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H_∞ Optimal Control Problems for Jump Linear Equations

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Abstract

We consider a set of continuous algebraic Riccati equations with indefinite quadratic parts that arise in H_∞ control problems. It is well known that the approach for solving such type of equations is proposed in the literature. Two matrix sequences are constructed. Three effective methods are described for computing the matrices of the second sequence, where each matrix is the stabilizing solution of the set of Riccati equations with definite quadratic parts. The acceleration modifications of the described methods are presented and applied. Computer realizations of the presented methods are numerically compared. In addition, a second iterative method is proposed. It constructs one matrix sequence which converges to the stabilizing solution to the given set of Riccati equations with indefinite quadratic parts. The convergence properties of the second method are commented. The iterative methods are numerically compared and investigated.

Keywords

H_∞ Optimal Control Problem, Generalized Riccati Equation, Indefinite Sign, Stabilizing Solution

1. Introduction

Recently the algebraic Riccati equations with indefinite quadratic part have been investigated intensively. The paper of Lanzon *et al*. [1] is the first where is investigated an algebraic Riccati equation with an indefinite quadratic part in the deterministic case. Further on, the Lanzon's approach has been extended and applied to the algebraic Riccati equations of different types [2]-[5] and for the stochastic case [6]. Many situations in management, economics and finance [7]-[9] are characterized by multiple decision makers/players who can enforce the decisions that have enduring consequences. The similar game models lead us to the solution of the Riccati equations with an indefinite quadratic part. The findings in [8] show how to model economic and financial applications using a discrete-time H_∞-approach to simulate optimal solutions under a flexible choice of system parameters. Here, a continuous H_∞-approach to jump linear equations is studied and investigated.

More precisely, how to find the stabilizing solution of the coupled algebraic Riccati equations of the optimal control problem for jump linear systems with indefinite quadratic part:

$$\begin{aligned}&\left(A_0(i)\right)^{\mathrm{T}} X(i)+X(i) A_0(i)+\sum_{k=1}^{r} A_k^{\mathrm{T}}(i) X(i) A_k(i)+\sum_{j=1}^{N} \lambda_{ij} X(j)\\&-X(i)\left(B_2(i) B_2^{\mathrm{T}}(i)-\gamma^{-2} B_1(i)\left(B_1(i)\right)^{\mathrm{T}}\right) X(i)+C^{\mathrm{T}}(i) C(i)=0, \quad i=1,\cdots,N,\end{aligned} \tag{1}$$

is considered. In the above equations the matrix coefficients $A_0(i), A_1(i),\cdots,A_r(i)$ are $n\times n$ real matrices, $B_1(i), B_2(i)$ are $n\times m$ real matrices, $C(i)$ is a $p\times n$ real matrix and the unknown $X(i)=\left(X(i)\right)^{\mathrm{T}}$ is a symmetric $n\times n$ matrix $\left(i=1,\cdots,N\right)$. The considered set of Riccati Equation (1) is connected to the stochastic controlled system with the continuous Markov process (see [2]), which is called a H_∞ control problem. The parameter $\gamma>0$ presents a level of attenuation of the corresponding H_∞ control problem. In order to solve a given H_∞ control problem, we have to find the control $\tilde{u}(t)$ which is given by

$$\tilde{u}(t)=-\left(B_2(\eta_t)\right)^{\mathrm{T}} \tilde{X}(\eta_t) x(t),$$

where $\{\eta_t\}_{t\ge 0}$ is a right continuous Markov process and $\tilde{\boldsymbol{X}}=\left(\tilde{X}(1),\cdots,\tilde{X}(N)\right)$ is the stabilizing solution to (1) (see [2]).

The stabilizing solution of the considered game theoretic Riccati equation is obtained as a limit of a sequence of approximations constructed based on stabilizing solutions of a sequence of algebraic Riccati equations of stochastic control with definite sign of the quadratic part. The main idea is to construct two matrix sequences such that the sum of corresponding matrices converges to the stabilizing solution of the set of Riccati Equation (1). Such approach is considered in [2]. The properties of this approach are considered in terms of the concept of mean square stabilizability and the assumption that the convex set $\mathcal{A}(\gamma)$ is not empty (see Dragan and coauthors in [2]).

Here we introduce the sufficient conditions for the existence of stabilizing solutions of the set of Riccati Equation (1). We will prove under these conditions some convergence properties of constructed matrix sequences in terms of perturbed Lyapunov matrix equations. In addition, we introduce a second iterative method where we construct one matrix sequence. We show that the second iterative method constructs a convergent matrix sequence. Moreover, if the sufficient conditions of the first approach are satisfied then the second iterative method converges.

2. Preliminary Facts

The notation $\mathcal{H}^n$ stands for the linear space of symmetric matrices of size n over the field of real numbers. For any $X, Y\in\mathcal{H}^n$, we write $X>Y$ or $X\ge Y$ if $X-Y$ is positive definite or $X-Y$ is positive semidefinite. We use notation $\boldsymbol{X}=\left(X(1),\cdots,X(N)\right)$. The notations $\boldsymbol{X}\in\mathcal{H}^n$ and the inequality $\boldsymbol{X}\ge\boldsymbol{Z}$ mean that for $i=1,\cdots,N,\ X(i)\in\mathcal{H}^n$ and $X(i)\ge Z(i)$, respectively. The linear space $\mathcal{H}^n$ is a Hilbert space with the Frobenius inner product $\langle X, Y\rangle=\mathbf{trace}(XY)$. A linear operator $\mathcal{L}$ on $\mathcal{H}^n$ is called asymptotically stable if the eigenvalues to $\mathcal{L}$ lie in the open left half plane and almost asymptotically stable if the eigenvalues to $\mathcal{L}$ lie in the closed left half plane.

