Journal of Mathematical Finance, 2016, 6, 260-268
Published Online May 2016 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2016.62022

Markov-Dependent Risk Model with Multi-Layer Dividend Strategy and Investment Interest under Absolute Ruin

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Received 1 February 2016; accepted 6 March 2016; published 9 March 2016

Abstract

In this paper, we consider the Markov-dependent risk model with multi-layer dividend strategy and investment interest under absolute ruin, in which the claim occurrence and the claim amount are regulated by an external discrete time Markov chain. We derive systems of integro-differential equations satisfied by the moment-generating function, the *n*th moment of the discounted dividend payments prior to absolute ruin and the Gerber-Shiu function. Finally, the matrix form of systems of integro-differential equations satisfied by the Gerber-Shiu function is presented.

Keywords

Markov-Dependent Risk Model, Absolute Ruin, Multi-Layer Dividend Strategy, Gerber-Shiu Function, Investment Interest

1. Introduction

The dividend problem has long been an important issue in finance and actuarial sciences. Due to the importance of the dividend problem, the study of the risk model with dividend strategy has received more and more attention. Most of the strategies considered are of two kinds: one is the barrier strategy; another is the threshold strategy. For more recent studies about dividend problems, see [1]-[4]. In these papers, they extend the threshold dividend strategy to the multiple case, and make in-depth study of the model by the probabilistic and differential equation approaches. Under such a dividend strategy, many authors have extensively studied the Gerber-Shiu function for both the classical and the renewal risk model.

In classical insurance theory, we usually say that ruin occurs when the surplus is below zero. But in reality, the insurer could borrow an amount of money equal to the deficits at a debit interest rate to continue his business

How to cite this paper: Li, B.L. and Ma, S.X. (2016) Markov-Dependent Risk Model with Multi-Layer Dividend Strategy and Investment Interest under Absolute Ruin. *Journal of Mathematical Finance*, **6**, 260-268.
http://dx.doi.org/10.4236/jmf.2016.62022

when the surplus falls below zero. Meanwhile, the insurer will repay the debts from his premium income. If debts are reasonable, the negative surplus may return to a positive level. However, when the negative surplus is below some certain level, the insurer is no longer allowed to run his business and absolute ruin occurs at this situation.

Absolute ruin probability has been frequently considered in recent research works. Dassios and Embrechts considered the absolute ruin, and by a martingale approach they derived the explicit expression for the probability of absolute ruin in the case of exponential individual claim in [5]. Cai defined Gerber-Shiu function at absolute ruin and derived a system of the integro-differential equations satisfied by the Gerber-Shiu function in [6].

Most of the literature in finance is based on the assumption that the inter-arrival time between two successive claims and the claim amounts are independent. However, the independence assumption can be inappropriate and unrealistic in practical contexts. So in recent years, the risk model with dependence structure between inter-arrival times and claim sizes has got more and more attention. For example, see [7]-[9]. Yu and Huang [8] studied the dividend payments prior to absolute ruin in a Markov-dependent risk process. Zhou *et al*. [9] proposed a Markov-dependent risk model with multi-layer dividend strategy.

To the best of our knowledge, Markov-dependent risk model with multi-layer dividend strategy and investment interest under absolute ruin has not been investigated. This motivates us to investigate such a risk model in this work. Generally, the authors only extensively consider Gerber-Shiu function in risk models with multi-layer dividend strategy. In this paper, we study not only Gerber-Shiu function, but also the moment-generating function and the *n*th moment of the discounted dividend payments prior to absolute ruin.

The rest of the paper is organized as follows. In Section 2, the model is described and basic concepts are introduced. In Sections 3, we get integro-differential equations for the moment-generating function of the discounted dividend payments prior to absolute ruin and boundary conditions. In Section 4, the integro-differential equations satisfied by higher moment of the discounted dividend payments prior to absolute ruin and boundary conditions are derived. In Section 5, we obtain the systems of integro-differential equations for the Gerber-Shiu function and its matrix form. Section 6 concludes the paper.

