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On the Stochastic Dominance of Portfolio Insurance Strategies

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Abstract

This paper compares the performance of the two main portfolio insurance strategies, namely the Option-Based Portfolio Insurance (OBPI) and the Constant Proportion Portfolio Insurance (CPPI). For this purpose, we use the stochastic dominance approach. We provide several explicit sufficient conditions to get stochastic dominance results. When taking account of specific constraints, we use the consistent statistical test proposed by Barret and Donald [1]. It is similar to the Kolmogrov-Smirnov test with a complete set of restrictions related to the various forms of stochastic dominance. We find that the CPPI method can perform better than the OBPI one at the third order stochastic dominance.

Keywords

Stochastic Dominance, Portfolio Insurance, CPPI, OBPI, Barret and Donald Test

1. Introduction

The goal of portfolio insurance is to provide a guarantee against portfolio downside risk (usually 100% of the initial invested amount) while allowing to benefit from significant gains for bullish markets. The two standard portfolio insurance methods are the *Option Based Portfolio Insurance* (OBPI), introduced by Leland and Rubinstein [2] and the *Constant Proportion Portfolio Insurance* (CPPI) considered by Perold [3]. Basically, the OBPI portfolio is a combination of a risky asset S (usually a financial index such as the S&P) and a put written on it. Whatever the value of S at the terminal date T, the portfolio value will be always higher than the strike of the put. Therefore this strike is chosen in order to provide the desired guaranteed level. The standard CPPI method consists in a simplified strategy to allocate assets dynamically over time. A floor is initially determined

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such as it is equal to the lowest acceptable value of the portfolio. Then, the amount allocated to the risky asset (called the exposure) is defined as follows: the cushion, which is equal to the excess of the portfolio value over the floor is multiplied by a predetermined multiple. The remaining funds are usually invested in the reserve asset, usually T-bills. As the cushion approaches zero, exposure approaches zero too. In continuous time, this keeps portfolio value from falling below the floor.

To compare these two main portfolio strategies, we search for stochastic dominance (SD) properties since SD takes account of the entire return distribution. The major argument for stochastic dominance is that it does not require any specific knowledge about the preferences of investors. Indeed, the first stochastic dominance order is related to investors with an increasing utility function. Stochastic dominance of order two focuses on investors having an increasing and concave utility, meaning that they are risk-averse[1]. However, at a given order (for example 1 or 2), the stochastic dominance criterion cannot always allow to rank all portfolios. There exist cases where no stochastic dominance is observable. But there exists a stochastic dominance criteria at each order and, the higher the order, the less stringent the criterion. Thus it is reasonable to expect that there exists an order for which a portfolio strategy dominates another one (or *vice versa*). De Giorgi [6] shows that, in a market without friction, the market portfolio can be efficient according to the criterion of the second order stochastic dominance. Therefore the test of stochastic dominance is consistent with the theory of portfolio choice. To compare with alternative approaches such as those based on performances measures, note that Darsinos and Satchell [7] show that n-order stochastic dominance implies Kappa ($n-1$) dominance. It means for example that the second order stochastic dominance implies the Omega dominance while the third order SD implies dominance according to the Sortino measure.

For the portfolio insurance strategies, Bertrand and Prigent [8] proved that the stochastic dominance at the first order is a too strong condition, meaning that neither the CPPI nor the OBPI dominates the other strategy for this criterion[2]. However, as proved theoretically by Zagst and Kraus [10], stochastic dominance of portfolio insurance strategies can be obtained mainly from the third order. Our main purpose is to extend previous results when taking account of quite general share values and/or of specific constraints such as capped strategies introduced to limit financial risk exposures.

The paper is organized as follows. In Section 2, we briefly introduce the basic properties of the CPPI and the OBPI strategies. In Section 3, we examine the stochastic dominance (SD) framework to compare portfolio insurance strategies. First, we provide several sufficient conditions to get stochastic dominance properties for the standard portfolio insurance methods. Second, to extend previous results, we introduce specific statistical tests and simulation methods for computing p-values when examining SD^j with j larger than one. We use the test considered by Barret and Donald [1], based on the multiplier central limit theory discussed in Van der Vaart and Wellner [11].

2. Basic Properties of the CPPI and the OBPI Strategy

2.1. The Financial Market

We consider two basic assets that are traded in continuous time during the investment period $[0,T]$. The "risk-free" asset (a money market account for example) is denoted by B. Denote by r the constant continuous interest rate $r>0$. We get:

$$B_t = B_0 \cdot \mathrm{e}^{rT}, \tag{1}$$

with initial value $B_0>0$. The risky asset (for example a financial market index) is denoted by S. It is assumed to be a geometric Brownian motion given by:

$$\mathrm{d}S_t = S_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\right), \tag{2}$$

with non negative initial value $S_0>0$ and where $W=\left(W_t\right)_{(0\le t\le T)}$ is a standard Brownian motion. There exists a constant drift term parameterized by $\mu>r$ and a volatility denoted by σ.

To price options, we use the Black and Scholes formula while taking account of the spread between the em-

[1]See Levy ([4] [5]) for details about stochastic dominance and expected utility, with applications to investment strategies.
[2]For details about various comparisons of CPPI and OBPI strategies, see Prigent [9].

pirical and the implied volatility[3].

2.2. Constant Proportion Portfolio Insurance (CPPI)

The standard CPPI method consists in a simplified strategy to allocate assets dynamically over time so that its value V_t^{CPPI} never falls below the floor F_t. This latter one is equal to the lowest acceptable value of the portfolio and is defined as a percentage p ($1 \geq p \geq 0$) of the initial investment V_0^{CPPI}, *i.e.*:

$$F_t = p e^{rt} V_0^{CPPI}. \tag{3}$$

The excess of the portfolio value over the floor is called the cushion C_t. It is equal to:

$$C_t = \max\left\{V_t^{CPPI} - F_t, 0\right\}. \tag{4}$$

Then, the amount allocated on the risky asset (called the exposure E_t) is equal to the cushion multiplied by a constant multiple m. Therefore, the exposure $(E_t)_{(0\leq t\leq T)}$ satisfies:

$$E_t = m \cdot C_t = m \cdot \max\left\{V_t^{CPPI} - F_t, 0\right\}. \tag{5}$$

The interesting case is when $m > 1$, that is when the payoff function of the portfolio value at maturity V_T^{CPPI} is a convex function with respect to the risky asset S_T.

