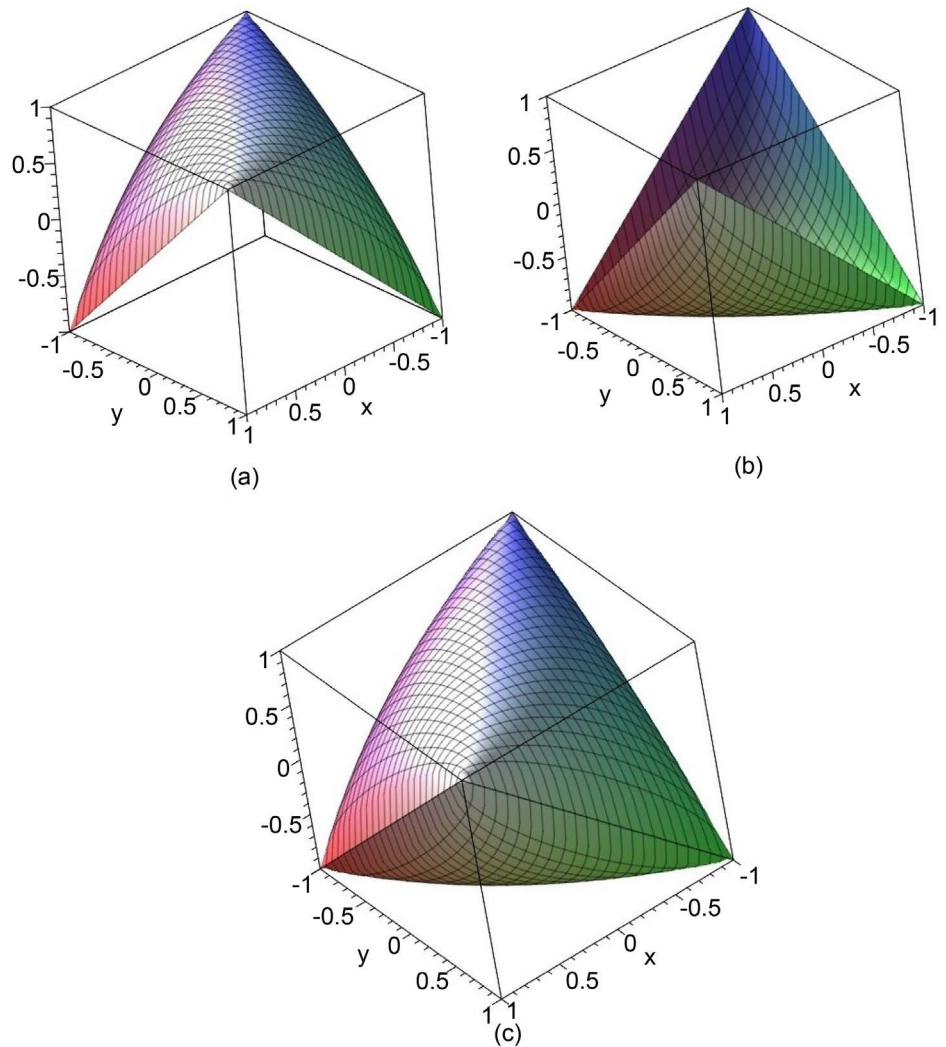
Figure 3. (a) Kummer superior subsurface $\Sigma_0^+ : z^+ = xy + \sqrt{x^2 y^2 - x^2 - y^2 + 1}$, (b) Kummer inferior subsurface $\Sigma_0^- : z^- = xy - \sqrt{x^2 y^2 - x^2 - y^2 + 1}$, (c) complete Kummer subsurface Σ_0. Note that the Kummer subsurface Σ_0 is closed and its is completely inside the hypercube in this case. Thus the region between Σ_0 and the hypercube has negative ρ determinant for the three assets system.

and

$$\lambda_2^+ (x, y) = \frac{3}{2} - \frac{1}{2}\sqrt{1 + 8x^2 y^2 + 8xy\sqrt{x^2 y^2 - x^2 - y^2 + 1}} \tag{41}$$

Figure 4 gives the eigenvalues $\lambda_1^+ (x, y)$ and $\lambda_2^+ (x, y)$ as functions of x and y.