We denote $\boldsymbol{X}=\left(X(1),\cdots,X(N)\right)$ and define the matrix function $\mathcal{R}_i(\boldsymbol{X})$ as follows:

$$\begin{aligned}\mathcal{R}_i(\boldsymbol{X}):=&\left(A_0(i)\right)^{\mathrm{T}} X(i)+X(i) A_0(i)+\sum_{l=1}^{r} A_l^{\mathrm{T}}(i) X(i) A_l(i)+\sum_{j=1}^{N} \lambda_{ij} X(j)\\&-X(i)\left(B_2(i)\left(B_2(i)\right)^{\mathrm{T}}-\gamma^{-2} B_1(i)\left(B_1(i)\right)^{\mathrm{T}}\right) X(i)+C^{\mathrm{T}}(i) C(i), \quad i=1,\cdots,N.\end{aligned} \tag{2}$$

We will rewrite the function $\mathcal{R}_i(\boldsymbol{X}), i=1,\cdots,N$ in the form:

$$\begin{aligned}\mathcal{R}_i(\boldsymbol{X}):=&\left(A(i)\right)^{\mathrm{T}} X(i)+X(i) A(i)+\Pi_i(X)+C^{\mathrm{T}}(i) C(i)\\&-X(i)\left(B_2(i)\left(B_2(i)\right)^{\mathrm{T}}-\gamma^{-2} B_1(i)\left(B_1(i)\right)^{\mathrm{T}}\right) X(i), \quad i=1,\cdots,N.\end{aligned} \tag{3}$$

where $A(i)=A_0(i)+0.5\lambda_{ii}I_n$ and $\Pi_i(X)=\sum_{l=1}^{r}A_l^{\mathrm{T}}(i)X(i)A_l(i)+\sum_{j\neq i}\lambda_{ij}X(j)$.

Note that transition coefficients $\lambda_{ij}\geq 0$ if $i\neq j$ and $\sum_j\lambda_{ij}=0$ for all *i*. Thus if $X(j)\geq 0$, we have $\Pi_i(X)\geq 0, i=1,\cdots,N$.

We introduce the following perturbed Lyapunov operator

$$\mathcal{L}_{A(i);\Pi_i}(\boldsymbol{X})=(A(i))^{\mathrm{T}}X(i)+X(i)A(i)+\Pi_i(\boldsymbol{X}),\quad i=1,\cdots,N,$$

and will present the solvability of (1) through properties if the perturbed Lyapunov operator.

Proposition 1: [10] The following are equivalent:

1) The matrix $\boldsymbol{X}=(X(1),\cdots,X(N))$ is the stabilizing solution to (1);

2) The perturbed Lyapunov operator $\mathcal{L}_{\tilde{A}_i;\Pi_i}, i=1,\cdots,N$ is asymptotically stable where:

$$\begin{cases}F_1(i)=\gamma^{-2}(B_1(i))^{\mathrm{T}}X(i),\ F_2(i)=-(B_2(i))^{\mathrm{T}}X(i),\\ \tilde{A}_i=A(i)+B_1(i)F_1(i)+B_2(i)F_2(i).\end{cases}$$

The above proposition presents a deterministic characterization of a stabilizing solution to set of Riccati Equation (1).

A matrix $\boldsymbol{X}\in\mathcal{H}^n$ is called stabilizing for $\mathcal{L}_{\tilde{A}_i;\Pi_i}, i=1,\cdots,N$ if eigenvalues of $\mathcal{L}_{\tilde{A}_i;\Pi_i}(\boldsymbol{X}), i=1,\cdots,N$ lie in the open left half plane. In order words the stabilizing $\tilde{\boldsymbol{X}}$ to (1) stabilizes the operators $\mathcal{L}_{\tilde{A}_i;\Pi_i}, i=1,\cdots,N$.

Knowing the stabilizing solution $\tilde{\boldsymbol{X}}=(\tilde{X}(1),\cdots,\tilde{X}(N))$ to (1) we consider $\tilde{F}_1(i)=\gamma^{-2}(B_1(i))^{\mathrm{T}}\tilde{X}(i)$ and $\tilde{F}_2(i)=-(B_2(i))^{\mathrm{T}}\tilde{X}(i)$ and therefore the matrix $\tilde{A}(i)=A(i)+B_1(i)\tilde{F}_1(i)+B_2(i)\tilde{F}_2(i)$ builds a perturbed Lyapunov operator which is asymptotically stable.

Dragan *et al*. [2] have introduced the following iteration scheme for finding the stabilizing solution to set of algebraic Riccati Equation (1). They construct two matrix sequences $\{\boldsymbol{X}^{(k)}\}$ and $\{\boldsymbol{Z}^{(k)}\}, k=0,1,\cdots$ as follows:

$$X^{(k+1)}(i)=X^{(k)}(i)+Z^{(k)}(i),\ \text{ with } X^{(0)}(i)=0,\ i=1,\cdots,N,\ k=0,1,2,\cdots. \tag{4}$$

Each matrix $Z^{(k)}(i), i=1,\cdots,N, k=0,1,2,\cdots$ is computed as the stabilizing solution of the algebraic Riccati equation with definite quadratic part:

$$\begin{aligned}&\mathcal{G}_i(Z^{(k)})(A^{(k)}(i))^{\mathrm{T}}Z^{(k)}(i)+Z^{(k)}(i)A^{(k)}(i)+\Pi_i(\boldsymbol{Z}^{(k)})+\mathcal{R}_i(\boldsymbol{X}^{(k)})\\ &-Z^{(k)}(i)B_2(i)B_2^{\mathrm{T}}(i)Z^{(k)}(i)=0,\quad i=1,\cdots,N.\end{aligned} \tag{5}$$

where

$$\begin{cases}F_1(X^{(k)}(i))=\gamma^{-2}(B_1(i))^{\mathrm{T}}X^{(k)}(i),\\ F_2(X^{(k)}(i))=-(B_2(i))^{\mathrm{T}}X^{(k)}(i),\\ A^{(k)}(i)=A(i)+B_1(i)F_1(X^{(k)}(i))+B_2(i)F_2(X^{(k)}(i)).\end{cases}$$

However, it is not explained in [2] how Equation (5) has to be solved.

In our investigation we present a few iterative methods for finding the stabilizing solution to (5). Convergence properties of the matrix sequence $\{\boldsymbol{X}^{(k)}\}, k=0,1,\cdots$ will be derived. A second iterative method is derived. The second aim of the paper is to provide a short numerically survey on iterative methods for computing the stabilizing solution to the given set of Riccati equations. Results from the numerical comparison are given on a family of numerical examples.