Then the cushion value C_t must satisfy:

$$\mathrm{d}C_t = \mathrm{d}(V_t - F_t) = (V_t - E_t)\frac{\mathrm{d}B_t}{B_t} + E_t \frac{\mathrm{d}S_t}{S_t} - \mathrm{d}F_t.$$

By applying Itô's lemma, we obtain:

$$C_t = C_0 \cdot \exp\left[\left(r + m(\mu - r)\right)t - \frac{1}{2}m^2\sigma^2 t\right] \cdot \exp\left[m\sigma W_t\right]. \tag{6}$$

By using the relation:

$$S_t^m = S_0^m \cdot \exp\left[m\sigma W_t + m\left(\mu - \frac{1}{2}\sigma^2\right)t\right],$$

we deduce:

$$\exp\left[m\sigma W_t\right] = \frac{S_t^m}{S_0^m}\exp\left[-m\left(\mu t - \frac{1}{2}m^2\sigma^2 t\right)\right].$$

Substituting this expression for $\exp\left[m\sigma W_t\right]$ into the expression for C_t leads to:

$$C_t(m, S_t) = C_0 \cdot \left(\frac{S_t}{S_0}\right)^m \exp\left[rt(1-m) + \frac{1}{2}m\sigma^2 t - \frac{1}{2}m^2\sigma^2 t\right] = \alpha_t S_t^m, \tag{7}$$

with

$$\alpha_t = C_0 \cdot \left(\frac{1}{S_0}\right)^m \exp\left[rt(1-m) + \frac{1}{2}m\sigma^2 t - \frac{1}{2}m^2\sigma^2 t\right].$$

We deduce that the value of the CPPI portfolio V_t^{CPPI} at any time t is given by:

$$V_t^{CPPI}(m, S_t) = F_t e^{rt} + \alpha_t S_t^m. \tag{8}$$

Note that, for the CPPI method, the two key management parameters are the initial floor value $F_0 = pV_0 e^{-rT}$

[3]We consider that the two strategies operate in different market environments. While the CPPI strategy operates on the financial market with its empirical market volatility (the so-called "local volatility"), the OBPI uses options that have to be priced using the implied volatility. In the financial market, we observe a spread between the empirical volatility and the implied volatility corresponding to the usual "smile" effect.

and the multiple m.

Remark 2.1. (Capped CPPI) The manager may want to increase his profits, from usual performances of asset S while potentially discarding very high values of S. In that case, the exposure is determined by:

$$E_t = \inf\left\{mC_t, \lambda V_t^{CPPI}\right\}, \tag{9}$$

where λ denotes the gearing coefficient. Its usual value is equal to 0.9.

2.3. Option-Based Portfolio Insurance (OBPI)

In what follows, we describe the option-based portfolio insurance strategy. It provides a guarantee equal to $p \cdot V_0^{OBPI}$ whatever the market fluctuations. Indeed, for a given share q, we have:

$$V_T^{OBPI} = q\left[K + \left(S_T - K\right)^+\right], \tag{10}$$

which implies that $V_T^{OBPI} \geq p \cdot V_0^{OBPI}$ if $qK = p \cdot V_0^{OBPI}$.

This relation shows that the insured amount at maturity is the exercise price multiplied by the number of shares, *i.e.* qK. The value V_T^{OBPI} of this portfolio at any time t in the period $\left[0, T\right]$ is equal to:

$$V_t^{OBPI} = q\left[K\mathrm{e}^{(-r(T-t))} + C\left(t, K\right)\right],$$

where $C\left(t, K\right)$ denotes the Black-Scholes value of the European call option with strike K, calculated under the risk neutral probability Q.

The portfolio value V_t^{OBPI}, for all dates t before T, is always above the deterministic level $qK\mathrm{e}^{-r(T-t)}$. In order to guarantee the minimum terminal portfolio value $p \cdot V_0^{OBPI}$, the strike K of the European Call option must satisfy the following relation:

$$p \cdot V_0 = qK,$$

which implies that:

$$\frac{C\left(0, K\right)}{K} = \frac{1 - p\mathrm{e}^{-rT}}{p}. \tag{11}$$

Therefore, the strike K is an increasing function $K\left(p\right)$ of the percentage p, since in Equation (Equation (11)) both functions are decreasing respectively with respect to K and p. Then, the number of shares q is given by:

$$q = \frac{V_0}{K\mathrm{e}^{(-rT)} + C\left(0, K\right)}. \tag{12}$$

Thus, for any investment value V_0, the number of shares q is a decreasing function of the percentage p.

In what follows, we price the option using the implicit volatility σ_i.

We denote its price by $Call\left(S_0, K, r, \sigma^i\right)$.

Remark 2.2. (Capped OBPI) If the manager wants to increase his profit while potentially discarding very high value of S, the options are capped at a level L, as follows. Consider a parameter L higher than the strike K.

The terminal value of the capped OBPI with strike K and parameter L is defined by:

$$\begin{aligned} V_{cap,T}^{OBPI} &= qMin\left[K + \left(S_T - K\right)^+, L\right], \\ &= q\left[K + \left(S_T - K\right)^+ - \left(S_T - L\right)^+\right]. \end{aligned} \tag{13}$$

3. Stochastic Dominance of Portfolio Insurance Strategies

3.1. Stochastic Dominance: Theoretical Results

In what follows, we provide several sufficient conditions to get stochastic dominance results as in Zagst and Kraus [10] but without assuming as them that q is equal to 1 (see previous Relation 12).