2. The Model

In this section, we investigate the Markov-dependent risk model with multi-layer dividend strategy and investment interest under absolute ruin, in which the claim occurrence and the claim amount are regulated by an external discrete time Markov chain $\{J_n\}_{n\ge 0}$. First, let $\{J_n\}_{n\ge 0}$ be an irreducible discrete time Markov chain with finite state space $S=\{1,2,\cdots,d\}$ and transition matrix $\Lambda=(p_{ij})$. Similar to Albrecher and Boxma [7], we define the structure of a semi-Markov dependence type insurance problem as follows. Let W_i denote the time between the arrival of the $(i-1)$th and the *i*th claims and $W_0=X_0=0$ a.s., then

$$\begin{aligned}&P\left(W_{n+1}\le x, X_{n+1}\le y, J_{n+1}=j \mid J_n=i,(W_t,X_t,J_t),0\le t\le n\right)\\&=P\left(W_1\le x, X_1\le y, J_1=j\mid J_0=i\right)=\left(1-\mathrm{e}^{-\lambda_i x}\right)p_{ij}F_j(y),\end{aligned}\tag{2.1}$$

where X_n is the amount of the *n*th claim. Thus at each instant of a claim, the Markov chain jumps to a state *j* and the distribution F_j of the claim depends on the new state *j*, and has a positive mean μ_j. Then, the next interarrival time is exponentially distributed with parameter λ_j. Note that given the states J_{n-1} and J_n, the quantities W_n and X_n are independent, but there is an autocorrelation among consecutive claim sizes and among consecutive interclaim times as well as cross-correlation between W_n and X_n.

In our risk model, we assume that the insurer could borrow money with the amount equal to the deficit at a debit interest force $\beta>0$ when the surplus falls below zero or the company is on deficit. And when the surplus becomes positive, the insurer could earn interest at an investment rate $r\,(0<r<\beta)$. We also assume that the premium rate is a step function, instead of a constant, dependent on the current surplus level. More precisely, define N layers $0=b_0<b_1<\cdots<b_N=\infty$. When the surplus $U_g(t)$ is in layer *k*, *i.e.* $b_{k-1}\le U_g(t)<b_k$, premium is collected with rate $c(U_g(t))=c_k$ until a claim causes the surplus to a lower layer or the surplus grows to the next higher layer. Meanwhile, the premium will be collected with rate c_β when the surplus

becomes negative. In reality, we assume $c_\beta = c_1 \ge c_2 \ge \cdots \ge c_N$. When the surplus is in layer k, dividends are paid continuously at a constant rate $\varepsilon_k = c_1 - c_k$. Furthermore, we assume the net profit condition is fulfilled in each layer, that is

$$\sum_{i=1}^{d} \pi_i \mu_i < c_k \sum_{i=1}^{d} \pi_i \frac{1}{\lambda_i}, \tag{2.2}$$

where $\pi = (\pi_1, \cdots, \pi_d)$ is the stationary distribution of process $\{J_n\}_{n\ge 0}$. We denote the surplus by $U_g(t)$. Then the dynamics of $U_g(t)$ can be expressed as

$$\mathrm{d}U_g(t) = \begin{cases} \left(c_\beta + \beta U_g(t)\right)\mathrm{d}t - \mathrm{d}\sum_{n=1}^{N(t)} X_n, & U_g(t) < 0, \\ \left(c_k + rU_g(t)\right)\mathrm{d}t - \mathrm{d}\sum_{n=1}^{N(t)} X_n, & b_{k-1} \le U_g(t) < b_k, k = 1, 2, \cdots, N. \end{cases} \tag{2.3}$$

where $N(t)$ is the number of claims up to time t.

Note that the surplus is no longer able to become positive when the negative surplus attains the level $-c_\beta/\beta$ or is below $-c_\beta/\beta$, because the insurer cannot repay all his debts for his business. We denote the absolute ruin time of the model (2.3) by τ_g, which is defined by $\tau_g = \inf\{t \ge 0 : U_g(t) \le -c_\beta/\beta\}$, and $\tau_g = \infty$ if $U_g(t) \ge -c_\beta/\beta$, for all $t \ge 0$. Given the initial surplus u, and the force of interest δ, the present value of all dividends until time of absolute ruin τ_g is defined by

$$D_{u,g} = \int_0^{\tau_g} \mathrm{e}^{-\delta t}\mathrm{d}D(t), \tag{2.4}$$

where $D(t)$ is the cumulative amount of dividends paid out up to time t. In the sequel we will be interested in the moment-generating function

$$M_i(u,y) = E\left[\mathrm{e}^{yD_{u,g}} \mid J_0 = i\right], \ i \in S \tag{2.5}$$