For the Kummer inferior subsurface Σ_0^-, the diagonal form of the ρ matrix is instead

$$\begin{pmatrix} \lambda_1^- (x, y) & 0 & 0 \\ 0 & \lambda_2^- (x, y) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{42}$$

Figure 4. (a) $\lambda_1^+(x,y)$, (b) $\lambda_2^+(x,y)$.

where

$$\lambda_1^-(x,y) = \frac{3}{2} + \frac{1}{2}\sqrt{1 + 8x^2y^2 - 8xy\sqrt{x^2y^2 - x^2 - y^2 + 1}} \qquad (43)$$

and

$$\lambda_2^-(x,y) = \frac{3}{2} - \frac{1}{2}\sqrt{1 + 8x^2y^2 - 8xy\sqrt{x^2y^2 - x^2 - y^2 + 1}} \qquad (44)$$

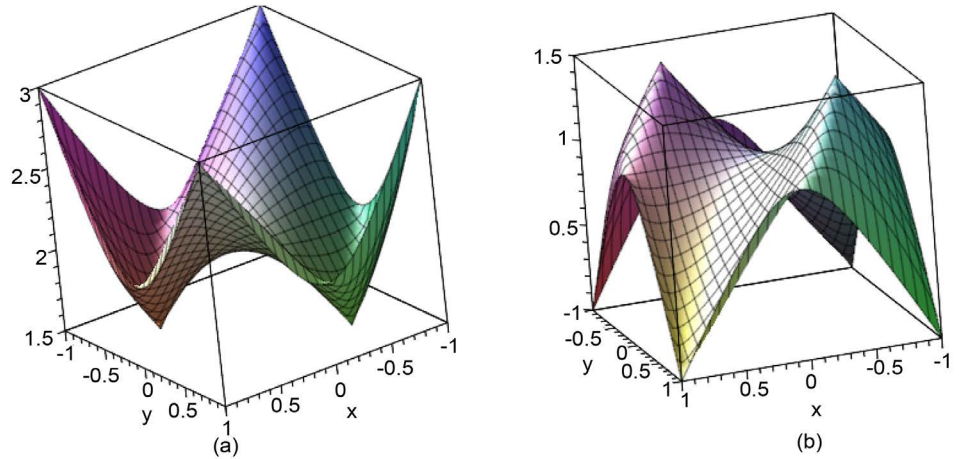**Figure 5** gives the eigenvalues $\lambda_1^-(x,y)$ and $\lambda_2^-(x,y)$ as functions of x and y.

Note that the eigenvalues $\lambda_1^+(x,y)$ and $\lambda_1^-(x,y)$ are always greater than zero, but $\lambda_2^+(x,y)$ and $\lambda_2^-(x,y)$ are zero for the extreme values of the correlation parameter $x = \pm 1$ and $y = \pm 1$. **Figure 6** shows both eigenvalues $\lambda_2^+(x,y)$ and $\lambda_2^-(x,y)$ in the same graph. It is possible to see clearly that the $\lambda_2(x,y)$ proper value becomes equal to zero only for the extreme correlations value cases

$$\boldsymbol{r} = (1,1,1), \boldsymbol{r} = (1,-1,-1), \boldsymbol{r} = (-1,1,-1), \boldsymbol{r} = (-1,-1,1) \qquad (45)$$

which are the vertexes of the Kummer Σ_0 subsurface in **Figure 3** or the four base points of **Figure 6**.

Thus, depending on which region of the three dimensional cube the vector $\boldsymbol{r} = (x,y,z)$ is lying, the correlation matrix ρ has two null eigenvalues, one null eigenvalue or it can be invertible. Thus the rank of the ρ matrix changes when $\boldsymbol{r}$ moves along the Kummer surface.

