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Conditional Law of the Hitting Time for a Lévy Process in Incomplete Observation

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Abstract

We study the default risk in incomplete information. That means we model the value of a firm by a Lévy process which is the sum of a Brownian motion with drift and a compound Poisson process. This Lévy process cannot be completely observed, and another process represents the available information on the firm. We obtain a stochastic Volterra equation satisfied by the conditional density of the default time given the available information. The uniqueness of solution of this equation is proved. Numerical examples of (conditional) density are also given.

Keywords

Conditional Density, Default Time, Lévy Processes, Filtering Theory, Stochastic Voltera Equations

1. Introduction

Here we consider a jump-diffusion process X which models the value of a firm. This is a Lévy process. Details on this class of processes can be found in [1] and [2]. Their use in financial modeling is well developed in [3]. We study the first passage time of process X at level $x > 0$ modeling the default time. We investigate the behavior of the default time under incomplete observation of assets. In the literature, there exists some papers in relation to this topic. Duffie and Lando [4] suppose that bond investors cannot observe the issuer's assets directly; instead, they only receive periodic and imperfect reports. For a setting in which the assets of the firm are geometric Brownian motion until informed equity holders optimally liquidate, they derive the conditional distribution of the assets, and give the available information. In a similar model, but with complete information, Kou and Wang [5] study the first passage time of a jump-diffusion process whose jump sizes follow a double exponential distribution. They obtain explicit solutions of the Laplace transform of the distribution of the first passage time. Laplace transform of the joint distribution of jump-diffusion and its running maximum,

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$S_t = \sup_{s\le t} X_s$, is too obtained. To finish, they give numerical examples. Bernyk *et al*. [6], for their part, consider stable Lévy process X of index $\alpha \in]1,2[$ with non negative jumps and its running maximum. They characterize the density function of S_t as the unique solution of a weakly singular Volterra integral equation of the first kind. This leads to an explicit representation of the density of the first passage time. To unify the noisy information in Duffie and Lando [4], X. Guo, R. A. Jarrow and Y. Zang [7] define a filtration which models incomplete information. By simple examples, they give the importance of this notion. Similarly to Kou and Wang, without specifying the jumps size law, Dorobantu [8] provides the intensity function of the default time. That is very important for investors, but the information brought by this intensity is low. Furthermore, Roynette *et al*. [9] prove that the Laplace transform of the random triplet (first passage time, overshoot, undershoot) satisfies an integral equation. After normalization of the first passage time, they show under some convenient assumptions that the random triplet converges in distribution as level x goes to ∞. Gapeev and Jeanblanc [10] study a model of a financial market in which the dividend rates of two risky asset's initial values change when certain unobservable external events occur. The asset price dynamics are described by a geometric Brownian motion, with random drift rates switching at independent exponential random times. These random times are independent of the constantly correlated driving Brownian motion. They obtain closed expressions for rational values of European contingent claims given the available information. Moreover, estimates of the switching times and their conditional probability density are provided. Coutin and Dorobantu [11] prove that the default time law has a density (defective when $\mathbb{E}(X_1)<0$) with respect to the Lebesgue measure in case of a stationary independent increment process built on a pair (compound Poisson process, Brownian motion).

We extend this approach studying the conditional law of the first passage time of Lévy process at level x given a partial information. We solve this problem using filtering theory inspired by Zakai [12], Pardoux [13], Coutin [14], Bain and Crisan [15], based on the so called "reference probability measure" method. The paper is organized as follows: Section 2 sets the model; Section 3 gives the results on the existence of the conditional density given the observed filtration and on the integro-differential equation satisfied by this conditional density; Section 4 gives the proofs of the results. To finish, we conclude and give some auxiliary results in Appendix.

2. Model and Motivations

This section defines the basic space in which we work and announces what we will do. Subsection 2.1 gives the model of the firm value and defines the default time. Subsection 2.2 recalls some important results in the complete information case. Subsection 2.3 defines the signal and observation process and the model for available information. Basically, it introduces the notion of filtering theory. Subsection 2.4 gives our motivation.

2.1. Construction of the Model

Let $\left(\Omega,\mathcal{F},\left(\mathcal{F}_t,t\ge 0\right),\mathbb{P}^0\right)$ be a filtered probability space satisfying the usual conditions on which we define a standard Brownian motion W, a sequence of independent and identically distributed random variables $(Y_i)_{i\in\mathbb{N}^*}$ with distribution function F_Y, a Poisson process N with intensity $\lambda>0$ and a stochastic process Q. We assume that all these elements are independent, (W,Q) is a Brownian motion and (Y,N) is a compound Poisson process with intensity ν under $\mathbb{P}^0$ defined for any Borel set A by $\nu(\mathrm{d}t,A)=\lambda\int_A F_Y(\mathrm{d}y)\mathrm{d}t$. On this probability space, we define a process X as follows:

$$X_t = mt + W_t + \sum_i^{N_t} Y_i. \tag{1}$$

X models a firm value and the default is modeled by the first passage time of X at a level $x>0$. Hence the default time is defined as

$$\tau_x = \inf\left\{t\ge 0 : X_t \ge x\right\}. \tag{2}$$

We suppose that X is not perfectly observable and that observation is modeled by process Q.

2.2. Some Results When *X* Is Perfectly Observed

Let $\left(\tilde{X}_t, t\ge 0\right)$ be a Brownian motion with drift $m\in\mathrm{R}$ ($\tilde{X}_t = mt + W_t$). For $z>0$, we let $\tilde{\tau}_z = \inf\left\{t\ge 0, \tilde{X}_t \ge z\right\}$.

By (5.12) page 197 of [16], $\tilde{\tau}_z$ has the following law on $\overline{\mathbb{R}}_+$:

$$\tilde{f}(u,z)\,\mathrm{d}u + \mathbb{P}^0(\tilde{\tau}_z = \infty)\delta_\infty(\mathrm{d}u) \tag{3}$$

where

$$\tilde{f}(u,z) = \frac{|z|}{\sqrt{2\pi u^3}}\exp\left[-\frac{1}{2u}(z-mu)^2\right]\mathbf{1}_{]0,+\infty[}(u) \text{ and } \mathbb{P}^0(\tilde{\tau}_z = \infty) = 1 - \mathrm{e}^{mz-|mz|}.$$

The function $\tilde{f}(.,z)$ is $\mathcal{C}^\infty$ on $[0,+\infty[$, and all its derivatives admit 0 as right limit at 0 and therefore belongs to $\mathcal{C}^\infty([0,+\infty[)$. For $\sigma > 0$, Roynette *et al.* [9] consider as a firm value the process $X_t = mt + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and as a default time the random variable $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$. They let $K_x := X_{\tau_x} - x$ namely overshoot and $L_x := x - X_{\tau_x^-}$ namely undershoot. They prove that the Laplace transform of (τ_x, K_x, L_x) satisfies an integral equation. After a suitable renormalization of τ_x that we can note here $\overline{\tau}_x$, they show that $(\overline{\tau}_x, K_x, L_x)$ converges in distribution as x goes to ∞. Overall they have obtained an asymptotic behavior of the defaut time, the overshoot and the undershoot.

For a general Lévy process, Doney and Kiprianou [17] give the law of the quintuplet $\left(\overline{G}_{\tau_x}, \tau_x - \overline{G}_{\tau_x^-}, X_{\tau_x} - x, x - X_{\tau_x^-}, x - \overline{X}_{\tau_x^-}\right)$ where $\overline{X}_t = \sup_{s\leq t} X_s$ and $\overline{G}_t = \sup\{s < t : \overline{X}_s = X_s\}$.

Coutin and Dorobantu [11] consider (1) and (2) and show that τ_x admits a density with respect to the Lebesgue measure. They give the following closed expression of this density

$$f(t,x) = \begin{cases} \lambda\mathbb{E}\left(1_{\tau_x > t}(1-F_Y)(x - X_t)\right) + \mathbb{E}\left(1_{\tau_x > T_{N_t}}\tilde{f}\left(t - T_{N_t}, x - X_{T_{N_t}}\right)\right) & \text{if } t > 0 \\ \frac{\lambda}{2}\left(2 - F_Y(x) - F_Y(x_-)\right) + \frac{\lambda}{4}\left(F_Y(x) - F_Y(x_-)\right) & \text{if } t = 0, \end{cases} \tag{4}$$

where $(T_i, i \in \mathbb{N}^*)$ is the sequence of the jump times of the process N.

2.3. The Incomplete Information

Our work is inspired and is in the same spirit as D. Dorobantu [8]. In her thesis, Dorobantu assumes that investors wishing to detain a part of the firm do not have complete information. They don't observe perfectly the process value X of the firm but a noisy value. She defined a process Q independent of W, N, Y and satisfying the following evolution equation

$$Q_t = \int_0^t h(X_s)\,\mathrm{d}s + B_t, \quad t \in \mathbb{R}_+$$

with h a Borel and bounded function and B a standard Brownian motion.

Definition 1. *The process X is called the signal. The process Q is called the observation and is perfectly observed by investors.*

This leads us to a filtering model and we introduce the filtering framework inspired of Zakai [12], Coutin [14] or Pardoux [13].

Since the function h is bounded, the Novikov condition, $\forall T > 0, \mathbb{E}^0\left(\mathrm{e}^{\frac{1}{2}\int_0^T h^2(X_s)\mathrm{d}s}\right) < \infty$, is satisfied and we define the following exponential martingale for the filtration $(\mathcal{F}_t)_{t\geq 0}$ by

$$L_t = \exp\left(\int_0^t h(X_s)\,\mathrm{d}Q_s - \frac{1}{2}\int_0^t h^2(X_s)\,\mathrm{d}s\right), \quad t \in \mathbb{R}_+.$$

For a fixed maturity $T > 0$, the process $(L_{t\wedge T}, t \in \mathbb{R}_+)$ is a uniformly integrable $\left(\mathbb{P}^0, (\mathcal{F}_t)_{t\geq 0}\right)$-martingale.