and the n th moment function

$$V_{n,i}(u) = E\left[D_{u,g}^n \mid J_0 = i\right], \ i \in S \tag{2.6}$$

with $V_{0,i}(u) = 1$, and the expected discounted penalty function, for $i \in S$

$$\Phi_i(u) = E\left[\mathrm{e}^{-\delta\tau_g} w\left(U_g(\tau_g -), \left|U_g(\tau_g)\right|\right) I_{\{\tau_g<\infty\}} \mid U_g(0) = u, J_0 = i\right], \tag{2.7}$$

where, $U_g(\tau_g -)$ is the surplus prior to absolute ruin and $\left|U_g(\tau_g)\right|$ is the deficit at absolute ruin. The penalty function $w(x_1, x_2)$ is an arbitrary nonnegative measurable function defined on $(-c_\beta/\beta, +\infty) \times (c_\beta/\beta, +\infty)$. For fix $i \in S$, throughout this paper we assume that $M_i(u,y)$, $V_{n,i}(u)$ and $\Phi_i(u)$ are sufficiently smooth functions in u and y in their respective domains.

3. Integro-Differential Equations for $M_i(u,y)$

In this section, we give the integro-differential equations for the moment-generating function $M_i(u,y)$. Clearly, the moment-generating function $M_i(u,y)$ behaves differently. For $i \in S$, we define

$$M_i(u,y) = \begin{cases} M_{\beta,i}(u,y), & -c_\beta/\beta < u < 0, \\ M_{k,i}(u,y), & b_{k-1} \le u < b_k, \ k = 1, 2, \cdots, N. \end{cases}$$

For notational convenience, let $h_\beta(u,t) = u\mathrm{e}^{\beta t} + c_\beta\left(\mathrm{e}^{\beta t} - 1\right)/\beta$, $h_k(u,t) = u\mathrm{e}^{rt} + c_k\left(\mathrm{e}^{rt} - 1\right)/r$, $k = 1, 2, \cdots, N$.

Theorem 3.1. *For* $i \in S$, $-c_\beta/\beta < u < 0$,

$$\left(\beta u+c_\beta\right)\frac{\partial M_{\beta,i}\left(u,y\right)}{\partial u}=\delta y\frac{\partial M_{\beta,i}\left(u,y\right)}{\partial y}+\lambda_i M_{\beta,i}\left(u,y\right)$$
$$-\lambda_i\sum_{j=1}^{d}p_{ij}\left[\int_0^{u+c_\beta/\beta}M_{\beta,j}\left(u-x,y\right)\mathrm{d}F_j\left(x\right)+\bar{F}_j\left(u+c_\beta/\beta\right)\right], \tag{3.1}$$

and, for $b_{k-1}\le u<b_k$, $k=1,2,\cdots,N$,

$$\left(ru+c_k\right)\frac{\partial M_{k,i}\left(u,y\right)}{\partial u}=\delta y\frac{\partial M_{k,i}\left(u,y\right)}{\partial y}+\left[\lambda_i-\left(c_1-c_k\right)y\right]M_{k,i}\left(u,y\right)$$
$$-\lambda_i\sum_{j=1}^{d}p_{ij}\left[\int_0^{u-b_{k-1}}M_{k,j}\left(u-x,y\right)\mathrm{d}F_j\left(x\right)+\cdots+\int_{u-b_1}^{u}M_{1,j}\left(u-x,y\right)\mathrm{d}F_j\left(x\right)\right. \tag{3.2}$$
$$\left.+\int_u^{u+c_\beta/\beta}M_{\beta,j}\left(u-x,y\right)\mathrm{d}F_j\left(x\right)+\bar{F}_j\left(u+c_\beta/\beta\right)\right].$$

Proof. Fix $i\in S$, and $-c_\beta/\beta<u<0$, let t_0 be the solution to the equation of $h_\beta\left(u,t\right)=0$, namely $t_0=\ln\left(\frac{c_\beta}{c_\beta+u\beta}\right)^{\frac{1}{\beta}}$, which is the time when the surplus returns to the level zero if no claim occurs to time t_0. Then $h_\beta\left(u,t\right)$ is the surplus at time $t\le t_0$ if no claim occurs prior to time t_0. We consider a small time interval $\left(0,t\right]$, where $t\le t_0$. In view of the strong Markov property of the surplus process $\left\{U_g\left(t\right)\right\}_{t\ge 0}$, we have

$$M_i\left(u,y\right)=E\left[M_i\left(U_g\left(t\right),y\mathrm{e}^{-\delta t}\right)\right]. \tag{3.3}$$