3.1.1. The Second Order Stochastic Dominance

Mosler [12] has stated a theorem for determining the second order stochastic dominance (denoted by $\succ_2$) between random variables based on the condition of intersection between the cumulative distribution functions.

Theorem 3.1. (Mosler [12]). Let V^* and V be two random variables with finite expectations. Denote for all $x \in [a,b]$, $H(x) = F_{V^*}(x) - F_V(x)$ the difference of their respective cumulative distributions functions. Then, we get:

$$H \in \mathrm{S}_1,\ E[V] \le E[V^*] \Rightarrow V^* \succ_2 V,$$

where S_k denotes the set of all real functions H, with k changes of sign:

$$\mathrm{S}_k = \Big\{ H : \mathbb{R} \to \mathbb{R} : \exists s_1, \cdots, s_k \in \mathbb{R},\ s_0 := -\infty, s_{k+1} := +\infty$$

$$\text{with } (-1)^j \cdot H(s) \ge 0, \forall s \in (s_j, s_{j+1}),\ j = 0, \cdots, k,\ H \ne 0 \Big\}.$$

We deduce that, if $H(x) \in \mathrm{S}_k$, then the two functions $F_{V^*}(x)$ and $F_V(x)$ intersect k times.

For example, we have:

$$\mathrm{S}_1 = \left\{ H : \mathbb{R} \to \mathbb{R} : \exists\ s_1 \in \mathbb{R} \text{ with } H(s) \begin{cases} \ge 0, s \in (-\infty, s_1) \\ \le 0, s \in (s_1, +\infty) \end{cases},\ H \ne 0 \right\}.$$

And

$$\mathrm{S}_2 = \left\{ H : \mathbb{R} \to \mathbb{R} : \exists\ s_1, s_2 \in \mathbb{R} \text{ with } H(s) \begin{cases} \ge 0, s \in (-\infty, s_1) \\ \le 0, s \in (s_1, s_2) \\ \ge 0, s \in (s_2, +\infty) \end{cases},\ H \ne 0 \right\}.$$

The second order stochastic dominance depends on the values taken by the multiple m, the historical volatility σ and the implied volatility σ_i used to price the Call for the OBPI strategy. The determination of the second order stochastic dominance requires understanding the behavior of the function $H(x) = F_{V_T^{OBPI}}(x) - F_{V_T^{CBPI}}(x)$ based on the values taken by the multiple m. If $m = 1$, then the function H is strictly decreasing and presents a single point of intersection with the horizontal axis, thus $H \in \mathrm{S}_1$. Therefore, for $m = 1$, $\sigma \le \sigma_i$, we can conclude, using theorem of Mosler [12], that, if $H(x) \in \mathrm{S}_1$ and $E\left[V_T^{OBPI}\right] \le E\left[V_T^{CPPI}\right]$, then the CPPI strategy stochastically dominates at the second order the OBPI strategy.

Theorem 3.2. Let $m = 1$ and $Call(S_0, K, r, \sigma_i) \ge Call(S_0, K, \mu, \sigma)$. Additionally, assume that $V_0 > Call(S_0, K, r, \sigma_i)$. Then, we deduce:

$$V_T^{CPPI} \succ_2 V_T^{OBPI}.$$

Proof. See Appendix A1.

Remark 3.1. Condition $V_0 > Call(S_0, K, r, \sigma_i)$ insures that $q > \alpha_T$ which allows proving the previous theorem. When $q = 1$ (as in Zagst and Kraus [10]), this condition is necessary satisfied.

3.1.2. The Third Order Stochastic Dominance

As mentioned by Zagst and Kraus [10], the third order stochastic dominance (denoted by $\succ_3$) can be deduced under some specific assumptions.

Theorem 3.3. (Karlin-Novikov; Mosler [12])

Let V, V^* be non-negative random variables with finite second moments.

Denote $H(x) = F_V(x) - F_{V^*}(x)$ for all $x \in \mathbb{R}$. Then:

$$H \in \mathrm{S}_2, E[V] \le E[V^*], E\left[V^{*2}\right] \le E\left[V^2\right] \Rightarrow V^* \succ_3 V.$$

The validation of the third order stochastic dominance requires the analysis of the condition $E\left[V^{*2}\right] \le E\left[V^2\right]$ of previous Karlin and Novikov theorem. We get:

Theorem 3.4. Assuming that $\left(V_0 e^{rT}\right)^2 \le E\left[V_T^{OBPI\,2}\right]$, there exists a value $m_{\max}$ of the multiple such that:

$$E\left[\left(V_T^{CPPI}\right)^2\right] \leq E\left[\left(V_T^{OBPI}\right)^2\right] \Leftrightarrow m \leq m_{\max}.$$

Proof. See Appendix A.2.

Using previous theorems, we deduce:

Theorem 3.5. *Let* $m_{\min}^1$ *defined by*:

$$m_{\min}^1 = 1 + \frac{1}{(\mu - r)T} \ln\left(\frac{Call(S_0, \mu, \sigma)}{Call(S_0, r, \sigma_i)}\right)$$

and $m_{\min} = \max\left\{1, m_{\min}^1\right\}$.

Then, we get:

$$m \in \left[m_{\min}, m_{\max}\right] \Rightarrow V_T^{CPPI} \succ_3 V_T^{OBPI}.$$

Proof. Condition $m \geq m_{\min}^1$ implies that $E\left[V_T^{CPPI}\right] \geq E\left[V_T^{OBPI}\right]$ (see Appendix A.1) while condition $m \leq m_{\max}$ implies that $E\left[V_T^{CPPI\,2}\right] \leq E\left[V_T^{OBPI\,2}\right]$. Therefore, using Karlin and Novikov theorem, we deduce the result.

To illustrate these theoretical results, we consider the following numerical example: $\mu = 7\%$, $\sigma = 15\%$, $\sigma_i = 18\%$, $r = 3\%$, $T = 5$ years, $V_0 = S_0 = 100$, and $p = 100\%$. Applying Relation (11), we deduce that $K \simeq 116$ and $q \simeq 0.86$. **Table 1** illustrates the results of the third degree stochastic dominance for different values of the multiple $m = 1, \cdots, 5$.