Definition 2. *For fixed* $t > 0$, *let us define a probability measure* $\mathbb{P}$ *on* $\mathcal{F}_t$ *by*

$$\mathbb{P}_{|\mathcal{F}_t} := L_t \mathbb{P}^0_{|\mathcal{F}_t}$$

We also note that the law of X, so the one of τ_x, under $\mathbb{P}^0$ is the same as under $\mathbb{P}$. Note that investors have additional information on the firm which is modeled at time *t* by

$$\mathcal{D}_t = \sigma\left(1_{\tau_x \le u}, u \le t\right).$$

Then all the available information is represented by the filtration

$$\mathcal{G} := \left(\mathcal{G}_t = \mathcal{F}_t^Q \vee \mathcal{D}_t,\ t \ge 0\right)$$

where the σ-algebra $\mathcal{F}_t^Q$ is generated by the observation of the process Q up to time *t*.

2.4. Motivations

D. Dorobantu [8] obtains the $\mathcal{G}$-intensity of the default, namely the $\mathcal{G}$-predictable process $(\lambda_t)_{t\ge 0}$, such that

$$M_t = \mathbf{1}_{\tau_x > t} - \int_0^t \lambda_s \mathrm{d}s,\ t \ge 0$$

is a $\mathcal{G}$-martingale. With this result, using their available information, the investors can predict the default time. More precisely, given that default did not occur at time *t*, the probability that it occurs at time $t + \mathrm{d}t$ is approximated by $\lambda_t \mathrm{d}t$. But the information brought by the knowledge of $(\lambda_t)_{t\ge 0}$ is low. This motivates us to show that the conditional law of default time τ_x given $\mathcal{G}$ admits a density with respect to Lebesgue measure and to give its dynamic evolution.

This section presents our basic model of a firm with incomplete information about its assets. More generally, we treat a continuous time setting, staying with the work of D. Dorobantu [8] in her thesis second part. Next section gives our main results.

3. The Results

3.1. Existence of the Conditional Density

We recall that $\tau_x = \inf\{t \ge 0 : X_t \ge x\}$ is the default time of a firm and $\mathcal{G}_t$ is the available information of investors at time *t*. In this subsection, we prove that conditionally on the σ-algebra $\mathcal{G}_t$, τ_x admits a density with respect to the Lebesgue measure.

Proposition 1. *For all* $t > 0$, *on the set* $\{\tau_x > t\}$, *the* $\mathcal{G}_t$ *conditional law of* τ_x *has the following form*

$$\bar{f}(r,t,x)\mathrm{d}r + \mathbb{P}(\tau_x = \infty \mid \mathcal{G}_t)\delta_\infty(\mathrm{d}r) \text{ and } \mathbb{P}(\tau_x = \infty \mid \mathcal{G}_t) = \mathbf{1}_{\tau_x > t}\mathbb{E}\left(G(\infty, x - X_t) \mid \mathcal{G}_t\right), \tag{5}$$

where

$$\bar{f}(r,t,x) := \mathbb{E}\left[f(r-t, x - X_t) \mid \mathcal{G}_t\right].$$

And

$$G(t,x) := \mathbb{P}(\tau_x > t) = \mathbb{P}^0(\tau_x > t) = \int_t^\infty f(u,x)\mathrm{d}u.$$

Remark 1 *Referring to* [9], *for all* $x > 0$, *the passage time* τ_x *is finite almost surely if and only if* $m + \mathbb{E}(Y_1) \ge 0$.

3.2. Mixed Filtering-Integro-Differential Equation for Conditional Density

In this subsection, we give our main results. Indeed, we first show that the conditional law of the hitting time τ_x given the filtration $(\mathcal{G}_t)_{t\ge 0}$ satisfies a stochastic integro-differential equation. Afterwards, we give a uniqueness result. This type of equation is the same as the one studied in [18] with the only difference that here, we have more general Voltera random coefficients.

Theorem 1. *Let* $t > 0$ *be a real number. For any* $r > t$, *on the set* $\{\tau_x > t\}$, *the conditional density of* τ_x *given* $\mathcal{G}_t$ *satisfies the stochastic integro-differential equation*:

$$\begin{aligned}\bar{f}(r,t,x) = {} & \frac{f(r,x)}{\mathbb{P}(\tau_x > t)} + \int_0^t \Pi^1(h)(r,t,u)\mathrm{d}Q_u - \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}\left(\mathbf{1}_{\tau_x > u} G(t-u, x - X_u) \mid \mathcal{G}_u\right)} \Pi(h)(t,u)\mathrm{d}Q_u \\ & + \int_0^t \frac{\bar{f}(r,u,x)}{\mathbb{E}\left(\mathbf{1}_{\tau_x > u} G(t-u, x - X_u) \mid \mathcal{G}_u\right)} \left[\Pi(h)(t,u)\right]^2 \mathrm{d}u - \int_0^t \Pi^1(h)(r,t,u)\Pi(h)(t,u)\mathrm{d}u.\end{aligned} \tag{6}$$

where

$$\Pi^1(\Phi)(r,t,u)=\frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}\Phi(X_u)f(r-u,x-X_u)\,|\,\mathcal{G}_u\right)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)\,|\,\mathcal{G}_u\right)},$$

$$\Pi(\Phi)(t,u)=\frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}\Phi(X_u)G(t-u,x-X_u)\,|\,\mathcal{G}_u\right)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}G(t-u,x-X_u)\,|\,\mathcal{G}_u\right)}$$

and G is defined in Proposition 1.

Proposition 2. *If* Equation (6) *admits a solution, this one is unique.*

3.3. Some Technical Results

Here, we give some technical and auxiliary results which are useful to prove Theorem 1 and Proposition 2.

Proposition 3. *For any bounded function* φ *such that* $\varphi(\tau_x)$ *is* F_T^X *-measurable,* $\forall t\le T$

$$\mathbb{E}^0\left(\varphi(\tau_x)\mathbf{1}_{\tau_x>t}L_T\,|\,\mathcal{F}_t^Q\right)=\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_x>t}\right]+\int_0^t\mathbb{E}^0\left[L_u h(X_u)\mathbb{E}^0\left[\mathbf{1}_{\tau_x>t}\varphi(\tau_x)\,|\,\mathcal{F}_u\right]|\,\mathcal{F}_u^Q\right]\mathrm{d}Q_u. \tag{7}$$

By this proposition, we establish two corollaries which give a representation more accessible of the processes $t\mapsto\mathbb{E}\left(L_b\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_t^Q\right)$ and $t\mapsto\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\,|\,\mathcal{F}_t^Q\right)$: we apply Proposition 3 respectively to the functions $\phi:y\to\mathbf{1}_{\{]a,b[(y)\}}$ and $\phi:y\to\mathbf{1}_{\{]T,\infty)(y)\}}$, the second expressions being consequence of the fact that on the event $\{\tau_x>t\}\subset\{\tau_x>u\}$, $\tau_x=u+\tau_{x-X_u}\circ\theta_u$ (θ is the shift operator) and

$\mathbb{E}^0\left[\mathbf{1}_{\tau_x>t}\mathbf{1}_{a<\tau_x<b}|\,\mathcal{F}_u\right]=\mathbf{1}_{\tau_x>u}\mathbb{E}^0\left[\mathbf{1}_{a-u<\tau_{x-X_u}<b-u}|\,\mathcal{F}_u\right].$

Corollary 1. *For all* $t<a<b$, *we have* $\mathbb{P}^0-a.s$

$$1)\ \ \mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_t^Q\right)=\mathbb{P}^0\left(a<\tau_x<b\right)+\int_0^t\mathbb{E}^0\left(L_u h(X_u)\mathbb{E}^0\left[\mathbf{1}_{\tau_x>t}\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_u\right]|\,\mathcal{F}_u^Q\right)\mathrm{d}Q_u, \tag{8}$$

and equivalently

$$2)\ \ \mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_t^Q\right)=\mathbb{P}^0\left(a<\tau_x<b\right)+\int_0^t\mathbb{E}^0\left(L_u h(X_u)\mathbf{1}_{\tau_x>u}\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right]|\,\mathcal{F}_u^Q\right)\mathrm{d}Q_u. \tag{9}$$

Corollary 2. *For* $t\le T$,

$$1)\ \ \mathbb{E}^0\left(L_b\mathbf{1}_{\tau_x>T}\,|\,\mathcal{F}_t^Q\right)=\mathbb{P}^0\left(\tau_x>T\right)+\int_0^t\mathbb{E}^0\left(L_u h(X_u)\mathbb{E}^0\left[\mathbf{1}_{\tau_x>t}\mathbf{1}_{\tau_x>T}\,|\,\mathcal{F}_u\right]|\,\mathcal{F}_u^Q\right)\mathrm{d}Q_u, \tag{10}$$

and equivalently

$$2)\ \ \mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\,|\,\mathcal{F}_t^Q\right)=\mathbb{P}^0\left(\tau_x>T\right)+\int_0^t\mathbb{E}^0\left(L_u h(X_u)\mathbf{1}_{\tau_{x>u}}G(T-u,x-X_u)\,|\,\mathcal{F}_u^Q\right)\mathrm{d}Q_u. \tag{11}$$