Thus conditioning on the time and the amount of the first claim, we obtain,

$$M_{\beta,i}\left(u,y\right)=\left(1-\lambda_i t\right)E\left[M_{\beta,i}\left(h_\beta\left(u,t\right),y\mathrm{e}^{-\delta t}\right)\right]+\lambda_i t\sum_{j=1}^{d}p_{ij}$$
$$\times E\left[\int_0^{h_\beta(u,t)+c_\beta/\beta}M_{\beta,j}\left(h_\beta\left(u,t\right)-x,y\mathrm{e}^{-\delta t}\right)\mathrm{d}F_j\left(x\right)+\bar{F}_j\left(h_\beta\left(u,t\right)+c_\beta/\beta\right)\right]+o\left(t\right). \tag{3.4}$$

By Taylor's expansion, we have

$$E\left[M_{\beta,i}\left(h_\beta\left(u,t\right),y\mathrm{e}^{-\delta t}\right)\right]=M_{\beta,i}\left(u,y\right)+\left(\beta u+c_\beta\right)t\frac{\partial M_{\beta,i}\left(u,y\right)}{\partial u}-\delta yt\frac{\partial M_{\beta,i}\left(u,y\right)}{\partial y}. \tag{3.5}$$

Substituting (3.5) into (3.4), and then dividing both sides of (3.4) by t and letting $t\to 0$, we get (3.1).

Similarly, when $b_{k-1}\le u<b_k, k=1,2,\cdots,N$, we still consider a small time interval $\left(0,t\right]$, where $t\left(t>0\right)$ is sufficiently small so that the surplus process will not reach b_k. Conditioning on the event occurring in the interval $\left(0,t\right]$, we obtain

$$M_{k,i}\left(u,y\right)=\left(1-\lambda_i t\right)\mathrm{e}^{(c_1-c_k)yt}E\left[M_{k,i}\left(h_k\left(u,t\right),y\mathrm{e}^{-\delta t}\right)\right]+\lambda_i\mathrm{e}^{(c_1-c_k)yt}t\sum_{j=1}^{d}p_{ij}$$
$$\times E\left[\int_0^{h_k(u,t)+c_\beta/\beta}M_j\left(h_k\left(u,t\right)-x,y\mathrm{e}^{-\delta t}\right)\mathrm{d}F_j\left(x\right)+\bar{F}_j\left(h_k\left(u,t\right)+c_\beta/\beta\right)\right]+o\left(t\right). \tag{3.6}$$

By Taylor's expansion, we have

$$E\left[M_{k,i}\left(h_k\left(u,t\right),y\mathrm{e}^{-\delta t}\right)\right]=M_{k,i}\left(u,y\right)+\left(ru+c_k\right)t\frac{\partial M_{k,i}\left(u,y\right)}{\partial u}-\delta yt\frac{\partial M_{k,i}\left(u,y\right)}{\partial y}. \tag{3.7}$$

Substituting (3.7) into (3.6), and then dividing both sides of (3.6) by t and letting $t\to 0$, we get (3.2). So the proof is completed.

Theorem 3.2. *For* $i \in S$, $M_{\beta,i}(u,y)$ *and* $M_{k,i}(u,y), k = 1,2,\cdots,N$, *satisfy*

$$M_{\beta,i}\left(-c_\beta/\beta, y\right) = 1, \tag{3.8}$$

$$M_{\beta,i}(0-,y) = M_{1,i}(0+,y), \tag{3.9}$$

$$M_{k,i}(b_k-,y) = M_{k+1,i}(b_k+,y), \quad k = 1,2,\cdots,N-1 \tag{3.10}$$

$$\lim_{u\to\infty} M_{N,i}(u,y) = \mathrm{e}^{(c_1-c_k)y/\delta}, \tag{3.11}$$

$$\begin{aligned}&(rb_k + c_k)\frac{\partial M_{k,i}(u,y)}{\partial u}\Big|_{u=b_k-}\\ &= (rb_k + c_{k+1})\frac{\partial M_{k+1,i}(u,y)}{\partial u}\Big|_{u=b_k+} + (c_k - c_{k-1})M_{k,i}(b_k-,y),\ k = 1,2,\cdots,N-1.\end{aligned} \tag{3.12}$$

Proof.