Lemma 1. *For the map* $\mathcal{R}_i(\boldsymbol{X}), i=1,\cdots,N$ *the following identities are valid:*

$$\text{i)}\quad \begin{aligned} \mathcal{R}_i(\boldsymbol{X}+\boldsymbol{Z}) &= \mathcal{R}_i(\boldsymbol{X}) + \left(A_i(X(i))\right)^{\mathrm{T}} Z(i) + Z(i) A_i(X(i)) + \Pi_i(\boldsymbol{Z}) \\ &\quad + \gamma^{-2} Z(i) B_1(i)\left(B_1(i)\right)^{\mathrm{T}} Z(i) - Z(i) B_2(i)\left(B_2(i)\right)^{\mathrm{T}} Z(i),\ i=1,\cdots,N \\ &\text{where } A_i(X(i)) = A(i) + B_1(i) F_1(X(i)) + B_2(i) F_2(X(i)), \\ &F_1(X(i)) = \gamma^{-2}\left(B_1(i)\right)^{\mathrm{T}} X(i),\quad F_2(X(i)) = -\left(B_2(i)\right)^{\mathrm{T}} X(i), \end{aligned} \tag{6}$$

for any symmetric matrices $\boldsymbol{X}, \boldsymbol{Z}$.

$$\text{ii)}\quad \begin{aligned} \mathcal{R}_i(W,V,\boldsymbol{X}) &= A_i(W,V)^{\mathrm{T}} X(i) + X(i) A_i(W,V) + C(i)^{\mathrm{T}} C(i) - \gamma^2 W^{\mathrm{T}} W + V^{\mathrm{T}} V \\ &\quad - \left(F_2(X(i)) - V\right)^{\mathrm{T}} \left(F_2(X(i)) - V\right) + \Pi_i(\boldsymbol{X}) \\ &\quad + \gamma^2 \left(F_1(X(i)) - W\right)^{\mathrm{T}} \left(F_1(X(i)) - W\right) \end{aligned} \tag{7}$$

with $A_i(W,V) = A(i) + B_1(i) W + B_2(i) V, W \in \mathbb{R}^{m\times n}, V \in \mathbb{R}^{m\times n}$.

Proof. The statements of Lemma 1 are verified by direct manipulations. □

Lemma 2. *Assume there exist positive definite symmetric matrices* $\boldsymbol{X}, \boldsymbol{Z}, \hat{\boldsymbol{X}}$ *with* $\hat{\boldsymbol{X}} \geq \boldsymbol{X}, \mathcal{R}_i(\hat{\boldsymbol{X}}) \leq 0, (i=1,\cdots,N)$ *and* $\boldsymbol{Z}$ *is the stabilizing solution to*

$$0 = \left(A_i(X(i))\right)^{\mathrm{T}} Z(i) + Z(i) A_i(X(i)) + \Pi_i(\boldsymbol{Z}) + \mathcal{R}_i(\boldsymbol{X}) - Z(i) B_2(i)\left(B_2(i)\right)^{\mathrm{T}} Z(i),\ i=1,\cdots,N.$$

Then

i) if $\mathcal{L}_{\tilde{A}_i(X(i),\hat{X}(i));\Pi_i}$ is asymptotically stable for $i=1,\cdots,N$ with $\tilde{A}_i(X(i),\hat{X}(i)) = A(i) + B_1(i) F_1(X(i)) + B_2(i) F_2(\hat{X}(i))$ then $\hat{\boldsymbol{X}} - \boldsymbol{X} - \boldsymbol{Z} \geq 0$;

ii) if $\hat{X}(i) - X(i) - Z(i) \geq 0, i=1,\cdots,N$ then the Lyapunov operator $\mathcal{L}_{\breve{A}_i(\boldsymbol{X}+\boldsymbol{Z},\hat{\boldsymbol{X}});\Pi_i}$ is asymptotically stable for $i=1,\cdots,N$.

Proof. Assume the index i is fixed. We have $(\hat{Q}(i) \geq 0)$ $\mathcal{R}_i(\hat{\boldsymbol{X}}) + \hat{Q}(i) = 0$. Applying some matrix manipulations we obtain the equation:

$$\begin{aligned} 0 &= \mathcal{L}_{\tilde{A}_i(X(i),\hat{X}(i));\Pi_i}(\hat{\boldsymbol{X}} - \boldsymbol{X} - \boldsymbol{Z}) + \hat{Q}(i) \\ &\quad + \left[\hat{X}(i) - X(i) - Z(i)\right] B_2(i)\left(B_2(i)\right)^{\mathrm{T}} \left[\hat{X}(i) - X(i) - Z(i)\right] \\ &\quad + \gamma^{-2}\left(\hat{X}(i) - X(i)\right) B_1(i)\left(B_1(i)\right)^{\mathrm{T}} \left(\hat{X}(i) - X(i)\right). \end{aligned}$$

Thus $\hat{\boldsymbol{X}} - \boldsymbol{X} - \boldsymbol{Z} \geq 0$. The statement 1) is proved.

In order to prove the statement 2) we derive:

$$\begin{aligned} 0 &= \mathcal{L}_{\breve{A}_i(\boldsymbol{X}+\boldsymbol{Z},\hat{\boldsymbol{X}});\Pi_i}(\hat{\boldsymbol{X}} - \boldsymbol{X} - \boldsymbol{Z}) \\ &\quad + \left[\hat{X}(i) - X(i) - Z(i)\right] B_2(i)\left(B_2(i)\right)^{\mathrm{T}} \left[\hat{X}(i) - X(i) - Z(i)\right] \\ &\quad + \hat{Q}(i) + \gamma^{-2} Z(i) B_1(i) B_1(i)^{\mathrm{T}} Z(i) \\ &\quad + \gamma^{-2}\left(\hat{X}(i) - X(i) - Z(i)\right) B_1(i) B_1(i)^{\mathrm{T}} \left(\hat{X}(i) - X(i) - Z(i)\right). \end{aligned} \tag{8}$$

Since the matrices $\hat{X}(i) - X(i) - Z(i)$ and $\hat{Q}(i) + \gamma^{-2} Z(i) B_1(i) B_1(i)^{\mathrm{T}} Z(i), i=1,\cdots,N$ are positive definite then the Lyapunov operator $\mathcal{L}_{\breve{A}_i(\boldsymbol{X}+\boldsymbol{Z},\hat{\boldsymbol{X}});\Pi_i}$ is asymptotically stable for $i=1,\cdots,N$ because Riccati Equation (8) has the stabilizing positive semidefinte solution.

$$\left(\breve{A}_i(\boldsymbol{X}+\boldsymbol{Z},\hat{\boldsymbol{X}}) = A(i) + B_1(i) F_1(X(i) + Z(i)) + B_2(i) F_{2,i}(\hat{X})\right).$$

The lemma is proved.