5. Pricing, the Wei-Norman Theorem, Propagators and Σ_K

The problem of pricing the multi-asset option Π is now tackled by taking into account the geometrical properties of the correlation ρ matrix analyzed in the Section 3. In order to do that one needs first to solve the Equation (23). For this, the Wei-Norman theorem [23]-[26] is applied. In this particular case this theorem establishes that the solution of (23) can be writing as

$$\Psi(\boldsymbol{\zeta},t) = U(t,0)\Psi(\boldsymbol{\zeta},0) \qquad (46)$$
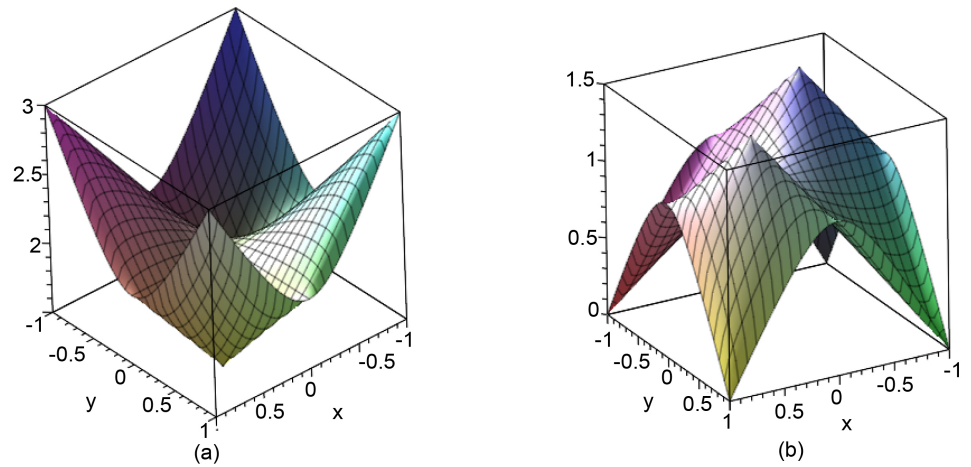
Figure 5. (a) $\lambda_1^-(x,y)$, (b) $\lambda_2^-(x,y)$.
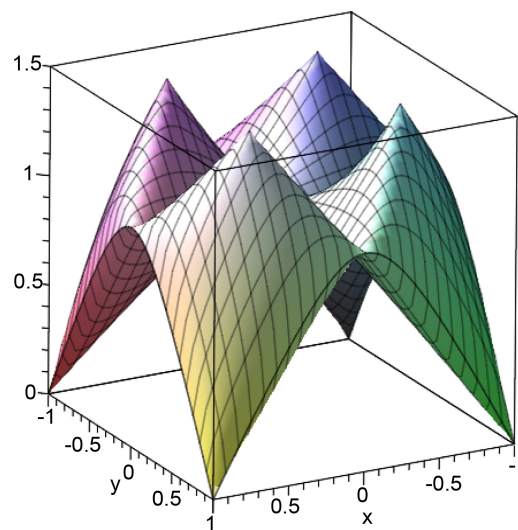
Figure 6. The λ_2 eigenvalue as function of (x,y).

where

$$U(t,0) = \prod_{k=1}^{N} e^{\left[a_k(t)L_k\right]} \tag{47}$$

with

$$a_k(t) = \int_0^t \frac{1}{2}\lambda_k(\boldsymbol{r})\,\mathrm{d}t = \frac{1}{2}\lambda_k(\boldsymbol{r})t \tag{48}$$

and

$$L_k = \frac{\partial^2}{\partial \zeta_k^2} \tag{49}$$

that is

$$\Psi(\boldsymbol{\zeta},t) = \prod_{k=1}^{N} e^{\left[\frac{1}{2}\lambda_k(\boldsymbol{r})t\frac{\partial^2}{\partial \zeta_k^2}\right]}\Psi(\boldsymbol{\zeta},0) \tag{50}$$

by inserting N one dimensional Dirac's deltas, one can write the above equation as

$$\Psi(\boldsymbol{\zeta},t) = \prod_{k=1}^{N} e^{\left[\frac{1}{2}\lambda_k(\boldsymbol{r})t\frac{\partial^2}{\partial\zeta_k^2}\right]} \left[\prod_{m=1}^{N}\int d\zeta_m' \delta(\zeta_m - \zeta_m')\right]\Psi(\boldsymbol{\zeta}',0) \tag{51}$$