Results of **Table 1** show third order stochastic dominance of the CPPI strategy for $m = 3 \in [2.99, 3.12]$.

Recall that, if the multiplier $m \geq m_{\min} = 2.99$, we have $E\left[V_T^{CPPI}\right] \geq E\left[V_T^{OBPI}\right]$. However, for $m \geq m_{\max} = 3.12$, we have $E\left[V_T^{CPPI\,2}\right] \geq E\left[V_T^{OBPI\,2}\right]$ and the sufficient condition of Karlin and Novikov theorem is no longer satisfied. The range of the multiple values, for which a stochastic dominance at the third order is verified, depends notably on the values of the implied volatility, the empirical volatility and the drift. **Figures 1-3** illustrate this dependence.

Table 1. The third order stochastic dominance for multipliers equal to 1, ···, 5.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$m_{\min}$			2.99		
$m_{\min} \prec m$	-	-	*	*	*
Condition S_2	-	*	*	*	*
$m_{\max}$			3.12		
$m \prec m_{\max}$	-	*	*	-	-
Third order SD if $m \in \left[m_{\min}, m_{\max}\right]$	-	-	*	-	-

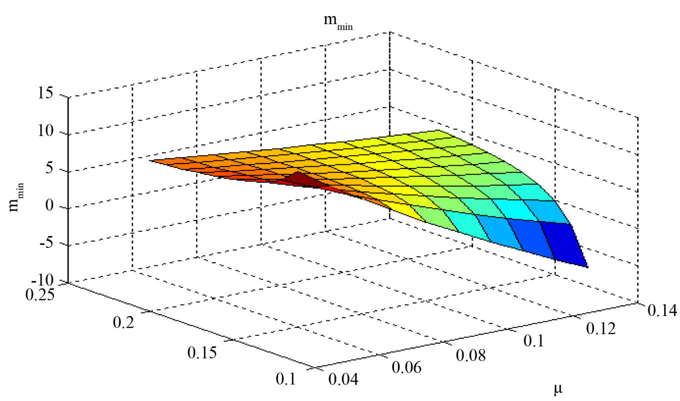

Figure 1. The value of $m_{\min}$ depending on the drift and the implied volatility.

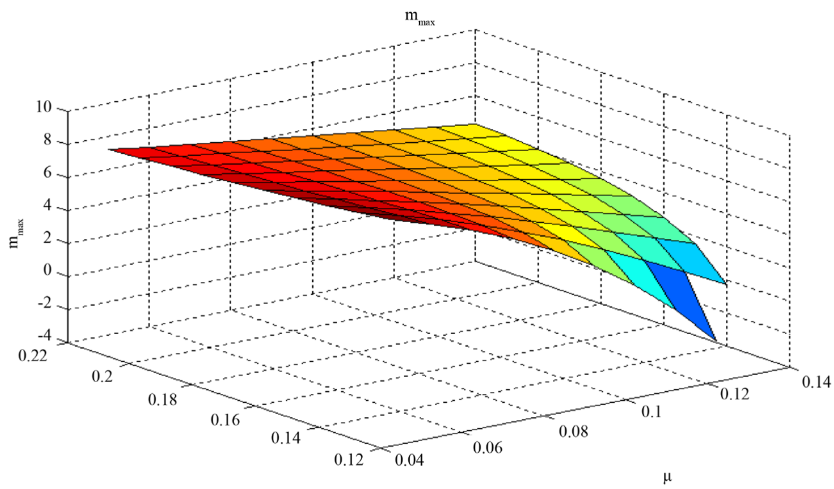

Figure 2. The value of $m_{\max}$ depending on the drift and the implied volatility.

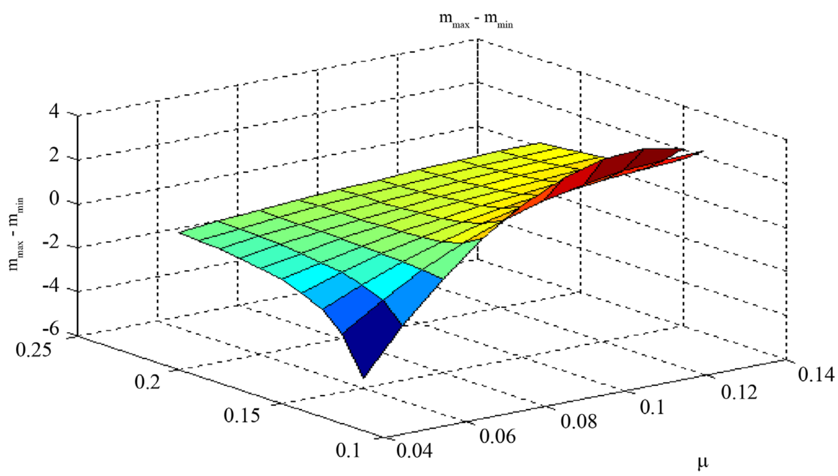

Figure 3. Difference of the upper and lower bounds on the multiplier.

The value of the lower bound $m_{\min}$ is a decreasing function with respect to the value of the drift μ. Indeed, when the drift increases, the expectation of the CPPI portfolio value increases more than that of the OBPI portfolio since the CPPI strategy is more allocated on the risky asset. The value of the lower bound $m_{\min}$ is not always an increasing function of the implied volatility.

The value of the upper bound $m_{\max}$ is a decreasing function with respect to the value of the drift μ. Indeed, when the drift increases, the expectation of the square of the CPPI portfolio value increases more than that of the OBPI portfolio since the CPPI strategy is more allocated on the risky asset. Therefore, the condition $E\left[V_T^{CPPI\,2}\right] \leq E\left[V_T^{OBPI\,2}\right]$ is more stringent when the multiple m increases. The value of the lower bound $m_{\min}$ is almost always an increasing function of the implied volatility.