1) If $u = -c_\beta/\beta$, the absolute ruin is immediate and no dividend is paid, so (3.8) holds.

2) For $-c_\beta/\beta < u < 0$, letting τ_0 be the time that the surplus reach 0 for the first time from $u < 0$ and using the Markov property of the surplus process $\{U_g(t), t \ge 0\}$, we have

$$\begin{aligned}M_{\beta,i}(u,y) &= E_i^u\left[I_{\tau_0<\tau_g}\mathrm{e}^{yD_{u,g}}\right] + E_i^u\left[I_{\tau_0\ge\tau_g}\mathrm{e}^{yD_{u,g}}\right]\\ &= E_i^u\left[I_{\tau_0<\tau_g}\exp\left\{y\int_0^{\tau_g-\tau_0}\mathrm{e}^{-\delta t}\mathrm{d}D(t+\tau_0)\right\}\right] + P\left(\tau_0 \ge \tau_g\right)\\ &= E_i^u\left[I_{\tau_0<\tau_g}\exp\left\{y\mathrm{e}^{-\delta\tau_0}\int_{\tau_0}^{\tau_g}\mathrm{e}^{-\delta t}\mathrm{d}D(t)\right\}\right] + P\left(\tau_0 \ge \tau_g\right)\\ &\le M_{1,i}(0,y) + P\left(\tau_0 \ge \tau_g\right).\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}M_{\beta,i}(u,y) &\ge E_i^u\left[I_{\tau_0<\tau_g,\tau_0=t_0}\mathrm{e}^{yD_{u,g}}\right] + E_i^u\left[I_{\tau_0\ge\tau_g}\mathrm{e}^{yD_{u,g}}\right]\\ &= M_{1,i}(0,y)E_i^u\left[\mathrm{e}^{-\delta\tau_0}I_{\tau_0<\tau_g,\tau_0=t_0}\right] + P\left(\tau_0 \ge \tau_g\right)\\ &= M_{1,i}(0,y)\mathrm{e}^{-\delta\tau_0}P(T_1 > t_0) + P\left(\tau_0 \ge \tau_g\right)\\ &= \mathrm{e}^{-(\lambda_i+\delta)t_0}M_{1,i}(0,y) + P\left(\tau_0 \ge \tau_g\right),\end{aligned} \tag{3.14}$$

where T_1 is the time of the first claim.

When $u \uparrow 0$, we notice that τ_0 and t_0 both go into zero. Letting $u \uparrow 0$ in (3.13) and (3.14) and in view of $\lim_{u\uparrow 0} P\left(\tau_0 \ge \tau_g\right) = 0$, we derive (3.9).

3) For Eq. (3.10), the method is similar to Equation (3.9), so we omit it here.

4) If $u \to \infty$, then $\tau_g = \infty$, so (3.11) holds.

5) For $k = m$, letting $u \uparrow b_m$ in (3.2), and $u \downarrow b_{m+1}$ in (3.2) when $k = m+1$, $m = 1,2,\cdots,N-1$, we can get (3.12).

The proof of Theorem 3.2 is complete.

4. Integro-Differential Equations for $V_{n,i}(u)$

In this section, we get the integro-differential equations for $V_{n,i}(u)$, $i \in S$. First, for $i \in S$, define

$$V_{n,i}(u) = \begin{cases}V_{n,\beta,i}(u), & -c_\beta/\beta < u < 0,\\ V_{n,k,i}(u), & b_{k-1} \le u < b_k, k = 1,2,\cdots,N,\end{cases}$$

with $V_{0,1,i}(u)=1$.

Using the representation

$$M_{\beta,i}(u,y)=1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n,\beta,i}(u), \tag{4.1}$$

$$M_{k,i}(u,y)=1+\sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n,k,i}(u),\ \ k=1,2,\cdots,N, \tag{4.2}$$

we have the following integro-differential equations.