3. Iterative Methods

In this section we are proving the some convergence properties of the matrix sequences $\{\boldsymbol{X}^{(k)}\}$ and $\{\boldsymbol{Z}^{(k)}\}, k=0,1,\cdots$ defined by iterative loop (4)-(5). We present the main theorem where the convergence properties for matrix sequences are derived.

Theorem 1. *Assume there exist symmetric matrices* $\hat{\boldsymbol{X}}$ *and* $\boldsymbol{X}^{(0)}$ *such that* $\mathcal{R}_i\left(\boldsymbol{X}^{(0)}\right)\geq 0$ *and* $\mathcal{R}_i\left(\hat{\boldsymbol{X}}\right)\leq 0$ *and* $\boldsymbol{X}^{(0)}\leq\hat{\boldsymbol{X}}$, *and the Lyapunov operator* $\mathcal{L}_{A^{(0)}(i);\Pi_i}, i=1,\cdots,N$ *is asymptotically stable. Then for the matrix sequences* $\left\{\boldsymbol{X}^{(k)}\right\}_{k=0}^{\infty},\left\{\boldsymbol{Z}^{(k)}\right\}_{k=0}^{\infty}$ *defined as the stabilizing solution of* (5) *satisfy*

i) The Lyapunov operator $\mathcal{L}_{A^{(k)}(i);\Pi_i}$ is asymptotically stable $i=1,\cdots,N$;

ii) $\mathcal{R}_i\left(\boldsymbol{X}^{(k+1)}\right)=\gamma^{-2}Z^{(k)}(i)B_1(i)\left(B_1(i)\right)^{\mathrm{T}}Z^{(k)}(i)\geq 0, i=1,\cdots,N$;

iii) The Lyapunov operator $\mathcal{L}_{\tilde{A}^{(k)}(i);\Pi_i}$ is asymptotically stable where

$\tilde{A}^{(k)}(i)=A(i)+B_1(i)F_1\left(X^{(k)}(i)\right)+B_2(i)F_2\left(X^{(k+1)}(i)\right), i=1,\cdots,N, k=0,1,\cdots$;

iv) $\hat{\boldsymbol{X}}\geq\boldsymbol{X}^{(k+1)}\geq\boldsymbol{X}^{(k)}\geq 0$ for $k=0,1,\cdots$.

Proof. The algorithm begins with $X^{(0)}(i)=0, i=1,\cdots,N$. Then $\mathcal{R}_i\left(\boldsymbol{X}^{(0)}\right)=C(i)^{\mathrm{T}}C(i)\geq 0$. The matrix $Z^{(0)}(i)$ is a solution of the Riccati equation:

$$\begin{aligned}&\left(A(i)\right)^{\mathrm{T}}Z^{(0)}(i)+Z^{(0)}(i)A(i)+\Pi_i\left(\boldsymbol{Z}^{(0)}\right)+C(i)^{\mathrm{T}}C(i)\\&-Z^{(0)}(i)B_2(i)B_2^{\mathrm{T}}(i)Z^{(0)}(i)=0,\quad i=1,\cdots,N.\end{aligned}\tag{9}$$

Under the assumption the Lyapunov operator $\mathcal{L}_{A^{(0)}(i);\Pi_i}, i=1,\cdots,N$ is asymptotically stable $\left(A^{(0)}(i)=A(i)\right)$. Thus, $Z^{(0)}(i)$ is the unique stabilizing solution of the above Riccati equation and $Z^{(0)}(i)\geq 0, i=1,\cdots,N$.

Using Lemma 1 1) and the fact that $Z^{(0)}(i)$ is a solution to (9) we have $\mathcal{R}_i\left(\boldsymbol{X}^{(0)}+\boldsymbol{Z}^{(0)}\right)=\gamma^{-2}Z^{(0)}(i)B_1(i)\left(B_1(i)\right)^{\mathrm{T}}Z^{(0)}(i)$. In addition, the operator

$$\mathcal{L}_{A^{(0)}(i)-B_2(i)B_2^{\mathrm{T}}(i)Z^{(0)}(i);\Pi_i}, i=1,\cdots,N$$

is asymptotically stable and

$$\begin{aligned}&A^{(0)}(i)-B_2(i)\left(B_2(i)\right)^{\mathrm{T}}Z^{(0)}(i)\\&=A(i)+B_1(i)F_1\left(X^{(0)}(i)\right)+B_2(i)F_2\left(X^{(0)}(i)\right)-B_2(i)\left(B_2(i)\right)^{\mathrm{T}}Z^{(0)}(i)A(i)\\&\quad+B_1(i)F_1\left(X^{(0)}(i)\right)+B_2(i)F_2\left(X^{(0)}(i)+Z^{(0)}(i)\right)\\&=\tilde{A}^{(0)}(i).\end{aligned}$$