or as

$$\Psi(\boldsymbol{\zeta},t) = \int K_\Psi(\boldsymbol{\zeta},t\,|\,\boldsymbol{\zeta}'0)\Psi(\boldsymbol{\zeta}',0)d\boldsymbol{\zeta}' \tag{52}$$

where the propagator K_Ψ is defined by

$$K_\Psi(\boldsymbol{\zeta},t\,|\,\boldsymbol{\zeta}'0) = \prod_{k=1}^{N} e^{\left[\frac{1}{2}\lambda_k(\boldsymbol{r})t\frac{\partial^2}{\partial\zeta_k^2}\right]}\left[\delta^{(N)}(\boldsymbol{\zeta}-\boldsymbol{\zeta}')\right] \tag{53}$$

with $\delta^{(N)}(\boldsymbol{\zeta}-\boldsymbol{\zeta}')$ the N dimensional Dirac's delta. Now using the Fourier expansion

$$\delta^{(N)}(\boldsymbol{\zeta}-\boldsymbol{\zeta}') = \int \frac{d\boldsymbol{p}}{(2\pi)^N} e^{i\boldsymbol{p}\cdot(\boldsymbol{\zeta}-\boldsymbol{\zeta}')} \tag{54}$$

the propagator can be written finally as the product

$$K_\Psi(\boldsymbol{\zeta},t\,|\,\boldsymbol{\zeta}'0) = \prod_{k=1}^{N}\left[\int \frac{dp_k}{2\pi} e^{-\frac{1}{2}\lambda_k(\boldsymbol{r})tp_k^2 + ip_k(\zeta_k-\zeta_k')}\right] \tag{55}$$

The Propagator Inside Σ_0

When $\boldsymbol{r}$ is inside of Σ_0, all eigenvalues $\lambda_k(\boldsymbol{r})$ are positive, so the N integrations in (55) can be performed to give [27] [28]

$$K_\Psi(\boldsymbol{\zeta},t\,|\,\boldsymbol{\zeta}'0) = \prod_{k=1}^{N}\left[\frac{1}{\sqrt{2\pi\lambda_k t}} e^{-\frac{(\zeta_k-\zeta_k')^2}{2\lambda_k t}}\right] \tag{56}$$

or

$$K_\Psi(\boldsymbol{\zeta},t\,|\,\boldsymbol{\zeta}'0) = \frac{1}{\sqrt{(2\pi t)^N \lambda_1\lambda_2\cdots\lambda_N}} e^{\sum_{k=1}^{N}-\frac{(\zeta_k-\zeta_k')^2}{2\lambda_k t}} \tag{57}$$

By using transformations (15), (16), (17) and (18) one can write the propagator for the option price $\Pi(S,\tau)$ in the (S,τ) space as

$$K_\Pi(\boldsymbol{S},\tau\,|\,\boldsymbol{S}'T) = \frac{\exp(-r(T-\tau))}{\sqrt{(2\pi(T-\tau))^N \det(\rho)}\,\sigma_1\sigma_2\cdots\sigma_N S_1'S_2'\cdots S_N'} e^{-\left[\frac{(\boldsymbol{\alpha}'\rho^{-1}\boldsymbol{\alpha})}{2(T-\tau)}\right]} \tag{58}$$

with

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_i(S_i,S_i') = \frac{\ln\left(\dfrac{S_i}{S_i'}\right)+\left(r-\dfrac{1}{2}\sigma_i^2\right)(T-\tau)}{\sigma_i} \tag{59}$$

which is the usual form of the propagator in the S space (see for example [3] [7]). Note this form of the propagator is valid only when $\det(\rho) > 0$. So (58) can be applied inside the closed subsurface Σ_0 or some region between Σ_0 and the interior of the

hypercube that verifies $\det(\rho) > 0$ and have only positive eigenvalues.