Proposition 4. *For any* $0<t<a<b$, *we have on the set* $\{\tau_x>t\}$,

$$\begin{aligned}
\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\,|\,\mathcal{F}_t^Q\right)}&=\frac{\mathbb{P}^0\left(a<\tau_x<b\right)}{\mathbb{P}^0\left(\tau_x>t\right)}+\int_0^t\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right]|\,\mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\,|\,\mathcal{F}_u^Q\right)}\mathrm{d}Q_u\\
&-\int_0^t\frac{\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_u^Q\right)\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(t-u,x-X_u)\,|\,\mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\,|\,\mathcal{F}_u^Q\right)\right]^2}\mathrm{d}Q_u\\
&+\int_0^t\frac{\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}\,|\,\mathcal{F}_u^Q\right)\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(t-u,x-X_u)\,|\,\mathcal{F}_u^Q\right)\right]^2}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\,|\,\mathcal{F}_u^Q\right)\right]^3}\mathrm{d}u\\
&-\int_0^t\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right]|\,\mathcal{F}_u^Q\right)\times\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(t-u,x-X_u)\,|\,\mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\,|\,\mathcal{F}_u^Q\right)\right]^2}\mathrm{d}u.
\end{aligned} \tag{12}$$

Remark 2. Equation (12) *of Proposition* 4 *can be rewriten as*:

$$\overline{\Gamma}_t = \frac{\mathbb{P}^0\left(a<\tau_x<b\right)}{\mathbb{P}^0\left(\tau_x>t\right)} + \int_0^t \sigma^1(h)(t,u)\,\mathrm{d}Q_u - \int_0^t \overline{\Gamma}_u \sigma(h)(t,u)\,\mathrm{d}Q_u$$
$$+\int_0^t \overline{\Gamma}_u\left[\sigma(h)(t,u)\right]^2 \mathrm{d}u - \int_0^t \sigma^1(h)(t,u)\sigma(h)(t,u)\,\mathrm{d}u.$$

Where

$$\overline{\Gamma}_t = \frac{\mathbb{E}^0\left(L_b \mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_t \mid \mathcal{F}_t^Q\right)},$$

$$\sigma^1(h)(t,u) = \mathbf{1}_{\{\tau_x>t\}} \frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h(X_u)\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right] \mid \mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)},$$

$$\sigma(h)(t,u) = \mathbf{1}_{\{\tau_x>t\}} \frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h(X_u) G(t-u,x-X_u) \mid \mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)}.$$

This equation is similar to the non normalized conditional distribution Equation (3.43) in A. Bain and D. Crisan [15], called Zakai equation.

In the same way, Equation (6) which is derived from (12) is similar to the normalized conditional distribution Equation (3.57) in A. Bain and D. Crisan [15], called Kushner-Stratonovich equation.

3.4. Numerical Examples

We simulate the density of the first passage time respectively in complete information and in incomplete information. We suppose that the jump size follows a double exponential distribution, *i.e*, the common density of Y is given by $f_Y(y) = p\cdot\eta 1\cdot \mathrm{e}^{-\eta 1 y}\mathbf{1}_{y\geq 0} + q\cdot\eta 2\cdot \mathrm{e}^{\eta 2 y}\mathbf{1}_{y<0}$ where $p,q\geq 0$ are constants, $p+q=1$ and $\eta 1,\eta 2>0$. Here, $\eta 1=\frac{1}{0.02}, \eta 2=\frac{1}{0.03}, p=\frac{1}{2}$ and $x=0.1$. The difference between the figures is on one hand due to the information and on another hand to the values taken by the parameters *m* and λ.

These four first figures (**Figue 1** and **Figure 2**) represent the densities of the first passage time for a jump

λ	CPU time
3	438.03805

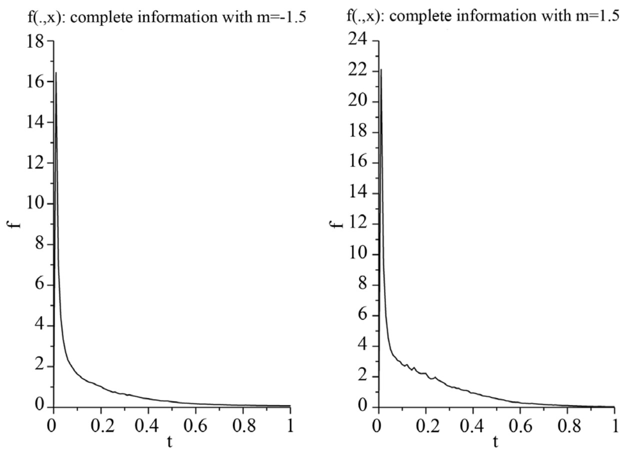

Figure 1. Densities for $\lambda=3$.

λ	CPU time
0.1	376.6704

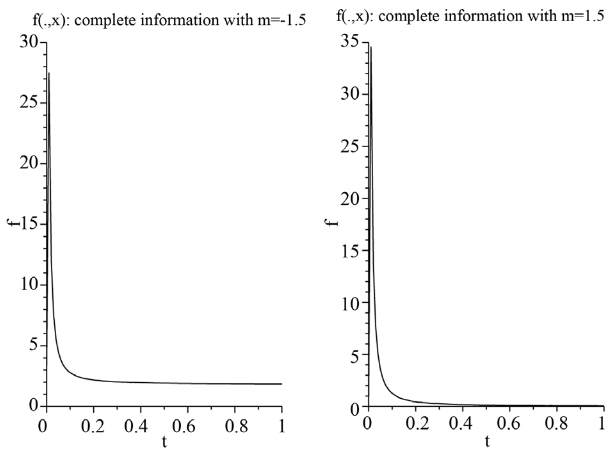

Figure 2. Densities for $\lambda = 0.1$.

diffusion process (case of complete information). The variable $t \in [0,1]$ and Monte Carlo results are based on 5000 simulation runs.

Figure 3, **Figure 4** and **Figure 6** are those of the conditional density $\overline{f}(r,t,x)$ (case of incomplete information), for fixed $t = 0.1$ and the variable r is such that $r \in]0.1, 0.6]$. Part II of A. Bain and D. Crisan [15], namely Numerical Algorithms, where the authors give some tools to solve the filtering problem is really useful. The class of the numerical method used is the particle method for continuous time framework.Here, the Monte Carlo results are based on 120 simulation runs.

We observe that the maximum reached is greater if the drift *m* is positive, meaning the positive level *x* is more probably reached in a shorter time.

In incomplete information, the distance between the curve and axis is greater than in complete information case, this would mean that in case of incomplete information, the level *x* is more difficult to be reached in a short time.

The choice of the small value of λ serves to compare the results with the limiting Brownian motion case ($\lambda = 0$). In complete information case, the formulae for the first passage times of Brownian motion can be found in [16].

A large value of λ implies a lot of jumps, a large computing time and less regular curve.

In these last four figures (**Figure 5** and **Figure 6**), the maximum reached is greater if the drift *m* is negative, meaning the positive level *x* is more probably reached in a shorter time. This is due to the very small value of λ.

4. Proofs

Proposition 1

Proof. First note that, since *X* is a $(\mathcal{F}, \mathbb{P})$-Markov process and $\mathcal{G}_t \subset \mathcal{F}_t$, we have

$$\mathbb{E}\left(\mathbf{1}_{\tau_x=\infty} \mid \mathcal{G}_t\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\tau_x=\infty} \mid \mathcal{F}_t\right) \mid \mathcal{G}_t\right) = \mathbb{E}\left[\mathbf{1}_{\tau_x>t}\mathbb{E}^t\left(1_{\tau_{x-X_t}=\infty}\right) \mid \mathcal{G}_t\right]$$

$$= \mathbf{1}_{\tau_x>t}\mathbb{E}\left(G(\infty, x - X_t) \mid \mathcal{G}_t\right), \text{ where } \mathbb{E}^t(.) = \mathbb{E}(. \mid \mathcal{F}_t).$$

The fact that τ_x is a $(\mathcal{G}, \mathbb{P})$-stopping time justifies the last equality.

Secondly, for any $b \ge a > t$ the $(\mathbb{P}, \mathcal{F})$ Markov property of the process *X* and the fact that on the set $\{\tau_x > t\}$, $\tau_x = t + \tau_{x-X_t} \circ \theta_t$ ensure

$$\mathbb{E}\left(\mathbf{1}_{a\le\tau_x<b} \mid \mathcal{G}_t\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{a\le\tau_x<b} \mid \mathcal{F}_t\right) \mid \mathcal{G}_t\right) = \mathbb{E}\left(\mathbf{1}_{\tau_x>t}\mathbb{E}^t\left(\mathbf{1}_{a-t\le\tau_{x-X_t}<b-t}\right) \mid \mathcal{G}_t\right).$$

λ	CPU time
2	358.11432

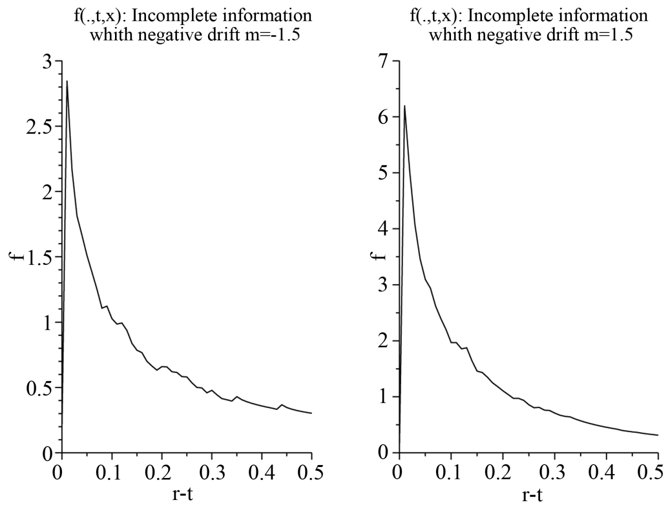

Figure 3. Conditional densities for $\lambda = 2$.