The Lyapunov operator $\mathcal{L}_{\tilde{A}^{(0)}(i);\Pi_i}$ is asymptotically stable. In addition, $\hat{\boldsymbol{X}}$ is a solution to $\mathcal{R}_i\left(\hat{\boldsymbol{X}}\right)+\hat{Q}(i)=0, i=1,\cdots,N\left(\hat{Q}(i)\geq 0\right)$ and applying Lemma 1 we obtain:

$$0 = \mathcal{R}_i\left(\hat{\boldsymbol{X}}\right) + \hat{Q}(i) = \mathcal{R}_i\left(0, F_2\left(\hat{X}(i)\right), \hat{\boldsymbol{X}}\right) + \hat{Q}(i)$$

$$0 = A_i\left(0, F_2\left(\hat{X}(i)\right)\right)^{\mathrm{T}} \hat{X}(i) + \hat{X}(i) A_i\left(0, F_2\left(\hat{X}(i)\right)\right) + C(i)^{\mathrm{T}} C(i) + \Pi_i\left(\hat{\boldsymbol{X}}\right)$$
$$+ \hat{Q}(i) + \left(F_2\left(\hat{X}(i)\right)\right)^{\mathrm{T}} F_2\left(\hat{X}(i)\right) + \gamma^2 \left(F_1\left(\hat{X}(i)\right)\right)^{\mathrm{T}} F_1\left(\hat{X}(i)\right).$$

Since $\hat{\boldsymbol{X}}$ is the stabilizing solution to the latest equation, then the Lyapunov operator $\mathcal{L}_{A\left(0,F_2\left(\hat{X}(i)\right),X\right);\Pi_i}$ is asymptotically stable with

$$A_i\left(0, F_2\left(\hat{X}(i)\right)\right) = A(i) + B_2(i) F_2\left(\hat{X}(i)\right) = A(i) + B_1(i) F_1\left(X^{(0)}(i)\right) + B_2(i) F_2\left(\hat{X}(i)\right).$$

Thus, following Lemma 2, 1) we conclude that $\hat{\boldsymbol{X}} \geq \boldsymbol{X}^{(0)} + \boldsymbol{Z}^{(0)} = \boldsymbol{X}^{(1)}$.

Thus, the properties 1), 2), 3) and 4) are true for $k = 0$. We compute $\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \boldsymbol{Z}^{(0)} \geq \boldsymbol{X}^{(0)}$.

Combining iteration (5) with equality $\boldsymbol{X}^{(k+1)} = \boldsymbol{X}^{(k)} + \boldsymbol{Z}^{(k)}$ we construct the following matrix sequences:

$$\boldsymbol{X}^{(0)} \leq \boldsymbol{X}^{(1)} \leq \cdots \leq \boldsymbol{X}^{(k)} \leq \cdots \leq \hat{\boldsymbol{X}} \quad \text{and} \quad \boldsymbol{Z}^{(0)}, \boldsymbol{Z}^{(1)}, \cdots, \boldsymbol{Z}^{(k)}, \cdots,$$

we prove by induction the following for $k = 0, 1, 2, \cdots$:

(α_k): The Lyapunov operator $\mathcal{L}_{A^{(k)}(i);\Pi_i}$ is asymptotically stable, $i = 1, \cdots, N$;

(β_k): $\mathcal{R}_i\left(\boldsymbol{X}^{(k+1)}\right) = \gamma^{-2} Z^{(k)}(i) B_1(i) \left(B_1(i)\right)^{\mathrm{T}} Z^{(k)}(i) \geq 0, i = 1, \cdots, N$;

(γ_k): The Lyapunov operator $\mathcal{L}_{\tilde{A}^{(k)}(i);\Pi_i}$ is asymptotically stable where

$\tilde{A}^{(k)}(i) = A^{(k)}(i) + B_1(i) F_1\left(X^{(k)}(i)\right) + B_2(i) F_2\left(X^{(k+1)}(i)\right), i = 1, \cdots, N$;

(δ_k): $\hat{\boldsymbol{X}} \geq \boldsymbol{X}^{(k+1)} \geq \boldsymbol{X}^{(k)} \geq 0$.

We have seen the statements (α_0), (β_0), (γ_0) and (δ_0)) are true. We assume the statements (α_k), (β_k), (γ_k) and (δ_k) are true for $k = r - 1$. We prove the same statements for $k = r$.

We know $\boldsymbol{X}^{(r)} = \boldsymbol{X}^{(r-1)} + \boldsymbol{Z}^{(r-1)}$. We compute $F_1\left(X^{(r)}(i)\right), F_2\left(X^{(r)}(i)\right)$, and $A^{(r)}(i), i = 1, \cdots, N$. We have to find $Z^{(r)}(i), i = 1, \cdots, N$ as a unique stabilizing solution to (5) with $k = r$. The matrix $\mathcal{R}_i\left(X^{(r)}\right)$ is positive semidefinite because $\left(\beta_{r-1}\right)$ is true. It remains to show that $\mathcal{L}_{A^{(r)}(i),\Pi_i}$ is asymptotically stable $i = 1, \cdots, N$.

Following Lemma 2, 2) the operator $\mathcal{L}_{\tilde{A}_i\left(\boldsymbol{X}^{(r-1)} + \boldsymbol{Z}^{(r-1)}, \hat{\boldsymbol{X}}\right);\Pi_i}$ is asymptotically stable because

$\hat{\boldsymbol{X}} - \boldsymbol{X}^{(r-1)} - \boldsymbol{Z}^{(r-1)} \geq 0$

$\left(A_i\left(X^{(r-1)}(i) + Z^{(r-1)}(i), \hat{X}(i)\right) = A(i) + B_1(i) F_1\left(X^{(r-1)}(i) + Z^{(r-1)}(i)\right) + B_2(i) F_2\left(\hat{X}(i)\right), i = 1, \cdots, N\right)$. Thus the operator $\mathcal{L}_{A(i) + B_1(i) F_1\left(X^{(r-1)}(i) + Z^{(r-1)}(i)\right);\Pi_i}$ is asymptotically stable. In addition,

$A(i) + B_1(i) F_1\left(X^{(r)}(i)\right) = A^{(r)}(i) - B_2 F_2\left(X^{(r)}(i)\right)$. Thus the operator $\mathcal{L}_{A^{(r)}(i);\Pi_i}$ is asymptotically stable, $i = 1, \cdots, N$. There exists a unique positive semidefinite solution $Z^{(r)}(i)$ to (5) with $k = r$. The last fact in combination of the presentation of $\mathcal{R}_i\left(\boldsymbol{X}^{(r)} + \boldsymbol{Z}^{(r)}\right)$ from Lemma 1, 1) we conclude that

$\mathcal{R}_i\left(\boldsymbol{X}^{(r)} + \boldsymbol{Z}^{(r)}\right) = \gamma^{-2} Z^{(r)}(i) B_1(i) \left(B_1(i)\right)^{\mathrm{T}} Z^{(r)}(i)$ and moreover is positive semidefnite. The assertions (α_r) and (β_r) are proved.