6. The Propagator for the Kummer Surface Σ_K

In this section, an expression for the propagator over the Kummer surface Σ_K is obtained. It is assumed that a region $\Sigma_{K_{N_B}}$ of Σ_K that has N_A non zero eigenvalues and $N_B = N - N_A$ null eigenvalues. Due to it is on the Σ_K surface, the Equation (26) implies that one of the coordinates of the r vector, is determined by the other $M-1$ coordinates. These independent coordinates are called $x_1, x_2, \cdots, x_{M-1}$. Thus in this section, the vector r is an M dimensional vector that depends on $M-1$ independent coordinates. In this situation the propagator in (55) gives

$$K_\Psi\left(\boldsymbol{\zeta}, t \mid \boldsymbol{\zeta}'0\right) = \left[\prod_{k=1}^{N_A} \int \frac{\mathrm{d}p_k}{2\pi} \mathrm{e}^{-\frac{1}{2}\lambda_k(r)tp_k^2 + ip_k(\zeta_k - \zeta_k')}\right]\left[\prod_{j=1}^{N_B} \int \frac{\mathrm{d}p_j}{2\pi} \mathrm{e}^{ip_j(\zeta_j - \zeta_j')}\right] \tag{60}$$

By performing the integrations

$$K_\Psi\left(\boldsymbol{\zeta}, t \mid \boldsymbol{\zeta}'0\right) = \frac{\mathrm{e}^{\sum_{k=1}^{N_A} -\frac{(\zeta_k - \zeta_k')^2}{2\lambda_k t}}}{\sqrt{(2\pi t)^{N_A} \lambda_1 \lambda_2 \cdots \lambda_{N_A}}} \prod_{j=1}^{N_B} \delta\left(\zeta_j - \zeta_j'\right) \tag{61}$$

If the N dimensional vector $\boldsymbol{\zeta}$ is separated in two parts as

$$\boldsymbol{\zeta} = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N_A} \\ \zeta_{N_A+1} \\ \vdots \\ \zeta_N \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N_A} \end{pmatrix} \\ \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N_B} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\zeta}_A \\ \boldsymbol{\zeta}_B \end{pmatrix} \tag{62}$$

the above propagator can be written in a more compact form as

$$K_\Psi\left(\boldsymbol{\zeta}, t \mid \boldsymbol{\zeta}'0\right) = \frac{1}{\sqrt{(2\pi t)^{N_A} \det(D_A)}} \mathrm{e}^{\frac{(\boldsymbol{\zeta}_A - \boldsymbol{\zeta}_A')^t D_A^{-1}(\boldsymbol{\zeta}_A - \boldsymbol{\zeta}_A')}{2t}} \delta^{(N_B)}\left(\boldsymbol{\zeta}_B - \boldsymbol{\zeta}_B'\right) \tag{63}$$

where

$$D_A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N_A} \end{pmatrix} \tag{64}$$

is the reduced diagonal ρ matrix on the Kummer surface Σ_K. If one separates the vector $\boldsymbol{\chi}$ in A and B components as

$$\boldsymbol{\chi} = \begin{pmatrix} \boldsymbol{\chi}_A \\ \boldsymbol{\chi}_B \end{pmatrix} \tag{65}$$

then relation (20) induces the transformation

$$\begin{pmatrix} \boldsymbol{\zeta}_A \\ \boldsymbol{\zeta}_B \end{pmatrix} = \begin{pmatrix} U_{AA}^{-1} & U_{AB}^{-1} \\ U_{BA}^{-1} & U_{BB}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\chi}_A \\ \boldsymbol{\chi}_B \end{pmatrix} \tag{66}$$

where U_{AA}^{-1}, U_{AB}^{-1}, U_{BA}^{-1} and U_{BB}^{-1} are the matrices that result from sectioning U^{-1} into A and B components.

The quadratic term in the exponential of (61) can be expressed in the χ_A and χ_B components as

$$
\begin{aligned}
&\left(\zeta_A - \zeta'_A \right)^t D_A^{-1} \left(\zeta_A - \zeta'_A \right) \\
&= \left(\chi_A - \chi'_A \right)^t \left[U_{AA}^{-1} \right]^t D_A^{-1} U_{AA}^{-1} \left(\chi_A - \chi'_A \right) + \left(\chi_B - \chi'_B \right)^t \left[U_{AB}^{-1} \right]^t D_A^{-1} U_{AA}^{-1} \left(\chi_A - \chi'_A \right) \\
&\quad + \left(\chi_A - \chi'_A \right)^t \left[U_{AA}^{-1} \right]^t D_A^{-1} U_{AB}^{-1} \left(\chi_B - \chi'_B \right) + \left(\chi_B - \chi'_B \right)^t \left[U_{AB}^{-1} \right]^t D_A^{-1} U_{AB}^{-1} \left(\chi_B - \chi'_B \right)
\end{aligned}
\tag{67}
$$