λ	CPU time
0.1	353.00736

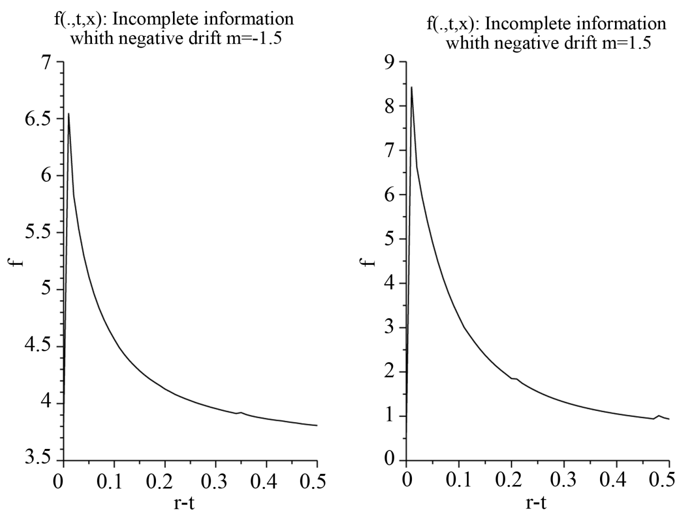

Figure 4. Conditional densities for $\lambda = 0.1$.

The $\mathcal{F}_t$-conditional law of τ_{x-X_t} has the density (possibly defective) $f(.-t, x-X_t)$, thus

$$\mathbb{E}\left(\mathbf{1}_{a\le\tau_x<b} \mid \mathcal{G}_t\right) = \mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)\,\mathrm{d}r \mid \mathcal{G}_t\right].$$

By hypothesis, we have $r-t \ge a-t > 0$. It follows from Lemma 3 of Appendix that

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f(r-t, x-X_t)\,\mathrm{d}r\right] < \infty.$$

Then, we have for any $b \ge a > t$,

λ	CPU time
0.01	373.16157

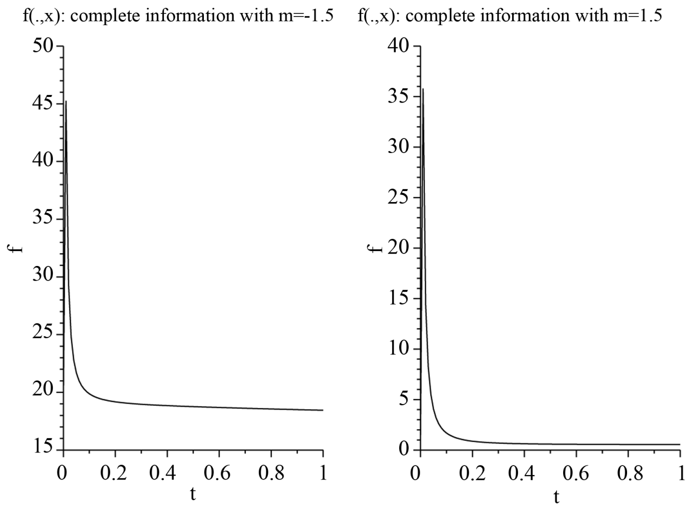

Figure 5. Densities for $\lambda = 0.01$.

λ	CPU time
0.01	358.96784

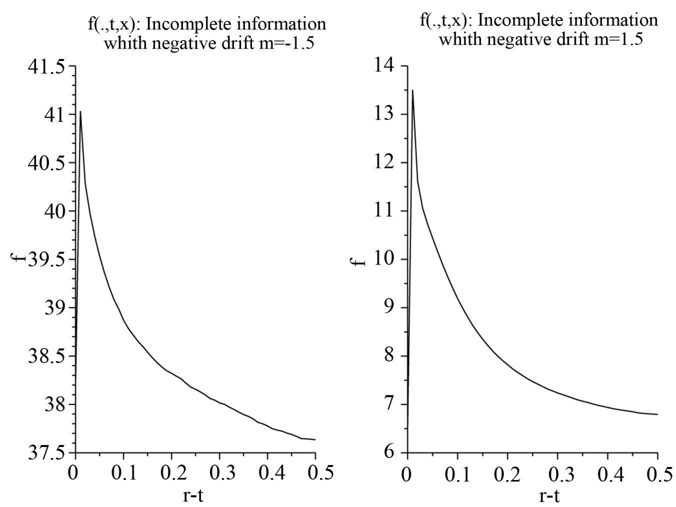

Figure 6. Conditional densities for $\lambda = 0.01$.

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f\left(r-t,x-X_t\right)\mathrm{d}r \mid \mathcal{G}_t\right]=\int_a^b\mathbb{E}\left[\mathbf{1}_{\tau_x>t} f\left(r-t,x-X_t\right)\mid \mathcal{G}_t\right]\mathrm{d}r \ \ a.s.$$

Now, we show the equality almost surely for all $b\ge a>t$. Let M_1 and M_2 be the processes defined by

$$M_1 : b\mapsto \mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f\left(r-t,x-X_t\right)\mathrm{d}r \mid \mathcal{G}_t\right] \text{ and } M_2 : b\mapsto \int_a^b\mathbb{E}\left[\mathbf{1}_{\tau_x>t} f\left(r-t,x-X_t\right)\mid \mathcal{G}_t\right]\mathrm{d}r.$$

These processes are increasing, then they are sub-martingales with respect to the filtration $\tilde{\mathcal{G}}_b=\mathcal{G}_t \ \forall b\ge t$. Note that $b\mapsto \mathbb{E}\left(M_1(b)\right)$ and $b\mapsto \mathbb{E}\left(M_2(b)\right)$ are too continuous. Using Revuz-Yor Theorem 2.9 p. 61 [19], they have same càd-làg modification for all b, meaning that

$$\mathbb{E}\left[\mathbf{1}_{\tau_x>t}\int_a^b f\left(r-t,x-X_t\right)\mathrm{d}r \mid \mathcal{G}_t\right]=\int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t} f\left(r-t,x-X_t\right) \mid \mathcal{G}_t\right]\mathrm{d}r \quad a.s. \forall b.$$

We conclude that, almost surely, for all $b \geq a > t$,

$$\mathbf{1}_{\tau_x>t}\mathbb{E}\left(1_{a<\tau_x\leq b} \mid \mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}\int_a^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t} f\left(r-t,x-X_t\right) \mid \mathcal{G}_t\right]\mathrm{d}r.$$

Taking $a = t + \frac{1}{n}$, letting n going to infinity and using monotone Lebesgue Theorem yield that, $\mathbb{P}-a.s\ \forall b \geq t$,

$$\mathbb{E}\left(\mathbf{1}_{t<\tau_x\leq b} \mid \mathcal{G}_t\right)=\int_t^b \mathbb{E}\left[\mathbf{1}_{\tau_x>t} f\left(r-t,x-X_t\right) \mid \mathcal{G}_t\right]\mathrm{d}r.$$

□

Proposition 2

Proof. Let $\bar{f}$ and $\bar{g}$ be two solutions of Equation (6) and $\bar{\delta} = \bar{f} - \bar{g}$. It follows that

$$\bar{\delta}(r,t,x)=-\int_0^t \bar{\delta}(r,u,x)K(t,u,x)\mathrm{d}Q_u+\int_0^t \bar{\delta}(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u \tag{13}$$

where

$$K(t,u,x)=\frac{\Pi(h)(t,u)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}. \tag{14}$$

We recall the expression

$$\Pi(h)(t,u)=\frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}h\left(X_u\right)G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}$$

and remark that $\left|\Pi(h)(t,u)\right| \leq \|h\|_\infty$. Then

$$\left|K(t,u,x)\right| \leq \mathbf{1}_{\tau_x>u}\frac{\|h\|_\infty}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u}G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}.$$

Markov property implies

$$\left|K(t,u,x)\right| \leq \mathbf{1}_{\tau_x>u}\frac{\|h\|_\infty}{\mathbb{E}\left(\mathbf{1}_{\tau_x>t} \mid \mathcal{G}_u\right)}.$$

We use Lemma 4 with $t = uY = \mathbf{1}_{\tau_x>t}$ and $b = t$ and it follows that

$$\left|K(t,u,x)\right| \leq \|h\|_\infty \mathbf{1}_{\tau_x>u}\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u \mid \mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_u^Q\right)}$$

and Lemma 7 (22) with the pair (t,u) gets

$$\left|K(t,u,x)\right| \leq \|h\|_\infty \mathbf{1}_{\tau_x>u}\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u \mid \mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_u^Q\right)}.$$

All computations are done on the set $\{\tau_x > t\}$. We observe too $u \to \frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_u^Q\right)}$ is a positive submartingale. Then for all $T \geq t \geq u$, we obtain by Lemma 7 (22) with the pair (t,T), Doob's inequality and $\{\tau_x > T\} \subset \{\tau_x > t\}$,

$$\mathbb{E}^0\left(\left[\sup_{u\leq T}\frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_u^Q\right)}\right]^2\right) \leq 4\mathbb{E}^0\left(\left[\frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_T^Q\right)}\right]^2\right) \leq 4\mathbb{E}^0\left(\left[\frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T \mid \mathcal{F}_T^Q\right)}\right]^2\right).$$

Thanks to Jensen inequality and Lemma 8 with $\alpha=2$ and $t=T$, it follows that

$$\mathbb{E}^0\left(\sup_{u\le T}\frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\mid\mathcal{F}_u^Q\right)}\right)<\infty.$$

Concerning the numerator, $\mathbb{E}^0\left(1_{\tau_x>u}L_u\mid\mathcal{F}_u^Q\right)\le\mathbb{E}^0\left(L_u\mid\mathcal{F}_u^Q\right)$. Since Novikov condition $\mathbb{E}^0\left(\mathrm{e}^{\frac{1}{2}\int_0^T h^2(X_s)\mathrm{d}s}\right)<\infty$ is satisfied then $\mathbb{E}^0\left(L_u\mid\mathcal{F}_u^Q\right)$ is a locally square integrable $\left(\mathbb{P}^0,\mathcal{F}^Q\right)$-martingale. Once again Doob's inequality gets

$$\mathbb{E}^0\left(\sup_{u\le T}\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u\mid\mathcal{F}_u^Q\right)\right)<\infty.$$

So finally

$$\sup_{u\le T}\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u\mid\mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\mid\mathcal{F}_u^Q\right)}<\infty\quad\mathbb{P}-a.s. \tag{15}$$

Let $T_n(\omega)=\inf\left\{t\ge0:\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u\mid\mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_u\mid\mathcal{F}_u^Q\right)}\ge n\right\}$, and $\Omega_n=\left\{\omega:\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u\mid\mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\mid\mathcal{F}_u^Q\right)}\le n\right\}$. On the set Ω_n, $\left|K(t,u,x)\right|\le n\|h\|_\infty$, $T_n(\omega)\ge t$. Moreover (15) proves that $T_n\to\infty$ so $\bigcup_n\Omega_n=\Omega$.