Theorem 4.1. *For* $i\in S$, $-c_\beta/\beta<u<0$,

$$(\beta u+c_\beta)V'_{n,\beta,i}(u)=(n\delta+\lambda_i)V_{n,\beta,i}(u)-\lambda_i\sum_{j=1}^{d}p_{ij}\int_0^{u+c_\beta/\beta}V_{n,\beta,j}(u-x)\mathrm{d}F_j(x), \tag{4.3}$$

and, for $b_{k-1}\le u<b_k$, $k=1,2,\cdots,N$,

$$\begin{aligned}(ru+c_k)V'_{n,k,i}(u)&=(n\delta+\lambda_i)V_{n,k,i}(u)-n(c_1-c_k)V_{n-1,k,i}(u)\\&-\lambda_i\sum_{j=1}^{d}p_{ij}\Big[\int_0^{u-b_{k-1}}V_{n,k,j}(u-x)\mathrm{d}F_j(x)+\int_{u-b_{k-1}}^{u-b_{k-2}}V_{n,k-1,j}(u-x)\mathrm{d}F_j(x)\\&+\cdots+\int_{u-b_1}^{u}V_{n,1,j}(u-x)\mathrm{d}F_j(x)+\int_u^{u+c_\beta/\beta}V_{n,\beta,j}(u-x)\mathrm{d}F_j(x)\Big].\end{aligned} \tag{4.4}$$

Proof. Substituting (4.1) into (3.1), and then equating the coefficients of y^n, we can get (4.3). Similarly, substituting (4.1) and (4.2) into (3.2), and then equating the coefficients of y^n, we obtain (4.4).

Theorem 4.2. *For* $i\in S$, $V_{n,\beta,i}(u)$ *and* $V_{n,k,i}(u), k=1,2,\cdots,N, n=1,2,\cdots,$ *satisfy*

$$V_{n,\beta,i}(-c_\beta/\beta)=0, \tag{4.5}$$

$$V_{n,\beta,i}(0-)=V_{n,1,i}(0+), \tag{4.6}$$

$$V_{n,k,i}(b_k-)=V_{n,k+1,i}(b_k+),\ \ k=1,2,\cdots,N-1 \tag{4.7}$$

$$\lim_{u\to\infty}V_{n,N,i}(u)=\left(\frac{c_1-c_k}{\delta}\right)^n, \tag{4.8}$$

$$\begin{aligned}&(rb_k+c_k)V'_{n,k,i}\big|_{u=b_k-}\\&=(rb_k+c_{k+1})V'_{n,k+1,i}\big|_{u=b_k+}+(c_k-c_{k+1})V_{n-1,k,i}(b_k-),\ \ k=1,2,\cdots,N-1.\end{aligned} \tag{4.9}$$

Proof. This method is similar to Theorem 3.2.

5. The Gerber-Shiu Function

In this section, systems of integro-differential equations for the Gerber-Shiu function are presented. For $i\in S$, define

$$\Phi_i(u)=\begin{cases}\Phi_{i,\beta}(u), & -c_\beta/\beta<u<0,\\ \Phi_{i,k}(u), & b_{k-1}\le u<b_k, k=1,2,\cdots,N,\end{cases}$$

Theorem 5.1. *For* $i\in S$, $-c_\beta/\beta<u<0$,

$$(\beta u+c_\beta)\Phi'_{\beta,i}(u)=(\delta+\lambda_i)\Phi_{\beta,i}(u)-\lambda_i\sum_{j=1}^{d}p_{ij}\Big[\int_0^{u+c_\beta/\beta}\Phi_{\beta,j}(u-x)\mathrm{d}F_j(x)+A_j(u)\Big], \tag{5.1}$$

and, for $b_{k-1}\le u<b_k$, $k=1,2,\cdots,N$,

$$\left(ru+c_k\right)\Phi'_{k,i}\left(u\right)=\left(\delta+\lambda_i\right)\Phi_{k,i}\left(u\right)-\lambda_i\sum_{j=1}^{d}p_{ij}\left[\int_0^{u+c_\beta/\beta}\Phi_j\left(u-x\right)\mathrm{d}F_j\left(x\right)+A_j\left(u\right)\right], \tag{5.2}$$

with boundary conditions

$$\Phi_{\beta,i}\left(0-\right)=\Phi_{1,i}\left(0+\right), \tag{5.3}$$

$$\Phi_{k,i}\left(b_k-\right)=\Phi_{k+1,i}\left(b_k+\right),\quad k=1,2,\cdots,N-1 \tag{5.4}$$

$$\left(rb_k+c_k\right)\Phi'_{k,i}\Big|_{u=b_k-}=\left(rb_k+c_{k+1}\right)\Phi'_{k+1,i}\Big|_{u=b_k+},\quad k=1,2,\cdots,N-1 \tag{5.5}$$

where $A_j\left(u\right)=\int_{u+c_\beta/\beta}^{\infty}w\left(u,x-u\right)\mathrm{d}F_j\left(x\right)$.