Now, from (66)

$$
\left(\zeta_B - \zeta'_B \right) = U_{BA}^{-1} \left(\chi_A - \chi'_A \right) + U_{BB}^{-1} \left(\chi_B - \chi'_B \right)
\tag{68}
$$

The Dirac's delta in (63) implies that

$$
0 = U_{BA}^{-1} \left(\chi_A - \chi'_A \right) + U_{BB}^{-1} \left(\chi_B - \chi'_B \right)
\tag{69}
$$

The above equation permits writing the vector $\left(\chi_B - \chi'_B \right)$ in terms of $\left(\chi_A - \chi'_A \right)$ as

$$
\left(\chi_B - \chi'_B \right) = -U_{BB} U_{BA}^{-1} \left(\chi_A - \chi'_A \right)
\tag{70}
$$

replacing in (67) one can write the quadratic term as

$$
\left(\zeta_A - \zeta'_A \right)^t D_A^{-1} \left(\zeta_A - \zeta'_A \right) = \left(\chi_A - \chi'_A \right)^t \rho_{\Sigma_K}^{-1} \left(\chi_A - \chi'_A \right)
\tag{71}
$$

where $\rho_{\Sigma_K}^{-1}$ is defined by

$$
\begin{aligned}
\rho_{\Sigma_K}^{-1} = &\left[U_{AA}^{-1} \right]^t D_A^{-1} U_{AA}^{-1} + \left[U_{AB}^{-1} U_{BB} U_{BA}^{-1} \right]^t D_A^{-1} U_{AA}^{-1} \\
&+ \left[U_{AA}^{-1} \right]^t D_A^{-1} U_{AB}^{-1} U_{BB} U_{BA}^{-1} + \left[U_{AB}^{-1} U_{BB} U_{BA}^{-1} \right]^t D_A^{-1} U_{AB}^{-1} U_{BB} U_{BA}^{-1}
\end{aligned}
\tag{72}
$$

From (66)

$$
\mathrm{d}\zeta = \mathrm{d}\zeta_A \mathrm{d}\zeta_B = \mathrm{d}\chi_A \mathrm{d}\chi_B = \mathrm{d}\chi
\tag{73}
$$

Using (68) and (71) in (52), the option price can be written as

$$
\Psi\left(\chi_A, \chi_B, t \right) = \int \frac{\mathrm{e}^{\frac{\left(\chi_A - \chi'_A \right)^t \rho_{\Sigma_K}^{-1} \left(\chi_A - \chi'_A \right)}{2t}}}{\sqrt{\left(2\pi t \right)^{N_A} \det\left(D_A \right)}} \delta^{(N_B)} \left(U_{BA}^{-1} \left(\chi_A - \chi'_A \right) + U_{BB}^{-1} \left(\chi_B - \chi'_B \right) \right) \Psi\left(\chi'_A, \chi'_B, 0 \right) \mathrm{d}\chi'_A \mathrm{d}\chi'_B
\tag{74}
$$

Integrating over $\mathrm{d}\chi'_B$ gives

$$
\Psi_{(r)}\left(\chi_A, \chi_B, t \right) = \int \frac{\mathrm{e}^{\frac{\left(\chi_A - \chi'_A \right)^t \rho_{\Sigma_K}^{-1} \left(\chi_A - \chi'_A \right)}{2t}}}{\sqrt{\left(2\pi t \right)^{N_A} \det\left(D_A \right)}} \frac{1}{\det\left(U_{BB}^{-1} \right)} \Psi\left(\chi'_A, \chi'_B, 0 \right) \mathrm{d}\chi'_A
\tag{75}
$$

where χ'_B must be evaluated from (70) in terms of χ_B and $\left(\chi_A - \chi'_A \right)$ as