It follows using (13) that

$$\begin{aligned}\bar{\delta}(r,t,x)\mathbf{1}_{\Omega_n}&=\mathbf{1}_{\Omega_n}\left[-\int_0^t\bar{\delta}(r,u,x)K(t,u,x)\mathrm{d}Q_u+\int_0^t\bar{\delta}(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u\right]\\&=\mathbf{1}_{\Omega_n}\left[-\int_0^{t\wedge T_n}\bar{\delta}(r,u,x)K(t,u,x)\mathrm{d}Q_u+\int_0^{t\wedge T_n}\bar{\delta}(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u\right]\\&=\mathbf{1}_{\Omega_n}\left[-\int_0^t 1_{u\le T_n}\bar{\delta}(r,u,x)K(t,u,x)\mathrm{d}Q_u+\int_0^t\mathbf{1}_{u\le T_n}\bar{\delta}(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u\right].\end{aligned}$$

Taking $\bar{\Delta}_n(r,t,x)=\bar{\delta}(r,t,x)\mathbf{1}_{\Omega_n}$, we obtain

$$\bar{\Delta}_n(r,t,x)=-\int_0^t\bar{\Delta}_n(r,u,x)K(t,u,x)\mathrm{d}Q_u+\int_0^t\bar{\Delta}_n(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u. \tag{16}$$

Then

$$\begin{aligned}\mathbb{E}\left[\left|\bar{\Delta}_n(r,t,x)\right|^2\right]&\le2\mathbb{E}\left[\left|\int_0^t\bar{\Delta}_n(r,u,x)K(t,u,x)\mathrm{d}Q_u\right|^2\right]\\&\quad+2\mathbb{E}\left[\left|\int_0^t\bar{\Delta}_n(r,u,x)K(t,u,x)\Pi(h)(t,u)\mathrm{d}u\right|^2\right].\\&\le2n\|h\|_\infty^2\left(1+\|h\|_\infty^2\right)\int_0^t\mathbb{E}\left[\left|\bar{\Delta}_n(r,u,x)\right|^2\right]\mathrm{d}u\end{aligned}$$

By Gronwall's lemma, we deduce that $\bar{\Delta}_n(r,t,x)=0$ is the unique solution of (16) on the set Ω_n, so $\forall n$ $\bar{\delta}(r,t,x)\mathbf{1}_{\Omega_n}=0$. Uniqueness of solution of (6) is a consequence of $\Omega=\bigcup_n\Omega_n$. □

Proposition 3

Proof. Let be a process $S\in\mathcal{S}$ where the set of processes $\mathcal{S}$ is defined in Lemma 5 and a time *t*. Lemma 7 applied to $Y=\varphi(\tau_x)\mathbf{1}_{\tau_x>t}$ which belongs to $L^\infty\left(\Omega,\mathbb{P}^0,F_T^X\right)$ implies

$$\mathbb{E}^0\left(\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}L_TS_t\right)=\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}\right]+\mathbb{E}^0\left(\int_0^t\varphi(\tau_x)\mathbf{1}_{\tau_x>t}L_uS_u\rho_uh(X_u)\mathrm{d}u\right).$$

Conditioning by $\mathcal{F}_u^Q$ under the time integral, it follows that

$$\mathbb{E}^0\left(\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}L_T S_t\right)=\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}\right]+\mathbb{E}^0\left(\int_0^t S_u\rho_u\mathbb{E}^0\left(L_u h(X_u)\mathbf{1}_{\tau_x>t}\varphi(\tau_x)\mid\mathcal{F}_u^Q\right)\mathrm{d}u\right).$$

Conversely compute the expectation of the product of $S_t=1+\int_0^t S_u\rho_u\mathrm{d}Q_u$ by right hand of (7):

$$\begin{aligned}&\mathbb{E}^0\left[S_t\left(\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}\right]+\int_0^t\mathbb{E}^0\left(L_u h(X_u)\mathbf{1}_{\tau_x>t}\varphi(\tau_x)\mid\mathcal{F}_u^Q\right)\mathrm{d}Q_u\right)\right]\\&=\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}\right]+\mathbb{E}^0\left(\int_0^t S_u\rho_u\mathbb{E}^0\left(L_u h(X_u)\mathbf{1}_{\tau_x>t}\varphi(\tau_x)\mid\mathcal{F}_u^Q\right)\mathrm{d}u\right)\end{aligned}.$$

Since $\mathcal{S}$ is dense in $L^2\left(\Omega,\mathcal{F}^Q,\mathbb{P}^0\right)$,

$$\mathbb{E}^0\left(\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}L_T\mid\mathcal{F}_t^Q\right)=\mathbb{E}^0\left[\varphi(\tau_x)\mathbf{1}_{\tau_{x>t}}\right]+\int_0^t\mathbb{E}^0\left[L_u h(X_u)\mathbf{1}_{\tau_{x>t}}\varphi(\tau_x)\mid\mathcal{F}_u^Q\right]\mathrm{d}Q_u.$$

Finally we could replace $\mathbf{1}_{\tau_{x>t}}\varphi(\tau_x)$ by its $\mathcal{F}_u$ conditional expectation since $\mathcal{F}_u^Q\subset\mathcal{F}_u$. □

Proposition 4

Proof. Applying Lemma 4, it follows that

$$\mathbb{E}\left(\mathbf{1}_{a<\tau_x<b}\mid\mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\mid\mathcal{F}_t^Q\right)}.\tag{17}$$

But, since the condition $\int_0^t\mathbb{E}^0\left(f^2(t-u,x-X_u)\right)\mathrm{d}u<\infty$ is not necessarily satisfied, we are not able to prove that $\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\mid\mathcal{F}_t^Q\right)$ is a semi martingale (e.g. see Protter's Theorem 65 Chapter 4 [20]). This leads us to consider for $t<T\le t+1$, the expression $\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\mid\mathcal{F}_t^Q\right)$ instead of $\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t\mid\mathcal{F}_t^Q\right)$ at denominator of (17). But Lemma 7 of Appendix ensures that

$$\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\mid\mathcal{F}_t^Q\right)}=\frac{\mathbb{E}^0\left(L_t\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_t\mid\mathcal{F}_t^Q\right)}.$$

We apply Ito formula to the ratio of processes $\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_t\mid\mathcal{F}_t^Q\right)}$. For this end, we let two processes satisfying the stochastic equations respectively (9) and (11):

$$X_t=\mathbb{E}^0\left(L_t\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right),\ Y_t=\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_t\mid\mathcal{F}_t^Q\right)\text{ and }f(x,y)=\frac{x}{y}$$

The Itô's formula applied to $f(X_.,Y_.)$ from 0 to t gives us

$$\begin{aligned}\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\mid\mathcal{F}_t^Q\right)}&=\frac{\mathbb{P}^0\left(a<\tau_x<b\right)}{\mathbb{P}^0\left(\tau_x>T\right)}+\int_0^t\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right]\mid\mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_u\mid\mathcal{F}_u^Q\right)}\mathrm{d}Q_u\\&-\int_0^t\frac{\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_u^Q\right)\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(T-u,x-X_u)\mid\mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_u\mid\mathcal{F}_u^Q\right)\right]^2}\mathrm{d}Q_u\\&+\int_0^t\frac{\mathbb{E}^0\left(L_u\mathbf{1}_{a<\tau_x<b}\mid\mathcal{F}_u^Q\right)\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(T-u,x-X_u)\mid\mathcal{F}_u^Q\right)\right]^2}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_u\mid\mathcal{F}_u^Q\right)\right]^3}\mathrm{d}u\\&-\int_0^t\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)\left[G(a-u,x-X_u)-G(b-u,x-X_u)\right]\mid\mathcal{F}_u^Q\right)\times\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u}L_u h(X_u)G(T-u,x-X_u)\mid\mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_u\mid\mathcal{F}_u^Q\right)\right]^2}\mathrm{d}u.\end{aligned}$$

We achieve the proof letting $T \to t$ using the monotonous Lebesgue theorem since $\mathbf{1}_{\tau_x > T}$ increases to $\mathbf{1}_{\tau_x > t}$ when $T \to t$. □

Theorem 1

Proof. Let us now find a mixed filtering-integro-differential equation satisfied by the conditional probability density process defined from the representation

$$\mathbb{E}\left(\mathbf{1}_{a<\tau_x<b} \mid \mathcal{G}_t\right) = \int_a^b \overline{f}\left(r,t,x\right)\mathrm{d}r \quad \text{for some } a > t. \tag{18}$$

We fix a and t such that $a > t$. Let be $u \le t$, recalling the $\left(\mathbb{P}^0, \mathcal{F}\right)$-Markov property of X at point u and the fact that $\mathcal{F}^Q \subset \mathcal{F}$ justify

$$\begin{aligned}&\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_u^Q\right)\\ &= \mathbb{E}^0\left[L_u \mathbf{1}_{\tau_x>u} \mathbb{E}^0\left(\mathbf{1}_{a-u<\tau_{x-X_u}<b-u} \mid \mathcal{G}_u\right) \mid \mathcal{F}_u^Q\right].\end{aligned}$$