We have to prove the operator $\mathcal{L}_{\tilde{A}^{(r)}(i);\Pi_i}$ is asymptotically stable and

$\tilde{A}^{(r)}(i) = A(i) + B_1(i) F_1\left(X^{(r)}(i)\right) + B_2(i) F_2\left(X^{(r+1)}(i)\right), i = 1, \cdots, N$. In addition, the operator

$\mathcal{L}_{A^{(r)}(i)-B_2(i)B_2^{\mathrm{T}}(i)Z^{(r)}(i);\Pi_i}$, $i=1,\cdots,N$ is asymptotically stable because (α_r). Moreover, $A^{(r)}(i)-B_2(i)B_2^{\mathrm{T}}(i)Z^{(r)}(i)=A(i)+B_1(i)F_1\left(X^{(r)}(i)\right)+B_2(i)F_2\left(X^{(r)}(i)+Z^{(r)}(i)\right)$. Thus the ($\gamma_r$) is true for $k=r$.

Further on, we have $\boldsymbol{X}^{(r+1)}=\boldsymbol{X}^{(r)}+\boldsymbol{Z}^{(r)}\geq\boldsymbol{X}^{(r)}$ and $\hat{\boldsymbol{X}}\geq\boldsymbol{X}^{(r-1)}+\boldsymbol{Z}^{(r-1)}$ and thus

$$\mathcal{L}_{A(i)+B_1(i)F_1\left(X^{(r-1)}(i)+Z^{(r-1)}(i)\right)+B_2(i)F_2\left(\hat{X}(i)\right);\Pi_i}$$

is asymptotically stable by Lemma 2, 2) Using again Lemma 2, 1) we conclude $\hat{\boldsymbol{X}}\geq\boldsymbol{X}^{(r)}+\boldsymbol{Z}^{(r)}$. Hence $\hat{\boldsymbol{X}}\geq\boldsymbol{X}^{(r+1)}\geq\boldsymbol{X}^{(r)}$. All statements are proved for $k=r$.

The theorem is proved. □

The problem is to find the stabilizing solution $Y(i)$ to the general equation

$$\left(A^{(k)}(i)\right)^{\mathrm{T}}Y(i)+Y(i)A^{(k)}(i)+\Pi_i(\boldsymbol{Y})+\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)-Y(i)B_2(i)B_2^{\mathrm{T}}(i)Y(i)=0,\quad i=1,\cdots,N. \tag{10}$$

The Riccati Iterative Method. We choose $Y_0(i)=0$ and $Y_s(i),s=1,2,\cdots$ is the stabilizing solution to $(i=1,\cdots,N)$

$$\left(A^{(k)}(i)\right)^{\mathrm{T}}Y_s(i)+Y_s(i)A^{(k)}(i)+Q_{R,s-1}(i)-Y_s(i)B_2(i)B_2^{\mathrm{T}}(i)Y_s(i)=0, \tag{11}$$

with $\boldsymbol{Y}_{s-1}=\left(Y_{s-1}(1),\cdots,Y_{s-1}(N)\right)$. Note that the matrix $Q_{R,s-1}(i)=\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)+\Pi_i\left(\boldsymbol{Y}_{s-1}\right)$ is a positive semidefinite matrix for $Y_{s-1}(i)\geq 0, i=1,\cdots,N$.

It is well know that if the matrix pair $\left(A^{(k)}(i),B_2(i)\right),i=1,\cdots,N$ is stabilizable and the matrix $Q_{R,s-1}(i),i=1,\cdots,N$ is positive semidefinite, then there exists a semidefinite solution $\tilde{Y}(i)$ to the "perturbed" Riccati Equation (10).

Based on Riccati iteration (11) we consider the improved modification given by:

$$\left(A^{(k)}(i)\right)^{\mathrm{T}}Y_s(i)+Y_s(i)A^{(k)}(i)+\tilde{Q}_{R,s-1}(i)-Y_s(i)B_2(i)B_2^{\mathrm{T}}(i)Y_s(i)=0,\quad i=1,\cdots,N, \tag{12}$$

with

$$\tilde{Q}_{R,s-1}(i)=\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)+\sum_{l=1}^{r}A_l^T(i)Y_{s-1}(i)A_l(i)+\sum_{j<i}\lambda_{ij}Y_s(j)+\sum_{j>i}\lambda_{ij}Y_{s-1}(j).$$

The Lyapunov Iterative Method. We choose $Y_0(i)=0$ and $Y_s(i),s=1,2,\cdots$ is the stabilizing solution to

$$\left(\tilde{A}(i)\right)^{\mathrm{T}}Y_s(i)+Y_s(i)\tilde{A}(i)+Q_{L,s-1}(i)=0,\qquad i=1,\cdots,N, \tag{13}$$

with $\tilde{A}(i)=A^{(k)}(i)-B_2(i)B_2^{\mathrm{T}}(i)Y_{s-1}(i)$ and

$$Q_{L,s-1}(i)=\Pi_i\left(\boldsymbol{Y}_{s-1}\right)+\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)+Y_{s-1}(i)B_2(i)B_2^{\mathrm{T}}(i)Y_{s-1}(i).$$

We consider the Lyapunov iteration (13) as a special case of the Lyapunov iteration introduced and investigated by Ivanov [11]. Following the numerical experience in [11] we improve iteration (13) and introduce the improved Lyapunov iteration

$$\left(\tilde{A}(i)\right)^{\mathrm{T}}Y_s(i)+Y_s(i)\tilde{A}(i)+\tilde{Q}_{L,s-1}(i)=0,\qquad i=1,\cdots,N, \tag{14}$$

where

$$\tilde{Q}_{L,s-1}(i)=+\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)+Y_{s-1}(i)B_2(i)B_2^{\mathrm{T}}(i)Y_{s-1}(i)+\sum_{l=1}^{r}A_l^{\mathrm{T}}(i)Y_{s-1}(i)A_l(i)$$
$$+\sum_{j<i}\lambda_{ij}Y_s(j)+\sum_{j>i}\lambda_{ij}Y_{s-1}(j).$$

Convergence properties of the matrix sequence defined by (14) are given with Theorem 2.1 [11].