Previous stochastic dominance results have been established for the standard cases, *i.e*. the strategies are not capped. To deal with capped strategies as defined in Remarks (Capped CPPI) and (Capped OBPI), we have to conduct a numerical analysis. In a first step, we simulate the portfolios values using standard Monte Carlo methods; in a second step, we test the stochastic dominance properties.

4. Stochastic Dominance of Portfolio Insurance Strategies

4.1. Stochastic Dominance: Numerical and Empirical Tests

In the empirical framework, the stochastic dominance has been pioneered for example by Kroll and Levy [13].

To avoid sampling errors due to i.i.d. assumptions, general stochastic dominance tests have been developed (e.g. Davidson and Duclos [14]; Barrett and Donald [1]; Post [15]; Linton *et al*. [16]; Scaillet and Topaloglou [17]). The tests introduced by Barrett and Donald [1] and Linton *et al*. [16] are based on a comparison of the cumulative density functions of studied perspectives. They are based on the Kolmogorov-Smirnov type tests. Barrett and Donald [1] examine the application of tests for any predetermined orders of stochastic dominance, SD_j, using several simulation and bootstrap methods to estimate an asymptotic *p*-value.

4.1.1. Stochastic Dominance and Hypothesis Formulation

Due to the characterizations of stochastic dominance, it is convenient to represent the various orders of stochastic dominance using the integral operators, $\mathscr{T}_j(.;G)$, corresponding to successive integrations of the cumulative distribution function G to order $j-1$, namely:

$$\mathscr{T}_1(z;G)=G(z),$$

$$\mathscr{T}_2(z;G)=\int_0^z G(t)\,\mathrm{d}t=\int_0^z \mathscr{T}_1(t;G)\,\mathrm{d}t,$$

$$\mathscr{T}_3(z;G)=\int_0^z\int_0^s G(t)\,\mathrm{d}t=\int_0^z \mathscr{T}_2(s;G)\,\mathrm{d}s,$$

and so on.

The general hypotheses for testing stochastic dominance of G with respect to F at order j can be written compactly as:

$$H_0^j:\mathscr{T}_j(z;G)\le\mathscr{T}_j(z;F)\ \text{ for all } z\in(\underline{z},\overline{z}),$$

$$H_1^j:\mathscr{T}_j(z;G)\succ\mathscr{T}_j(z;F)\ \text{ for at least one } z\in(\underline{z},\overline{z}).$$

4.1.2. Test Statistics and Asymptotic Properties

In this paper, we test for stochastic dominance using the empirical distribution functions estimated from simulation of the two insurance portfolio strategies. The test of Linton *et al*. [16] allows for dependence in the data, and can be conducted with a limited number of assumptions. Suppose two prospects X and Y. Let N be the number of the realizations for the two prospects $\{X_i;i=1,\cdots,N\}$ and $\{Y_i;i=1,\cdots,N\}$. The null hypothesis is that a particular prospect X dominates the other one.

The empirical distributions used to construct the tests are respectively given by:

$$\widehat{F}_N(z)=\frac{1}{N}\sum_{i=1}^N 1(X_i\le z),\widehat{G}_N(z)=\frac{1}{N}\sum_{i=1}^N 1(Y_i\le z),$$

where j denotes the order of dominance and $1(.)$ denotes the indicator function.

The statistical test T_{Nj} for the full sample is defined by:

$$T_{Nj}=\sqrt{N}\sup_z\left(\mathscr{T}_j\left(z;\widehat{G}_N\right)-\mathscr{T}_j\left(z;\widehat{F}_N\right)\right).$$

The linear operator F_j is written as:

$$\mathscr{T}_j\left(z;\widehat{F}_N\right)=\frac{1}{N}\sum_{i=1}^N\mathscr{T}_j\left(z;1_{X_i}\right)=\frac{1}{N}\sum_{i=1}^N\frac{1}{(j-1)!}1(X_i\le z)(z-X_i)^{j-1}.$$

The second term of the linear operator is derived from Davidson and Duclos [14].

We have also to define a method in order to obtain the critical value of the test. The standard bootstrap does not work because we need to impose the null hypothesis in that case, which is difficult because it is defined by a complicated system of inequalities. According to Linton *et al*. [16], we apply the sub sampling method which is very simple to define and yet provide consistent critical values. Following the circular block method of Kläver [18], we have to compute again the test statistic for the sub sample of size b for each of the $N-b+1$ different subsamples $\{W_i,\cdots,W_{(i+b-1)}\}$, where $W_i=\{X_{ki};k=1,2\}$ and $i=1,\cdots,N-b+1$, and for the subsamples

$\{W_i,\cdots,W_N,W_1,\cdots,W_{(i+b-N-1)}\}$ where $i=N-b+2,\cdots,N$. Let $t_{(N,b,i)}$ be equal to the statistic t_b evaluated at the subsample $\{W_i,\cdots,W_{(i+b-1)}\}$ of size b. We have:

$$t_{(N,b,i)}=t_b\left(W_i,\cdots,W_{(i+b-1)}\right) \text{ for } i=1,\cdots,N-b+1,$$

with

$$t_{(N,b,j)}=\sqrt{b}\sup_z\left(\mathscr{T}_j\left(z.;\widehat{G}_b\right)-\mathscr{T}_j\left(z;\widehat{F}_b\right)\right).$$

The underlying rationale is that one can approximate the sampling distribution of T_N using the distribution of the values of $t_{(N,b,j)}$ computed over $N-b+1$ different subsamples of size b, when $b/N\to 0$ and $b\to\infty$ as $N\to\infty$.

We consider that each of these sub samples is also a sample of the true sampling distribution of the original data.

Following Kläver [6], we consider a sub sample size $b(N)=10\sqrt{N}$.

Let $\widehat{p}_j$ denote the empirical p-value:

$$\widehat{p}_j=\frac{1}{N}\sum_{i=1}^{N}1\left(t_{(N,b,j)}>T_{Nj}\right).$$

For $j=1,2,3$, we reject the null hypothesis at α significance according to the following rule:

- If $\widehat{p}_N^{(j)}(k)\le\alpha$, we reject the null hypothesis of j-order stochastic dominance of variable X with respect to the variable Y.
- If $\widehat{p}_N^{(j)}(k)\succ\alpha$, the variable X stochastically dominates the variable Y at the j-order.