$$
\chi'_B = \chi_B + \gamma \left(\chi_A - \chi'_A \right)
\tag{76}
$$

where the rectangular $N_B \times N_A$ matrix γ is defined by

$$
\gamma = U_{BB} U_{BA}^{-1}
\tag{77}
$$

It must be noted that U^{-1}, the eigenvalues λ_i, and the rectangular matrix γ are functions of the vector r that lies on the null surface Σ_K. Thus the option price is also a function of r. Using (15), (16), (17) and (18) one can write the option price in the (S, τ) space as $\Pi_{(r)}(S, \tau)$ and is given by

$$\Pi_{(r)}(S_A, S_B, \tau) = \int \frac{e^{\frac{(\boldsymbol{\alpha}_A)^t \rho_{\Sigma_K}^{-1}(\boldsymbol{\alpha}_A)}{2(T-\tau)}}}{\sqrt{(2\pi(T-\tau))^{N_A} \det(D_A)}} \frac{e^{-r(T-\tau)}}{\det(U_{BB}^{-1})} \frac{\Psi(S_A', S_B', T)}{\sigma_1 \cdots \sigma_{N_A}} \frac{dS_A'}{S_1' \cdots S_{N_A}'} \quad (78)$$

where the components of the $\boldsymbol{\alpha}$ are given by

$$\alpha_A^j = \frac{\ln\left(\frac{S_A^j}{S_A'^j}\right) + \left(r - \frac{1}{2}\sigma_j^2\right)(T-\tau)}{\sigma_j}, \qquad j = 1, \cdots, N_A \quad (79)$$

and the components of the vector S_B' are given in terms of S_A, S_A' and S_B according to

$$S_B'^i = S_B^i \left[\prod_{j=1}^{N_A} \left(\frac{S_A^j}{S_A'^j}\right)^{\frac{\sigma_i \gamma_{ij}}{\sigma_j}} \right] e^{\left[\left(r - \frac{1}{2}\sigma_i^2\right) + \sum_{j=1}^{N_A} \frac{\sigma_i}{\sigma_j}\gamma_{ij}\left(r - \frac{1}{2}\sigma_j^2\right)\right](T-\tau)}, \quad i = 1, \cdots, N_B \quad (80)$$

with γ_{ij} the components of the rectangular matrix γ

$$\gamma_{ij} = \left[U_{BB}U_{BA}^{-1}\right]_{ij}, \quad i = 1, \cdots, N_B, \quad j = 1, \cdots, N_A \quad (81)$$

When r moves over the Kummer surface Σ_K, the rank of the ρ matrix can change, so the dimensions of N_A and $N_B = N - N_A$ also change, but Equation (78) is always valid.

7. The Propagator Outside Σ_0

When the vector r is lying outside the Kummer subsurface Σ_0, there are regions where the determinant of the correlation matrix is negative. This implies that the propagator given in (58) becomes complex. But, worse than that, in this case one of the eigenvalues λ_k is negative, so the propagator given in (57) generates an exponential growth in the associated ζ_k coordinate. Then the convolution in (52) is not well defined. Thus, one cannot price the option in regions outside the Kummer subsurface Σ_0 that have negative ρ determinant.

8. Conclusions and Further Research

In this research, the existence of the solution of the multi-asset Black-Scholes model has been analyzed in detail. It has been shown that the correlation parameter space, which is equivalent to an N dimensional hypercube, limits the existence of a valid solution for the multi-asset Black-Scholes model. Particularly, it has been demonstrated that inside of this hypercube there is a surface, called the Kummer surface Σ_K, where the determinant of the correlation matrix ρ is zero, the usual formula for the propagator of the N asset Black-Scholes equation is no longer valid. In particular, the case for three

assets and its implied geometry has been studied in detail when the determinant of the correlation matrix is zero. Finally, by using the Wei-Norman theorem, the propagator over the variable rank surface Σ_K for the general N asset case has been computed, which is applicable over all the Kummer surface, whatever be the rank of the ρ matrix. This formulation corrects the past solution of this problem and its extensions.

As future research, most of the papers related to the multi-asset Black-Scholes model must be revisited in line of our results, as well as others where it is implicitly assumed that a well behaved multivariate Gaussian distribution must exist, as is the case of the stochastic volatility family (see for instance [29] [30]).

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