Further on, we consider an alternative iteration process where one matrix sequence is constructed. This sequence converges to the stabilizing solution of the given set of Riccati equations. We are proving that this introduced iteration is equivalent to the iteration loop (4)-(5). We substitute $\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)$ from (3) in recurrence Equation (5) and after matrix manipulations we obtain for $i=1,\cdots,N$:

$$\begin{aligned} 0 = &\left(A(i)+B_1(i)F_1\left(X^{(k)}(i)\right)\right)^{\mathrm{T}} X^{(k+1)}(i)+C^{\mathrm{T}}(i)C(i) \\ &+X^{(k+1)}(i)\left(A(i)+B_1(i)F_1\left(X^{(k)}(i)\right)\right)+\Pi_i(\boldsymbol{X}^{(k+1)}) \\ &-X^{(k+1)}(i)B_2(i)B_2^{\mathrm{T}}(i)X^{(k+1)}(i)-\gamma^{-2}X^{(k)}(i)B_1(i)\left(B_1(i)\right)^{\mathrm{T}}X^{(k)}(i). \end{aligned} \tag{15}$$

Thus, we can construct the matrix sequence $\left\{\boldsymbol{X}^{(k)}\right\}_{k=0}^{\infty}$ with $X^{(0)}(i)=0$ and each subsequent matrix is computed as a unique stabilizing solution to (15). In fact we just proved that the matrix sequence $\left\{\boldsymbol{X}^{(k)}\right\}_{k=0}^{\infty}$ defined by (15) is equivalent to the matrix sequence $\left\{\boldsymbol{X}^{(k)}\right\}_{k=0}^{\infty}$ defined by (4)-(5). In order to apply the iteration (15) we change the term $\Pi_i\left(\boldsymbol{X}^{(k+1)}\right)$ from (15) with $\Pi_i\left(\boldsymbol{X}^{(k)}\right)$.

The unknown matrix $X^{(k+1)}(i)$ is a solution to the set of continuous-time algebraic Riccati equation with the independent matrix

$$\Pi_i\left(\boldsymbol{X}^{(k)}\right)+C^{\mathrm{T}}(i)C(i)-\gamma^{-2}X^{(k)}(i)B_1(i)\left(B_1(i)\right)^{\mathrm{T}}X^{(k)}(i),\ i=1,\cdots,N.$$

4. Numerical Simulations

We have considered two iterative methods for computing the matrix sequence $\left\{\boldsymbol{Z}^{(k)}\right\}_{k=0}^{\infty}$: the Riccati iteration (15) and the Lyapunov iteration (14). In the begining we remark the LMI approach for finding the stabilizing solution to (5). Following similar investigations [12] [13] we conclude that the optimization problem (for given *k*)

$$\begin{aligned} &\max \sum_{i1}^{N}\left\langle I, Z^{(k)}(i)\right\rangle \\ &\text{subject to } i=1,\cdots,N \\ &\begin{pmatrix} \left(A^{(k)}(i)\right)^{\mathrm{T}} Z^{(k)}(i)+Z^{(k)}(i)A^{(k)}(i)+\Pi_i\left(\boldsymbol{Z}^{(k)}\right)+\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right) & Z^{(k)}(i)B_2(i) \\ \left(B_2(i)\right)^{\mathrm{T}}Z^{(k)}(i) & I \end{pmatrix} \geq 0 \\ &Z^{(k)}(i)=\left(Z^{(k)}(i)\right)^{\mathrm{T}} \end{aligned} \tag{16}$$

has a solution which is the stabilizing solution to (5).

We carry out experiments for solving a set of Riccati Equation (1). We construct two matrix sequences $\left\{\boldsymbol{X}^{(k)}\right\}$ and $\left\{\boldsymbol{Z}^{(k)}\right\}, k=0,1,\cdots$ for each example. The first matrix sequence is computed using iterative method (4)-(5). In order to form the second matrix sequence we apply Riccati iteration (15), Lyapunov iteration (14) and LMI approach (16). In addition, we construct a matrix sequence $\left\{\boldsymbol{X}^{(k)}\right\}, k=0,1,\cdots$ for each example using recurrence Equation (15) for this purpose.

The matrices $\left\{\boldsymbol{Z}^{(k)}\right\}, k=0,1,\cdots$ are computed in terms of the solutions of *N* Riccati equations for (15) and *N* algebraic Lyapunov equations for (14) at each step. For this purpose the MATLAB procedure *care* is applied where the flops are $81Nn^3$ per one iteration. Lyapunov iteration (14) solves *N* algebraic Riccati equations at each step. The MATLAB procedure *lyap* is used and the flops are $N\frac{27}{2}n^3$ per one iteration. In order to find the symmetric solution to (16) we adapt MATLAB's software functions of LMI Lab.

Our experiments are executed in MATLAB on a 2.20 GHz Intel(R) Core(TM) i7-4702MQ CPU computer. We use two variables *tolR* and *tol* for small positive numbers to control the accuracy of computations. We denote $Error_{\mathcal{G},k} = \max_{i=1,\cdots,N}\left\|\mathcal{G}_i\left(\boldsymbol{Z}^{(k)}\right)\right\|$ and $Error_{\mathcal{G},k} = \max_{i=1,\cdots,N}\left\|\mathcal{R}_i\left(\boldsymbol{X}^{(k)}\right)\right\|$. The iterations (15) and (14) stop when the inequality $Error_{\mathcal{G},k_0} \le tol$ is satisfied for some k_0. That is a practical stopping criterion for (15) and (14). The variable *It* means the maximal number of iterations for which the inequality $Error_{\mathcal{R},It} \le tolR$ holds. The last inequality is used as a practical stopping criterion for main iterative process (4)-(5). The tolerance *tol* controls accuracy of the procedure *mincx* which is used for numerical solution to (16).