4.1.3. Numerical Illustrations

In this subsection, we apply the tests of stochastic dominance in particular to check if the interval $[m_{\min},m_{\max}]$ provided for the third order stochastic dominance between the CPPI strategies and OBPI strategies in previous theoretical subsection can be enlarged. We consider the case of a guarantee equal to 100% of the initially invested amount. Our numerical base case corresponds to a drift equal to 4.5%, an investment horizon equal to 8 years, an historical volatility equal to 15%. Our goal is to determine an order of stochastic dominance between the two insured portfolios by varying the multiplier of the CPPI strategy into the interval $[2,9]$. We begin by varying the implicit volatility in $[20\%;32\%]$ as illustrated in **Table 2**.

We note that, for all cases in which the implied volatility far exceeds the historical volatility, the CPPI strategy, with a multiplier equal to 2, dominates the OBPI one.

We can also study the effect of the drift on the third order stochastic dominance (values of drift $\mu\in[4.5\%,7\%]$). We still fix the investment maturity to 8 years and consider a historical volatility equal to15%, a multiplier range in $[2,9]$, an implied volatility in $[27\%,32\%]$ and a guarantee level equal to 100%.

As shown in **Table 3**, the TSD is never verified, even if $m=2$ and $\sigma_i\in[27\%,32\%]$ when $\mu\ge 5\%$. We note that the CPPI strategy loses its attractiveness. Since $m=2$, we conclude that the CPPI strategy takes less advantage from the trend increase.

For lower trend levels and implicit volatility σ_i higher than the historical one σ, we get results given in **Table 4**.

Table 2. Third order stochastic dominance according to implicit volatility.

	m	μ	σ_i	σ	p-value	SD
$\sigma_i\le\sigma$	2 to 9	4.5%		15%		NTSD
$\sigma_i>\sigma$	2	4.5%	27%	15%	0.0187	TSD
	[3,9]	4.5%	27%	15%		NTSD
	2	4.5%	30%	15%	0.2990	TSD
	[3,9]	4.5%	30%	15%		NTSD

Table 3. No third order stochastic dominance cases.

	m	μ	σ_i	σ	p-value	SD
$\sigma_i < \sigma$	$[2,9]$	$[4.5\%,7\%]$	$[11\%,15\%]$	15%	0	NTSD
$\sigma_i > \sigma$	$[2,9]$	$[4.5\%,7\%]$	$[17\%,21\%]$	15%		NTSD
$\sigma_i > \sigma$	$[2,9]$	$[4.5\%,7\%]$	$[21\%,26\%]$	15%		NTSD
$\sigma_i > \sigma$	$[2,9]$	$[5\%,7\%]$	$[27\%,32\%]$	15%		NTSD

Table 4. Third order stochastic dominance (low trend).

m	μ	σ_i	σ	p-value	SD
4	1%	17%	15%	0.0305	TSD
$[2,\rightarrow,5]$	1%	17%	15%	0	NTSD
$[2,\rightarrow,5]$	1%	$[18\% \rightarrow 20\%]$	15%	0	NTSD
3	2%	18%	15%	0.3408	TSD
3	2%	19%	15%	0.0190	TSD
$[2,\rightarrow,5]$	2%	$[17\% \rightarrow 20\%]$	15%	0	NTSD
$[2,\rightarrow,5]$	2%	$[18\%,19\%]$	15%	0	NTSD
2	3%	18%	15%	0.0533	TSD
2	3%	19%	15%	0.3913	TSD
$[2,\rightarrow,5]$	$[3\%,\rightarrow,4\%]$	$[17\% \rightarrow 20\%]$	15%	0	NTSD

Remark 3.2. To summarize the numerical results:

-We have found that the CPPI method third order stochastically dominates the OBPI one for high implied volatility relatively to the empirical volatility;

-When the interval $[m_{\min}, m_{\max}]$ degenerates, we can find multiples for which the CPPI is stochastically dominated at the third order by OBPI;

-The implied volatility interval where the dominance relation is insured is larger for high values of implied volatility, for low values of the drift and for high values of the multiple.

-The TSD property of the CPPI strategy is rejected for the low values of σ_i with respect to σ.

-Through this numerical study, we can detect cases of third order stochastic dominance beyond the theoretical cases.

-Finally, when strategies are capped, the TSD property is generally not satisfied[4].

5. Conclusion

In the present paper, we have compared the CPPI and OBPI strategies, mainly with respect to the third stochastic dominance (TSD). We find that the CPPI method third order stochastically dominates the OBPI one for high implied volatility relatively to the empirical volatility. We have checked the TSD of the CPPI method compared to the OBPI method for low values of the drift weighted by high values of the multiplier. We have shown that the relation of SDT is rejected for the low values of the implicit volatility with respect to the statistical one. Further extensions could be based on the use of almost stochastic dominance as defined by Leshno and Levy [19], in order to extend the range of the multiple for which the CPPI dominates the OBPI.

References

[1] Barrett, G.F. and Donald, S.G. (2003) Consistent Tests for Stochastic Dominance. *Econometrica*, **71**, 71-104.

[4]Numerical details available on request.

http://dx.doi.org/10.1111/1468-0262.00390

[2] Leland, H.E. and Rubinstein, M. (1976) The Evolution of Portfolio Insurance. In: Luskin, D.L., Ed., *Portfolio Insurance*: *A Guide to Dynamic Hedging*, Wiley.

[3] Perold, A. (1986) Constant Portfolio Insurance. Working Paper, Harvard Business School.

[4] Levy, H. (1992) Stochastic Dominance and Expected Utility: Survey and Analysis. *Management Science*, **38**, 555-593. http://dx.doi.org/10.1287/mnsc.38.4.555

[5] Levy, H. (2015) Stochastic Dominance: Investment Decision Making under Uncertainty. 3rd Edition, Springer-Verlag.