Proof. Fix $i\in S$, and $-c_\beta/\beta<u<0$. Similar to argument as in Section 3, conditioning on the events that can occur in the small time interval $\left(0,t\right]$, we obtain,

$$\begin{aligned}\Phi_{\beta,i}\left(u,y\right)=\mathrm{e}^{-\delta t}\Bigg\{&\left(1-\lambda_i t\right)\Phi_{\beta,i}\left(h_\beta\left(u,t\right)\right)+\lambda_i t\sum_{j=1}^{d}p_{ij}\times\left[\int_0^{h_\beta(u,t)+c_\beta/\beta}\Phi_{\beta,j}\left(h_\beta\left(u,t\right)-x\right)\mathrm{d}F_j\left(x\right)\right.\\&\left.+\int_{h_\beta(u,t)+c_\beta/\beta}^{\infty}w\left(h_\beta\left(u,t\right),x-h_\beta\left(u,t\right)\right)\mathrm{d}F_j\left(x\right)\right]\Bigg\}+o\left(t\right),\end{aligned} \tag{5.6}$$

By Taylor's expansion, we have

$$\Phi_{\beta,i}\left(h_\beta\left(u,t\right)\right)=\Phi_{\beta,i}\left(u\right)+\left(\beta u+c_\beta\right)t\Phi'_{\beta,i}\left(u\right)+0\left(t\right). \tag{5.7}$$

Substituting (5.7) into (5.6), and then dividing both sides of (5.6) by *t* and letting $t\to0$, we get (5.1).

Similarly,when $b_{k-1}\le u<b_k, k=1,2,\cdots,N$, we still consider a small time interval $\left(0,t\right]$. We obtain

$$\begin{aligned}\Phi_{k,i}\left(u\right)=\mathrm{e}^{-\delta t}\{&\left(1-\lambda_i t\right)\Phi_{k,i}\left(h_k\left(u,t\right)\right)+\lambda_i t\sum_{j=1}^{d}p_{ij}\times\left[\int_0^{h_k(u,t)+c_\beta/\beta}\Phi_j\left(h_k\left(u,t\right)-x\right)\mathrm{d}F_j\left(x\right)\right.\\&\left.+\int_{h_k(u,t)+c_\beta/\beta}^{\infty}w\left(h_k\left(u,t\right),x-h_k\left(u,t\right)\right)\mathrm{d}F_j\left(x\right)\right]\Bigg\}+o\left(t\right).\end{aligned} \tag{5.8}$$

By Taylor's expansion, we have

$$\Phi_{k,i}\left(h_k\left(u,t\right)\right)=\Phi_{k,i}\left(u\right)+\left(ru+c_k\right)t\Phi'_{k,i}\left(u\right)+0\left(t\right). \tag{5.9}$$

Substituting (5.9) into (5.8), and then dividing both sides of (5.8) by *t* and letting $t\to0$, we get (5.2). For the boundary conditions (5.3)-(5.5), the method is similar to Theorem 3.2. So the proof is completed.

Integro-differential Equations (5.1) and (5.2) can be rewritten in matrix form.

Let

$$\Phi_{g,\beta}\left(u\right)=\left(\Phi_{\beta,1}\left(u\right),\Phi_{\beta,2}\left(u\right),\cdots,\Phi_{\beta,d}\left(u\right)\right)^{\mathrm{T}}$$

and

$$\Phi_{g,k}\left(u\right)=\left(\Phi_{k,1}\left(u\right),\Phi_{k,2}\left(u\right),\cdots,\Phi_{k,d}\left(u\right)\right)^{\mathrm{T}},k=1,2,\cdots,N$$

where *T* denoting transpose. We have the following theorem.