We consider a family of examples in case $N = 3, r = 2, n = 7, 8, \cdots, 14$ for two given values of $m_1 : m_1 = 4$ and $m_1 = n$. The coefficient real matrices are given as follows: $A_0(i), A_1(i), A_2(i), B_0(i)$, $B_1(i), B_2(i), C(i), i = 1, 2, 3$ were constructed using the MATLAB notations:

$$A_0(1) = randn(n,n)/8 - 0.45 * eye(n,n); \quad A_1(1) = randn(n,n)/8;$$
$$A_0(2) = randn(n,n)/8 - 0.45 * eye(n,n); \quad A_1(2) = randn(n,n)/8;$$
$$A_0(3) = randn(n,n)/8 - 0.45 * eye(n,n); \quad A_1(3) = randn(n,n)/8;$$
$$A_2(1) = full(sprand(n,n,0.6))/8; \quad A_2(2) = full(sprand(n,n,0.6))/8;$$
$$A_2(3) = full(sprand(n,n,0.6))/8;$$

and

$$B_1(1) = rand(n,m_1)/10; \quad B_1(2) = rand(n,m_1)/10; \quad B_1(3) = rand(n,m_1)/10;$$
$$B_2(1) = rand(n,m_1)/9; \quad B_2(2) = rand(n,m_1)/9; \quad B_2(3) = rand(n,m_1)/9;$$
$$C(1)_{n\times n} = dc\left[sqrt(0.08), sqrt(0.7)\right] := \begin{pmatrix} sqrt(0.08) & 0 & \dots & sqrt(0.7) \\ 0 & sqrt(0.08) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ sqrt(0.7) & \dots & 0 & sqrt(0.08) \end{pmatrix},$$

and

$$C(2)_{n\times n} = dc\left[sqrt(0.03), sqrt(0.5)\right], \text{ and } C(2)_{n\times n} = dc\left[sqrt(0.05), sqrt(0.9)\right].$$

In our definitions the functions randn (p, k) and sprand (q, m, 0.3) return a p-by-k matrix of pseudorandom scalar values and a q-by-m sparse matrix respectively (for more information see the MATLAB description). The following transition probability matrix

$$(\lambda_{ij}) = \begin{pmatrix} -0.33 & 0.17 & 0.16 \\ 0.30 & -0.53 & 0.23 \\ 0.26 & 0.10 & -0.36 \end{pmatrix}$$

is applied for all examples.

For our purpose we have executed hundred examples of each value of m for all tests. **Table 1** reports the average number of iterations for the main iterative process "It_M" and the average number of iterations for the second iterative process "It_S" needed for achieving the relative accuracy for all examples of each size. The column "CPU" presents the CPU time for executing the corresponding iterations. Results from experiments are given in **Table 1** with $tolR = 1e-7, tol = 1e-8$ for all tests. Results from experiments with the iteration (15) are given in **Table 2** with $tolR = 1e-7$ for all tests.

5. Conclusions

We have studied two iterative processes for finding the stabilizing solution to a set of continuous-time genera-

Table 1. Results from 50 runs for each value of *n*.

n	(4)-(5) with RI: (15)			(4)-(5) with LI: (14)			(4)-(5) with LMI: (16)		
	It_M	It_S	CPU	It_M	It_S	CPU	It_M	It_S	CPU
Test 1: $m_1 = 4$									
7	3	12.2	3.9 s	3	12.6	1.6 s	3	19.8	16.7 s
8	3	14.7	4.6 s	3	13.7	1.6 s	4	20.3	23.5 s
9	3	16.5	5.6 s	3	16.4	2.3 s	5	21.9	35.7 s
10	4	17.4	6.5 s	4	18.9	2.8 s	4	23.4	54.8 s
11	4	22.7	9.8 s	4	20.5	3.3 s	4	26.3	84.2 s
12	6	27.3	13.3 s	5	26.8	4.6 s	4	31.6	130.5 s
Test 2: $m_1 = n$									
7	3	12.7	4.0 s	3	12.5	1.3 s	4	20.7	20.8 s
8	4	13.2	4.4 s	3	14.9	1.8 s	3	22.0	28.0 s
9	4	15.6	6.2 s	4	16.2	2.1 s	3	22.3	39.8 s
10	4	17.7	7.8 s	4	18.4	2.5 s	3	26.2	66.0 s
11	4	20.5	9.7 s	5	21.0	3.0 s	4	37.2	125.3 s
12	4	23.3	11.5 s	4	22.8	3.3 s	4	36.9	163.2 s
13	4	25.1	11.6 s	4	25.6	4.0 s	5	57.0	371.0 s
14	4	28.6	15.3 s	4	27.3	4.7 s	4	73.8	636.5 s

Table 2. Results from 50 runs for each value of *n*.

n	the max number of iteration steps	the average number of iteration steps	CPU time
Iteration (15) for $m_1 = 4$			
7	30	17.0	1.8 s
8	38	18.6	2.1 s
9	28	19.7	2.5 s
10	58	23.2	3.1 s
11	56	27.5	4.2 s
12	61	31.6	5.2 s
Iteration (15) for $m_1 = n$			
7	24	16.1	1.8 s
8	27	17.7	2.1 s
9	30	18.8	2.6 s
10	29	20.9	3.3 s
11	40	24.3	4.0 s
12	51	25.5	4.3 s
13	40	27.9	4.5 s
14	46	31.6	5.2 s

lized Riccati Equation (1). We have made numerical experiments for computing this solution and we have compared the numerical results. In fact, it is a numerical survey on iterative methods for computing the stabilizing solution. We have compared the results from the experiments in regard of the number of iterations and CPU time for executing. Our numerical experiments confirm the effectiveness of proposed new method (15).

The application of all iterative methods shows that they achieve the same accuracy for different number of iterations. The executed examples have demonstrated that the two iterations "(4)-(5) with RI: (15)" and "(4)-(5) with LI: (14)" require very close average numbers of iterations (see the columns "It_S" for all tests). However, the CPU time is different for these iterations. In addition, by comparing iterations based on the solution, the linear matrix Lyapunov equations shows that iteration "(4)-(5) with LI: (14)" is slightly faster than the second iteration (15). This conclusion is indicated by numerical simulations. Based on the experiments, the main conclusion is that the Lyapunov iteration is faster than the Riccati iteration because these methods carry out the same number of iterations.

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