[6] De Giorgi, E. (2005) Reward-Risk Portfolio Selection and Stochastic Dominance. *Journal of Banking and Finance*, **29**, 895-926. http://dx.doi.org/10.1016/j.jbankfin.2004.05.027

[7] Darsinos, T. and Satchell, S. (2004) Generalizing Universal Performance Measures. *Risk*, **17**, 80-84.

[8] Bertrand, P. and Prigent, J.-L. (2005) Portfolio Insurance Strategies: OBPI versus CPPI. *Finance*, **26**, 5-32.

[9] Prigent, J.-L. (2007) Portfolio Optimization and Performance Analysis. Chapman & Hall, Boca Raton. http://dx.doi.org/10.1201/9781420010930

[10] Zagst, R. and Kraus, J. (2011) Stochastic Dominance of Portfolio Insurance Strategies: OBPI versus CPPI. *Annals of Operations Research*, **185**, 75-103. http://dx.doi.org/10.1007/s10479-009-0549-9

[11] Van Der Vaart, A.W. and Wellner, J.A. (1996) Weak Convergence and Empirical Processes with Applications to Statistics. Springer-Verlag, New York. http://dx.doi.org/10.1007/978-1-4757-2545-2

[12] Mosler, K.-C. (1982) Entscheidungsregeln bei Risiko-Multivariate stochastische Dominanz. Springer, Heidelberg. http://dx.doi.org/10.1007/978-3-642-95419-1

[13] Kroll, Y. and Levy, H. (1980) Sampling Errors and Portfolio Efficient Analysis. *Journal of Financial and Quantitative Analysis*, **15**, 655-688. http://dx.doi.org/10.2307/2330403

[14] Davidson, R. and Duclos, J.-Y. (2000) Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality. *Econometrica*, **68**, 1435-1464. http://dx.doi.org/10.1111/1468-0262.00167

[15] Post, T. (2003) Empirical Tests for Stochastic Dominance. *Journal of Finance*, **58**, 1905-1932. http://dx.doi.org/10.1111/1540-6261.00592

[16] Linton, O., Maasoumi, E. and Whang, Y.-J. (2005) Consistent Testing for Stochastic Dominance under General Sampling Schemes. *Review of Economic Studies*, **72**, 735-765. http://dx.doi.org/10.1111/j.1467-937X.2005.00350.x

[17] Scaillet, O. and Topaloglou, N. (2010) Testing for Stochastic Dominance Efficiency. *Journal of Business & Economic Statistics*, **28**, 169-180. http://dx.doi.org/10.1198/jbes.2009.06167

[18] Kläver, H. (2005) Testing for Stochastic Dominance Using Circular Block Methods. Graduate School of Risk Management, University of Köln.

[19] Leshno, M. and Levy, H. (2002) Preferred by All and Preferred by Most Decision Makers: Almost Stochastic Dominance. *Management Science*, **48**, 1074-1085. http://dx.doi.org/10.1287/mnsc.48.8.1074.169

Appendix

Appendix A.1. (Proof of Theorem 3.2)

The proof is similar to the proofs of Theorems 2, 3 and 4 of Zagst and Kraus [10] except that we take account of Relation (12)[5].

-The first step consists in proving the following equivalence:

$$E\left[V_T^{OBPI}\right] \leq E\left[V_T^{CPPI}\right] \Leftrightarrow Call\left(S_0, r, \sigma_i\right) \mathrm{e}^{(m-1)(\mu-r)T} \geq Call\left(S_0, \mu, \sigma\right),$$

which is also equivalent to:

$$m \geq m_{\min}^1 = 1 + \frac{1}{(\mu - r)T} \ln\left(\frac{Call\left(S_0, \mu, \sigma\right)}{Call\left(S_0, r, \sigma_i\right)}\right).$$

The proof is straightforward, using usual computations of both $E\left[V_T^{OBPI}\right]$ and $E\left[V_T^{CPPI}\right]$. Note this condition does not depend on q.

-The second step is to demonstrate that, for $m > 1$, the function $H(x) = F_{V_T^{OBPI}}(x) - F_{V_T^{CBPI}}(x)$ satisfies the following property:

$$\frac{1}{m-1}\left(\frac{\left(1 - p\mathrm{e}^{-rT}\right) \cdot m}{\mathrm{e}^{\frac{1}{2}(m-1)\sigma^2 \cdot T}}\right)^{\frac{m}{m-1}} \prec \frac{Call\left(S_0, r, \sigma_i\right)}{Kq^{\frac{1}{1-m}} \cdot \mathrm{e}^{-rT}} \Rightarrow H \in \mathrm{S}_2.$$

For this purpose, we can note that both the cumulative functions $F_{V_T^{CPPI}}(x)$ and $F_{V_T^{OBPI}}(x)$ can be written as follows:

$$F_{V_T^{OBPI}}(x) = P\left[pV_0 + q\left(S_T - K\right)^+ \leq x\right],$$

$$F_{V_T^{CPPI}}(x) = P\left[pV_0 + \alpha_T S_T^m \leq x\right].$$

Therefore, we deduce in particular that the sign of H does change on $\left(-\infty, pV_0\right]$ since $H(x) = 0$.

For $\left]pV_0, +\infty\right[$, we have to prove that the sign of H changes exactly twice. Therefore, we search the solutions of the equation $H(x) = 0$. Denote:

$$f_{V_T^{OBPI}}(s) = q\left(s - K\right)^+ \text{ and } f_{V_T^{CPPI}}(s) = \alpha_T s^m.$$

Then we get:

$$F_{V_T^{OBPI}}(x) = P\left\{f_{V_T^{OBPI}}(x) \leq x - pV_0\right\}$$

$$\text{and} \quad F_{V_T^{CPPI}}(x) = P\left\{f_{V_T^{CPPI}}(x) \leq x - pV_0\right\}.$$

Therefore, $F_{V_T^{OBPI}}$ and $F_{V_T^{CPPI}}$ intersect if and only $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ does, which is equivalent to

$$q\left(s - K\right)^+ = \alpha_T s^m.$$

Now, we introduce the function $h(s) = \alpha_T s^m - qs + qK$.