Theorem 5.2. $\Phi_{g,\beta}$ *and* $\Phi_{g,k}, k=1,2,\cdots,N$ *satisfy the following integro-differential equations*

$$\Phi'_{g,\beta}\left(u\right)=P_1\left(u\right)\Phi_{g,\beta}\left(u\right)+\int_0^{u+c_\beta/\beta}G_1\left(x\right)\Phi_{g,\beta}\left(u-x\right)\mathrm{d}x+A_1\left(u\right),\quad -c_\beta/\beta<u<0 \tag{5.10}$$

$$\begin{aligned}\Phi'_{g,k}\left(u\right)=&P_{2,k}\left(u\right)\Phi_{g,k}\left(u\right)+\int_0^{u-b_{k-1}}G_{2,k}\left(x\right)\Phi_{g,k}\left(u\right)\mathrm{d}x+\int_{u-b_{k-1}}^{u-b_{k-2}}G_{2,k}\left(x\right)\Phi_{g,k-1}\left(u-x\right)\mathrm{d}x+\cdots\\&+\int_{u-b_1}^{0}G_{2,k}\left(x\right)\Phi_{g,1}\left(u-x\right)\mathrm{d}x+\int_u^{u+c_\beta/\beta}G_{2,k}\left(x\right)\Phi_{g,\beta}\left(u-x\right)\mathrm{d}x+A_{2,k}\left(u\right),\quad b_{k-1}\le u<b_k\end{aligned} \tag{5.11}$$

with boundary conditions

$$\Phi_{g,\beta}(0-) = \Phi_{g,1}(0+), \tag{5.12}$$

$$\Phi_{g,k}(b_k -) = \Phi_{g,k+1}(b_k +), k = 1, 2, \cdots, N-1 \tag{5.13}$$

$$(rb_k + c_k)\Phi'_{g,k}\big|_{u=b_k-} = (rb_k + c_{k+1})\Phi'_{g,k+1}\big|_{u=b_k+}, k = 1, 2, \cdots, N-1 \tag{5.14}$$

where $P_1(u) = diag\left(\frac{\lambda_1+\delta}{\beta u + c_\beta}, \frac{\lambda_2+\delta}{\beta u + c_\beta}, \cdots, \frac{\lambda_d+\delta}{\beta u + c_\beta}\right)$, $P_{2,k}(u) = diag\left(\frac{\lambda_1+\delta}{ru + c_k}, \frac{\lambda_2+\delta}{ru + c_k}, \cdots, \frac{\lambda_d+\delta}{ru + c_k}\right)$,

$$G_1(u) = -\begin{pmatrix} \frac{\lambda_1}{\beta u + c_\beta} & & \\ & \ddots & \\ & & \frac{\lambda_d}{\beta u + c_\beta} \end{pmatrix} \Lambda \begin{pmatrix} f_1(x) & & \\ & \ddots & \\ & & f_d(x) \end{pmatrix},$$

$$G_{2,k}(u) = -\begin{pmatrix} \frac{\lambda_1}{ru + c_k} & & \\ & \ddots & \\ & & \frac{\lambda_d}{ru + c_k} \end{pmatrix} \Lambda \begin{pmatrix} f_1(x) & & \\ & \ddots & \\ & & f_d(x) \end{pmatrix}$$

are all $d \times d$ matrices, $A_1(u)$ and $A_{2,k}(u)$ defined by

$$A_1(u) = \int_{u+c_\beta/\beta}^{\infty} w(u, x-u) G_1(x) I\mathrm{d}x, \quad A_{2,k}(u) = \int_{u+c_\beta/\beta}^{\infty} w(u, x-u) G_{2,k}(x) I\mathrm{d}x,$$

are all *d*-dimensional vector, in which $I = (1, 1, \cdots, 1)^{\mathrm{T}}$ is an $d \times 1$ column vector.

6. Conclusions

In this paper, we investigate the Markov-dependent risk model with multi-layer dividend strategy and investment interest under absolute ruin. This complex model is more realistic. We derive systems of integro-differential equations satisfied by the moment-generating function, the *n*th moment of the discounted dividend payments prior to absolute ruin and the Gerber-Shiu function. Generally, many authors only extensively consider Gerber-Shiu function in risk models with multi-layer dividend strategy. However, due to the importance of the dividend problem, the problems considered by this paper are more important and interesting.

In addition that, we only obtain systems of integro-differential equations. As far as we know, it is not easy to derive the explicit expressions for the moment-generating function, the *n*th moment of the discounted dividend payments prior to absolute ruin and the Gerber-Shiu function. But, maybe we find some numerical method which can solve these equations. We leave it for the further research topic.

Acknowledgements

We would like to thank the referees for their constructive comments and suggestions which have improved the paper. This work was supported by the the Natural Sciences Foundation of China (grants 11301133 and 11471218).

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