By definition of G, we have

$$\begin{aligned}\mathbb{E}^0\left(\mathbf{1}_{a-u<\tau_{x-X_u}<b-u} / \mathcal{G}_u\right) &= G\left(a-u, x-X_u\right) - G\left(b-u, x-X_u\right)\\ &= \int_a^b f\left(r-u, x-X_u\right)\mathrm{d}r.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_u^Q\right)\\ &= \mathbb{E}^0\left(L_u \mathbf{1}_{\tau_x>u} \int_a^b f\left(r-u, x-X_u\right)\mathrm{d}r \mid \mathcal{F}_u^Q\right).\end{aligned}$$

By Tonelli Theorem,

$$\begin{aligned}&\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_u^Q\right)\\ &= \int_a^b \mathbb{E}^0\left(L_u \mathbf{1}_{\tau_x>u} f\left(r-u, x-X_u\right) \mid \mathcal{F}_u^Q\right)\mathrm{d}r.\end{aligned}$$

Similarly

$$\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right) = \mathbb{E}^0\left(L_u \mathbf{1}_{\tau_x>u} G\left(t-u, x-X_u\right) \mid \mathcal{F}_u^Q\right).$$

In Equation (12) of Proposition 4,

$$\begin{aligned}&\mathbb{E}^0\left(L_u \mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_u^Q\right) \text{ and}\\ &\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right)\left[G\left(a-u, x-X_u\right) - G\left(b-u, x-X_u\right)\right] \mid \mathcal{F}_u^Q\right)\end{aligned}$$

are respectively replaced by

$$\begin{aligned}&\int_a^b \mathbb{E}^0\left(L_u \mathbf{1}_{\tau_x>u} f\left(r-u, x-X_u\right) \mid \mathcal{F}_u^Q\right)\mathrm{d}r\\ &\text{and } \int_a^b \mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right) f\left(r-u, x-X_u\right) \mid \mathcal{F}_u^Q\right)\mathrm{d}r.\end{aligned}$$

By hypothesis, we have $r-u \ge a-u > 0$.

For $T = t$, Lemma 8 of Appendix ensures that

$$\mathbb{E}^0\left(\int_0^t \frac{\mathrm{d}u}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u G\left(t-u, x-X_u\right) \mid \mathcal{F}_u^Q\right)\right]^2}\right) < \infty.$$

The numerators being bounded by $\|h\|_\infty L_u$, we can apply stochastic Fubini's theorem to Equation (12) Proposition 4, which can be written again as

$$\mathbb{E}\left(\mathbf{1}_{a<\tau_x<b} \mid \mathcal{G}_t\right) = \frac{1}{\mathbb{E}^0\left(\tau_x > t\right)}\int_a^b f\left(r,x\right)\mathrm{d}r + \int_a^b\int_0^t \frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right) f\left(r-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)}\mathrm{d}Q_u\mathrm{d}r$$
$$-\int_a^b\int_0^t \frac{\mathbb{E}^0\left(L_u\mathbf{1}_{\tau_x>u} f\left(r-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right)G\left(t-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)\right]^2}\mathrm{d}Q_u\mathrm{d}r$$
$$+\int_a^b\int_0^t \frac{\mathbb{E}^0\left(L_u\mathbf{1}_{\tau_x>u} f\left(r-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right)G\left(t-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)\right]^2}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)\right]^3}\mathrm{d}u\mathrm{d}r$$
$$-\int_a^b\int_0^t \mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right) f\left(r-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)\frac{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u h\left(X_u\right)G\left(t-u,x-X_u\right) \mid \mathcal{F}_u^Q\right)}{\left[\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t} L_u \mid \mathcal{F}_u^Q\right)\right]^2}\mathrm{d}u\mathrm{d}r.$$

To express this result with $\mathbb{P}$ conditional expectation instead of $\mathbb{P}^0$ conditional expectation, each fraction under the integral is multiplied and divided by the same term $\mathbb{E}^0\left(\mathbf{1}_{\tau_x>u} L_u \mid \mathcal{F}_u^Q\right)$. To manage the indicator function, we use the filtration $\left(\mathcal{G}_t, t \ge 0\right)$ since τ_x is a $\mathcal{G}$-stopping time.

Therefore, using (20) in Lemma 4, on the set $\left\{\tau_x > t\right\}$, we obtain

$$\int_a^b \overline{f}\left(r,t,x\right)\mathrm{d}r = \frac{1}{\mathbb{P}\left(\tau_x > t\right)}\int_a^b f\left(r,x\right)\mathrm{d}r + \int_a^b\int_0^t \frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u} h\left(X_u\right) f\left(r-u,x-X_u\right) \mid \mathcal{G}_u\right)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>u} G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}\mathrm{d}Q_u\mathrm{d}r$$
$$-\int_a^b\int_0^t \overline{f}\left(r,u,x\right)\frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u} h\left(X_u\right)G\left(r-u,x-X_u\right) \mid \mathcal{G}_u\right)}{\left[\mathbb{E}\left(\mathbf{1}_{\tau_x>u} G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)\right]^2}\mathrm{d}Q_u\mathrm{d}r$$
$$+\int_a^b\int_0^t \overline{f}\left(r,u,x\right)\frac{\left[\mathbb{E}\left(\mathbf{1}_{\tau_x>u} h\left(X_u\right)G\left(r-u,x-X_u\right) \mid \mathcal{G}_u\right)\right]^2}{\left[\mathbb{E}\left(\mathbf{1}_{\tau_x>u} G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)\right]^3}\mathrm{d}u\mathrm{d}r$$
$$-\int_a^b\int_0^t \frac{\mathbb{E}\left(\mathbf{1}_{\tau_x>u} h\left(X_u\right) f\left(r-u,x-X_u\right) \mid \mathcal{G}_u\right)\mathbb{E}\left(\mathbf{1}_{\tau_x>u} h\left(X_u\right)G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)}{\left[\mathbb{E}\left(\mathbf{1}_{\tau_x>u} G\left(t-u,x-X_u\right) \mid \mathcal{G}_u\right)\right]^2}\mathrm{d}u\mathrm{d}r$$

which finishes the proof. □

5. Conclusion

This paper extends the study of the first passage time for a Lévy process in [5] from complete to incomplete information and D. Dorobantu's work in [8] from intensity to conditional density. Here, we are proving the existence of the density of τ_x law given an information set, giving a stochastic differential integral equation satisfied by it and some numerical examples. All this gives us a behavior of the default time. In future works, we will be interested by the same studies in discrete time, in another kind of information set or under another process modeling the firm value.

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Appendix

Lemma 1. *Let be* μ *and* σ *real numbers and G a Gaussian random variable with mean zero and variance one, then*

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right)=\frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}}.$$

Proof. Indeed using the law of *G*, we have

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right)=\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\mu+\sigma y)^2}{4}}e^{-\frac{y^2}{2}}dy.$$

Since $(\mu+\sigma y)^2+2y^2=\left(y\sqrt{2+\sigma^2}+\frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2+\frac{2\mu^2}{2+\sigma^2}$, then

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right)=e^{-\frac{\mu^2}{2(2+\sigma^2)}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(y\sqrt{2+\sigma^2}+\frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2}{4}}dy$$

By change of variable $x=y\sqrt{2+\sigma^2}$, it follows that

$$\mathbb{E}\left(e^{-\frac{(\mu+\sigma G)^2}{4}}\right)=\frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi}}e^{-\frac{\left(x+\frac{\mu\sigma}{\sqrt{2+\sigma^2}}\right)^2}{4}}dx=\frac{\sqrt{2}e^{-\frac{\mu^2}{2(2+\sigma^2)}}}{\sqrt{2+\sigma^2}}$$

□

Lemma 2. *If* $(T_i, i\in\mathbb{N}^*)$ *is the sequence of jump time of the process N, then*

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right)<\frac{1}{\sqrt{t}}+2\lambda\sqrt{t}.$$

Proof. We have

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right)=\sum_{n\geq 0}\mathbb{E}\left(\frac{1}{\sqrt{t-T_n}}\mathbf{1}_{T_n<t<T_{n+1}}\right)=\frac{e^{-\lambda t}}{\sqrt{t}}+\sum_{n\geq 1}\mathbb{E}\left(\frac{1}{\sqrt{t-T_n}}\mathbf{1}_{T_n<t<T_n+S_1}\right)$$

where S_1 is an exponential random variable with parameter λ and independent of T_n which follows a Gamma law with parameters *n* and λ. Therefore

$$\mathbb{E}\left(\frac{1}{\sqrt{t-T_{N_t}}}\right)=\frac{e^{-\lambda t}}{\sqrt{t}}+\sum_{n\geq 1}\int_0^t\frac{1}{\sqrt{t-u}}\frac{(\lambda u)^{n-1}}{(n-1)!}\lambda e^{-\lambda u}\int_{t-u}^{+\infty}\lambda e^{-\lambda v}dvdu$$

$$\leq\frac{1}{\sqrt{t}}+\lambda e^{-\lambda t}\sum_{n\geq 1}\frac{(\lambda t)^{n-1}}{(n-1)!}\int_0^t\frac{du}{\sqrt{t-u}}=\frac{1}{\sqrt{t}}+2\lambda\sqrt{t}.$$

□

Lemma 3. *There exists some constants* $\tilde{C}$ *and C such that* $\forall t>0, x\geq 0$,

$$f(t,x)\leq +\frac{C}{t}+\frac{|m|}{\sqrt{t}}+\tilde{C}+2\lambda|m|\sqrt{t}. \quad (19)$$