1) For $m > 1$, it reaches a minimum at a given value $s^* = \left(q/\left(\alpha_T m\right)\right)^{\frac{1}{m-1}}$.

Therefore, if $h\left(s^*\right) < 0$ the function h has exactly two zeros s_1 and s_2, which means that $f_{V_T^{OBPI}}$ and $f_{V_T^{CPPI}}$ intersect twice.

Standard calculus leads to the following condition:

$$\frac{1}{m-1}\left(\frac{\left(1 - p\mathrm{e}^{-rT}\right) \cdot m}{\mathrm{e}^{\frac{1}{2}(m-1)\sigma^2 .T}}\right)^{\frac{m}{m-1}} < \frac{Call\left(S_0, r, \sigma_i\right)}{Kq^{\frac{1}{1-m}} \cdot \mathrm{e}^{-rT}}.$$

[5]More details are available on request.

In that case, we have:

$$H(x) \geq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \geq F_{V_T^{CPPI}}(x), \forall x \leq s_1,$$
$$H(x) \leq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \leq F_{V_T^{CPPI}}(x), \forall s_1 < x < s_{2,}$$
$$H(x) \geq 0 \Leftrightarrow F_{V_T^{OBPI}}(x) \geq F_{V_T^{CPPI}}(x), \forall x \geq s_2,$$

which implies that $H \in \mathrm{S}_2$.

2) For $m = 1$, there exists only one intersection point equal to $\left(qK/(q-\alpha_T)\right)$ provided that $q > \alpha_T$.

This latter condition is equivalent to $V_0 > Call(S_0, r, \sigma_i)$. It is necessary satisfied for the special case $q = 1$ of Zagst and Kraus [10]. It implies that $H \in \mathrm{S}_1$.

Appendix A.2. (Proof of Theorem 3.4)

The proof is similar to the proof of Theorem 6 of Zagst and Kraus [10] but it takes account of Relation (12).

We have to examine the condition $E\left[V_T^{CPPI\,2}\right] \leq E\left[V_T^{OBPI\,2}\right]$.

-For the CPPI strategy, we get:

$$E\left[V_T^{CPPI}\right] = pV_0 + V_0\left(1 - p\mathrm{e}^{-rT}\right) \cdot \mathrm{e}^{(r+m(\mu-r)T)},$$
$$Var\left[V_T^{CPPI}\right] = V_0^2\left(1 - p\mathrm{e}^{-rT}\right)^2 \cdot \mathrm{e}^{2(r+m(\mu-r)T)}\left(\mathrm{e}^{m^2\sigma^2 T} - 1\right).$$

-For the OBPI strategy, we get:

$$E\left[V_T^{OBPI}\right] = pV_0 + qE\left[\max\{S_T - K; 0\}\right] = pV_0 + q\mathrm{e}^{\mu T} Call(S_0, K, \mu, \sigma),$$

and

$$Var[V_T^{OBPI}\ ] = q^2 \times$$

$$Var\left[V_T^{OBPI}\right] = q^2 \times S_0^2 \mathrm{e}^{2\mu T + \sigma^2 T} N\left[d_1 + \sigma\sqrt{T}\right] - 2KS_0\mathrm{e}^{\mu T} N\left[d_1\right] + K^2 N\left[d_2\right] - \mathrm{e}^{2\mu T} Call(S_0, K, \mu, \sigma)^2.$$

with $d_1 = \frac{Ln\left(\frac{S_0}{K}\right) + \left(\mu + \sigma^2/2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Then, we get:

$$E\left[V_T^{CPPI\,2}\right] = Var\left[V_T^{CPPI}\right] + E\left[V_T^{CPPI}\right]^2,$$

from which we deduce:

$$E[V_T^{CPPI\ 2}] =$$

$$E\left[V_T^{CPPI\,2}\right] = V_0^2\left(1 - p\mathrm{e}^{-rT}\right)^2 \cdot \mathrm{e}^{2(r+m(\mu-r))T} \mathrm{e}^{m^2\sigma^2 T} + \left(pV_0\right)^2 + 2pV_0^2\left(1 - p\mathrm{e}^{-rT}\right) \cdot \mathrm{e}^{2(r+m(\mu-r)T)}$$

We have also:

$$E\left[V_T^{OBPI\,2}\right] = Var\left[V_T^{OBPI}\right] + E\left[V_T^{OBPI}\right]^2,$$

which obviously does not depend on the multiple *m*.

Introduce now the function *g* defined by:

$$g(m) = E\left[V_T^{CPPI\,2}\right] - E\left[V_T^{OBPI\,2}\right] = V_0^2\left(1 - p\mathrm{e}^{-rT}\right)^2 \cdot \mathrm{e}^{2(r+m(\mu-r))T} \mathrm{e}^{m^2\sigma^2 T} + \left(pV_0\right)^2$$
$$+ 2pV_0^2\left(1 - p\mathrm{e}^{-rT}\right) \cdot \mathrm{e}^{2(r+m(\mu-r)T)} - E\left[V_T^{OBPI\,2}\right].$$

The function $g(.)$ is continuous and strictly increasing. It converges to infinity when *m* goes to infinity.

Therefore, assuming that $g(0) \leq 0$, there exists one and only one value $m_{\max}$ such that $g(m_{\max}) = 0$.

Finally, we deduce that:

$$E\left[V_T^{CPPI\,2}\right] \leq E\left[V_T^{OBPI\,2}\right] \Leftrightarrow m \leq m_{\max}.$$

Note that condition $g(0) \leq 0$ is equivalent to $\left(V_0 e^{rT}\right)^2 \leq E\left[V_T^{OBPI\,2}\right]$ since, for $m = 0$, the CPPI strategy corresponds to a whole investment on the risk free asset *B*.