Proof. The function *f* defined in (4) satisfies

$$f(t,x) \le \lambda + \mathbb{E}\left(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}\left(t - T_{N_t}, x - X_{T_{N_t}}\right)\right), \forall t > 0.$$

Using the fact that if $\tau_x > T_{N_t}$ then $x > X_{T_{N_t}}$, we have

$$\mathbb{E}\left(\mathbf{1}_{\tau_x > T_{N_t}} \tilde{f}\left(t - T_{N_t}, x - X_{T_{N_t}}\right)\right) \le \mathbb{E}\left(\mathbf{1}_{\left\{x - X_{T_{N_t}} > 0\right\}} \tilde{f}\left(t - T_{N_t}, x - X_{T_{N_t}}\right)\right).$$

Replacing $\tilde{f}$ by its expression, we obtain

$$\begin{aligned} f(t,x) &\le \lambda + \mathbb{E}\left(1_{x - X_{T_{N_t}} > 0} \frac{\left|x - X_{T_{N_t}}\right|}{\sqrt{2\pi\left(t - T_{N_t}\right)^3}} \exp\left[-\frac{\left(x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)\right)^2}{2\left(t - T_{N_t}\right)}\right]\right) \\ &\le \lambda + \mathbb{E}\left(\frac{\left[x - X_{T_{N_t}}\right]_+}{\sqrt{2\pi\left(t - T_{N_t}\right)^3}} \exp\left[-\frac{\left(x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)\right)^2}{2\left(t - T_{N_t}\right)}\right]\right) \\ &\le \lambda + \mathbb{E}\left(\frac{\left|x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)\right|}{\sqrt{2\pi\left(t - T_{N_t}\right)^3}} \exp\left[-\frac{(x - X_{T_{N_t}} - m(t - T_{N_t}))^2}{2\left(t - T_{N_t}\right)}\right]\right) + |m| \mathbb{E}\left(\frac{1}{\sqrt{2\pi\left(t - T_{N_t}\right)}}\right). \end{aligned}$$

Let $C_0 = \sup_{y \in \mathbb{R}} |y| e^{-\frac{y^2}{4}}$. We apply this bound to $y = \frac{x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)}{\sqrt{t - T_{N_t}}}$:

$$f(t,x) \le \lambda + |m| \mathbb{E}\left(\frac{1}{\sqrt{2\pi\left(t - T_{N_t}\right)}}\right) + \mathbb{E}\left(\frac{C_0}{\left(t - T_{N_t}\right)\sqrt{2\pi}} e^{-\frac{\left(x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)\right)^2}{4\left(t - T_{N_t}\right)}}\right).$$

Remark that conditionally to process N and the Y_i, the law of the random variable $\frac{x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)}{\sqrt{t - T_{N_t}}}$ is a Gaussian law with mean $\mu = \frac{x - mt - \sum_{i=1}^{N_t} Y_i}{\sqrt{t - T_{N_t}}}$ and variance $\sigma^2 = \frac{T_{N_t}}{t - T_{N_t}}$

Applying Lemma 1 we get the conditional expectation

$$\mathbb{E}\left(e^{-\frac{\left(x - X_{T_{N_t}} - m\left(t - T_{N_t}\right)\right)^2}{4\left(t - T_{N_t}\right)}} / N_t, Y_i, i = 1, \cdots, N_t\right) = \frac{\sqrt{2} e^{-\frac{\mu^2}{2\left(2 + \sigma^2\right)}}}{\sqrt{2 + \sigma^2}}.$$

Using the fact that $\sigma^2 = \frac{T_{N_t}}{t - T_{N_t}} \Rightarrow 2 + \sigma^2 = \frac{2t - T_{N_t}}{t - T_{N_t}}$, we obtain since $2 + \sigma^2 \ge \frac{t}{t - T_{N_t}}$

$$f(t,x) \le \lambda + \left(|m| + \frac{C_0}{\sqrt{t\pi}}\right) \mathbb{E}\left(\frac{1}{\sqrt{\left(t - T_{N_t}\right)}}\right).$$

The proof is completed with Lemma 2. □

The next lemma is inspired of Jeanblanc and Rutkovski [21] and Dorobantu [8].

Lemma 4. *For all* $t \in \mathbb{R}_+$, *for all a and b such that* $t < a < b$, *for all* $Y \in L^1\left(F_b, \mathbb{P}\right)$

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t} \mid \mathcal{F}_t^Q\right)>0,\ \mathbb{E}\left(Y\mathbf{1}_{t<\tau_x} \mid \mathcal{G}_t\right)=1_{\tau_x>t}\frac{\mathbb{E}^0\left(L_b Y\mathbf{1}_{t<\tau_x} \mid \mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_t^Q\right)}. \tag{20}$$

For instance with $Y=\mathbf{1}_{a<\tau_x<b}$, we get

$$\mathbb{E}\left(\mathbf{1}_{a<\tau_x<b} \mid \mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{a<\tau_x<b} \mid \mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_t^Q\right)}.$$

Proof. Assume that there exists t_0 such that $\mathbb{P}(\tau_x>t_0)=0$. Then for all $t\geq t_0, \mathbb{P}(\tau_x\leq t_0)=1$. It follows that the density function of τ_x f, defined in (4), is the zero function on $[t_0,+\infty[$. This means that $\forall t\in[t_0,\infty[$,

$$f(t,x)=\lambda\mathbb{E}\left(\mathbf{1}_{\tau_x>t}(1-F_Y)(x-X_t)\right)+\mathbb{E}\left(\mathbf{1}_{\tau_x>T_{N_t}}\tilde{f}\left(t-T_{N_t},x-X_{T_{N_t}}\right)\right)=0 \quad \mathbb{P}-a.s.$$

Then, $\mathbb{P}(\tau_x\leq t)=1$ implies that $\mathbb{E}\left(\mathbf{1}_{\tau_x>t}(\mathbf{1}-F_Y)(x-X_t)\right)=0$.

Thus $\mathbb{E}\left(\mathbf{1}_{\tau_x>T_{N_t}}\tilde{f}\left(t-T_{N_t},x-X_{T_{N_t}}\right)\right)=0$. But we have $t-T_{N_t}>0$ $\mathbb{P}-a.s$ and on the set $\{\tau_x>T_{N_t}\}$, $x-X_{T_{N_t}}>0$. Therefore, $\tilde{f}\left(t-T_{N_t},x-X_{T_{N_t}}\right)>0$ for all $t\geq t_0$ Hence, we obtain $\mathbf{1}_{\tau_x>T_{N_t}}=0, \forall t\geq t_0$ what is not possible. Indeed,

$$\mathbf{1}_{\tau_x>T_{N_t}}=0 \Leftrightarrow \sum_{n\geq 0}\mathbf{1}_{\tau_x>T_n}\mathbf{1}_{N_t=n}=0$$

That means for all $n\in\mathbb{N}, \mathbb{P}(T_n<t<T_{n+1},\tau_x>T_n)=0$. In particular, for $n=0$,

$$\mathbb{P}(T_1>t,\tilde{\tau}_x>0)=\mathbb{P}(\tilde{\tau}_x>0)\mathbb{P}(T_1>t)=\mathrm{e}^{\lambda t}\neq 0.$$

Thus for any t, t, $\mathbb{P}(\tau_x>t)>0$ and $\mathbb{E}\left(\mathbf{1}_{\tau_x>t} \mid \mathcal{F}_t^Q\right)>0$.

On the set $\{\tau_x>t\}$, any $\mathcal{G}_t$-measurable random variable coincides with some $\mathcal{F}_t^Q$-measurable random variable (cf. Jeanblanc and Rutkovski [21] p. 18). Then for all $Y\in L^1(F_b,\mathbb{P})$, there exists a $\mathcal{F}_t^Q$-measurable random variable Z such that

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t}Y \mid \mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}Z.$$

Taking the conditional expectation with respect to $\mathcal{F}_t^Q$, we get

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t}Y \mid \mathcal{F}_t^Q\right)=Z\mathbb{E}\left(\tau_x>t \mid \mathcal{F}_t^Q\right).$$

This implies that

$$\mathbb{E}\left(\mathbf{1}_{\tau_x>t}Y \mid \mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}\frac{\mathbb{E}\left(Y\mathbf{1}_{\tau_x>t} \mid \mathcal{F}_t^Q\right)}{\mathbb{E}\left(\mathbf{1}_{\tau_x>t} \mid \mathcal{F}_t^Q\right)}.$$

Using Kallianpur-Striebel formula (see Pardoux [13]) and $\mathbb{E}^0\left(L_b \mid \mathcal{F}_t^Q\right)=L_t$ we obtain

$$\mathbb{E}\left(Y\mathbf{1}_{\tau_x>t} \mid \mathcal{G}_t\right)=\mathbf{1}_{\tau_x>t}\frac{\mathbb{E}^0\left(L_b\mathbf{1}_{\tau_x>t}Y \mid \mathcal{F}_t^Q\right)}{\mathbb{E}^0\left(\mathbf{1}_{\tau_x>t}L_t \mid \mathcal{F}_t^Q\right)}.$$

□

The following is in [14].

Lemma 5. *The family of* $\mathcal{F}^Q$ *adapted processes*

$$\mathcal{S}=\left\{S_\cdot=\exp\left(\int_0^\cdot\rho_s\mathrm{d}Q_s-\frac{1}{2}\int_0^t\rho_s^2\mathrm{d}s\right),\rho\in L^2\left([0,T],\mathbb{R}\right)\right\}$$

is total in the set of processes taking their values in $L^2\left(\Omega,\mathcal{F}_t^Q,\mathbb{P}^0\right)$.

Let us denote by $\mathcal{F}^W$ (resp. $\mathcal{F}^N$ and $\mathcal{F}^X$) the completed, right continuous filtration generated by *W*, (resp. *N* or *X*)

Lemma 6. *Let* $\{U_t,t\geq 0\}$ *be an* $\mathcal{F}^W\otimes\mathcal{F}^N$*-progressively measurable process such that for all* $t\geq 0$*, we have*

$$\mathbb{E}^0\left[\int_0^t U_s^2\mathrm{d}s\right]<+\infty.$$

Then

$$\mathbb{E}^0\left[\int_0^t U_s\mathrm{d}Q_s\mid\mathcal{F}_t^W\otimes\mathcal{F}_t^N\right]=0. \tag{21}$$

Proof. As in Lemma 5, the family of processes

$$\mathcal{R}=\left\{r_\cdot=\mathcal{E}\left[\int_0^\cdot\gamma_s\mathrm{d}W_s+\int_0^\cdot\int_A\left(\mathrm{e}^{\beta_{s-}(x)}-1\right)\tilde{N}(\mathrm{d}s\mathrm{d}x)\right],\gamma\in L^2\left([0,T],\mathbb{R}\right),\beta\in L^\infty\left([0,T]\times A,\mathbb{R}\right)\right\}$$

is total in the set of processes taking their values in $L^2\left(\Omega,\mathcal{F}^W\otimes\mathcal{F}^N,\mathbb{P}^0\right)$, where $\tilde{N}$ is the compensated Poisson random measure on $\mathbb{R}\times\mathbb{R}$ and $A\subset\mathbb{R}$ is a Borel set.

Therefore, since $r_t=1+\int_0^t r_s\gamma_s\mathrm{d}W_s+\int_0^t\int_A r_{s-}\left(\mathrm{e}^{\beta_{s-}(x)}-1\right)\tilde{N}(\mathrm{d}s\mathrm{d}x)$, by Itô's formula, we have

$$\begin{aligned}\mathbb{E}^0\left(r_t\mathbb{E}^0\left[\int_0^t U_s\mathrm{d}Q_s\mid\mathcal{F}_t^W\otimes\mathcal{F}_t^N\right]\right)&=\mathbb{E}^0\left[r_t\int_0^t U_s\mathrm{d}Q_s\right]\\&=\mathbb{E}^0\left[\int_0^t r_{s-}\gamma_sU_s\mathrm{d}\langle W,Q\rangle_s\right]+\mathbb{E}^0\left[\int_0^t U_s\int_A r_{s-}\left(\mathrm{e}^{\beta_s(x)}-1\right)\mathrm{d}\langle\tilde{N},Q\rangle_s\right]=0.\end{aligned}$$

The equality is obtained from the fact that under $\mathbb{P}^0$, $\langle Q,W\rangle=\langle Q,\tilde{N}\rangle=0$ by independence. □

Lemma 7. *Let be a process* $S\in\mathcal{S}$ *such that for any t* $S_t=\exp\left(\int_0^t\rho_s\mathrm{d}Q_s-\frac{1}{2}\int_0^t\rho_s^2\mathrm{d}s\right)$, $\rho\in L^2\left([0,t],\mathbb{R}\right)$. *Let* $Y\in L^\infty\left(\Omega,\mathbb{P},\mathcal{F}_T^X\right)$ *and* $T\geq t$*, then*

$$\mathbb{E}^0\left(YL_TS_t\right)=\mathbb{E}^0(Y)+\mathbb{E}^0\left(\int_0^t\mathbb{E}^0\left(Y/\mathcal{F}_u\right)S_u\rho_uL_uh(X_u)\mathrm{d}u\right)$$

and

$$\mathbb{E}^0\left(YL_T\mid\mathcal{F}_t^Q\right)=\mathbb{E}^0\left(YL_t\mid\mathcal{F}_t^Q\right);\ \mathbb{E}^0\left(YL_T\mid\mathcal{F}_t\right)=\mathbb{E}^0\left(YL_t\mid\mathcal{F}_t\right). \tag{22}$$

For instance

$$\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_T\mid\mathcal{F}_t\right)=\mathbb{E}^0\left(\mathbf{1}_{\tau_x>T}L_t\mid\mathcal{F}_t\right).$$

Proof. Let be $S_\cdot=\int_0^\cdot S_u\rho_u\mathrm{d}Q_u\in\mathcal{S}$, $t\leq T$ and let us define the process *K*

$$K_\cdot=1+\int_0^\cdot S_u\rho_u\mathrm{d}Q_u.$$

The integration by parts Itô formula applied to the product $L_\cdot K_\cdot$ between 0 and *T* yields

$$L_TK_T=1+\int_0^T L_uS_u\rho_u dQ_u+\int_0^T\mathbf{1}_{u\leq t}K_uL_uh(X_u)\mathrm{d}Q_u+\int_0^T\mathbf{1}_{u\leq t}S_uL_u\rho_uh(X_u)\mathrm{d}u$$

and remark that $L_TK_T=L_TS_t$.

Since *X* and *Q* are independent under $\mathbb{P}^0$, we use Lemma 6 and it follows

$$\begin{aligned}\mathbb{E}^0\left(YL_TS_t\right)&=\mathbb{E}^0(Y)+\mathbb{E}^0\left(\int_0^{t\wedge T}\mathbb{E}^0\left(Y/\mathcal{F}_u\right)S_u\rho_uL_uh(X_u)\mathrm{d}u\right)\\&=\mathbb{E}^0(Y)+\mathbb{E}^0\left(\int_0^t YS_u\rho_uL_uh(X_u)\mathrm{d}u\right).\end{aligned} \tag{23}$$

Similarly, using first $\mathbb{E}^0\left[YL_tS_t\right]=\mathbb{E}^0\left[\mathbb{E}^0\left(Y/\mathcal{F}_t\right)L_tS_t\right]$, Itô's formula on product of processes $\mathbb{E}^0\left(Y/\mathcal{F}_.\right)L_.S_.$ and the independence between X and Q under $\mathbb{P}^0$ yields

$$\mathbb{E}^0\left(YL_tS_t\right)=\mathbb{E}^0\left(Y\right)+\mathbb{E}^0\left(\int_0^t YS_u\rho_uL_uh\left(X_u\right)\mathrm{d}u\right). \tag{24}$$

Equations (23) and (24) imply that

$$\mathbb{E}^0\left(YL_T\mid\mathcal{F}_t^Q\right)=\mathbb{E}^0\left(YL_t\mid\mathcal{F}_t^Q\right).$$

Now let be $f_t\left(X\right)\in L^\infty\left(\Omega,\mathbb{P}^0,\mathcal{F}_t^X\right)$ and apply the above equality to $Yf_t\left(X\right)$:

$$\mathbb{E}^0\left(Yf_t\left(X\right)L_T\mid\mathcal{F}_t^Q\right)=\mathbb{E}^0\left(Yf_t\left(X\right)L_t\mid\mathcal{F}_t^Q\right)$$

so

$$\mathbb{E}^0\left(Yf_t\left(X\right)L_TS_t\right)=\mathbb{E}^0\left(Yf_t\left(X\right)L_tS_t\right)$$

which concludes the proof. □

Lemma 8. *For all* $T\ge t$, $\forall\alpha>0$, $\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right)>0$ *almost surely and*

$$\mathbb{E}^0\left(\frac{1}{\left[\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right)\right]^\alpha}\right)\le\mathbb{P}^0\left(\tau_x>T\right)^{-\alpha}\mathrm{e}^{\frac{\alpha(\alpha+1)}{2}t\|h\|_\infty^2}.$$

Proof. The process $\left(\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right),t\le T\right)$ is a positive $\mathcal{F}^Q$ (upper) martingale, which converges to the non null random variable $\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_T^Q\right)$ (see Lemma 4) then it never vanishes.

From Corollary 2 (i), the process $M=\left(\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right),t\le T\right)$ is a $\left(\mathbb{P}^0,\mathcal{F}^Q\right)$ martingale with decomposition

$$\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right)=\mathbb{P}^0\left(\tau_x>T\right)+\int_0^t\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_uh\left(X_u\right)\mid\mathcal{F}_u^Q\right)\mathrm{d}Q_u.$$

Let $R_n=\inf\left\{t>0,\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_t^Q\right)<\frac{1}{n}\right\}$, using Itô's formula for $x\mapsto x^{-\alpha}$ between 0 and $t\wedge R_n$ and taking the expectation we derive

$$\begin{aligned}&\mathbb{E}^0\left[\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_{t\wedge R_n}^Q\right)^{-\alpha}\right]\\&=\mathbb{P}^0\left(\tau_x>T\right)^{-\alpha}+\frac{\alpha\left(\alpha+1\right)}{2}\mathbb{E}\left[\int_0^{t\wedge R_n}\frac{\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_uh\left(X_u\right)\mid\mathcal{F}_u^Q\right)^2}{\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_u\mid\mathcal{F}_u^Q\right)^{\alpha+2}}\mathrm{d}u\right]\\&\le\mathbb{P}^0\left(\tau_x>T\right)^{-\alpha}+\frac{\alpha\left(\alpha+1\right)}{2}\|h\|_\infty^2\int_0^t\mathbb{E}^0\left[\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_{u\wedge R_n}^Q\right)^{-\alpha}\right]\mathrm{d}u.\end{aligned}$$

Using Gronwall's Lemma

$$\mathbb{E}^0\left(\frac{1}{\mathbb{E}^0\left(\mathbf{1}_{\{\tau_x>T\}}L_T\mid\mathcal{F}_{u\wedge R_n}^Q\right)^\alpha}\right)\le\mathbb{E}^0\left(\tau_x>T\right)^{-\alpha}\mathrm{e}^{\frac{\alpha(\alpha+1)}{2}t\|h\|_\infty^2}.$$

The proof of Lemma 8 is achieved by letting n going to infinity. □