

Evaluation of Geometric Asian Power Options under Fractional Brownian Motion

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ABSTRACT

Modern option pricing techniques are often considered among the most mathematical complex of all applied areas of financial mathematics. In particular, the fractional Brownian motion is proper to model the stock dynamics for its long-range dependence. In this paper, we evaluate the price of geometric Asian options under fractional Brownian motion framework. Furthermore, the options are generalized to those with the added feature whose payoff is a power function. Based on the equivalent martingale theory, a closed form solution has been derived under the risk neutral probability.

KEYWORDS

Fractional Brownian Motion; Geometric Asian Options; Closed-Form Solution; Risk-Free Rate

1. Introduction

Estimating option pricing is a central topic in financial mathematics. A call (put) option is a contract which gives the holder the right but not the obligation, to buy (sell) a risky asset at a certain date with a predetermined price (called strike price). In terms of the execution time, options can be classified into three types: American options whose owner can choose to exercise at any time up to and including the expiration; Bermudan options which permit early exercise but only on a contractually specified finite set of dates; European options which can only be exercised at the expiration date.

The European calls and puts, which are with maturity T and strike price K , are often called vanilla options. And their payoffs at maturity, *i.e.*, $(S(T) - K)^+$ and $(K - S(T))^+$, respectively, depend only on the spot value of the underlying asset. On the other hand, there exist several kinds of exotic options such as Asian options, look-back options and knock-out options.

In 1987, Asian options were first introduced at a branch of an American bank in Tokyo, Japan. For an Asian option, its payoff is determined by the average value over some predetermined time interval. One advantage is that it can reduce the risk of market manipulation of the underlying instrument at maturity. Moreover, they offer better hedging possibilities for firms with a stream of exposures. Another benefit is that they are useful in protecting the owner from sudden short lasting price changes in the market, for example, due to order imbalances [1]. Because of the averaging property, Asian options reduce the volatility inherent in the option. And therefore, Asian options are usually cheaper comparing with its European counterparts.

Asian options have numerous permutations, such as fixed and floating strike price options. The payoff of fixed strike options is $(A(T) - K)^+$, $(K - A(T))^+$ for a call and put option, respectively, where K denotes the strike price and $A(T)$ is the average price of the underlying asset. For floating strike price options, the payoff takes the following form: $(S(T) - A(T))^+$ and $(A(T) - S(T))^+$ for a call and put option, respectively. In terms of the average price $A(T)$, Asian options can be classified into two categories: arithmetic average

Asians and geometric average Asians, and both these forms can be averaged on a weighted average basis. For the continuous case, arithmetic average is obtained by

$$A(T) = \frac{1}{T} \int_0^T S(t) dt.$$

And the geometric average is given by

$$A(T) = e^{\frac{1}{T} \int_0^T \ln S(t) dt}. \quad (1)$$

For the discrete case, we just need to change the integral into summation. In this sequel, we just consider the case of continuous geometric average with fixed strike price and leave the discrete case for readers.

An Asian option (also called average value option) is an important class of path-dependent options and its pricing has aroused much attention. The path integral formalism was created by Richard Feynman in quantum physics [2]. Norbert Wiener used this type of integral in his research of Brownian motion [3]. And Jan Dash first introduced this type of path integral into finance, who developed empirical studies related to the Black-Scholes-Merton model and the one-factor term structure constrained model. Till now, there is no known closed form solution for the arithmetic type, for it is difficult to analytically evaluate the sum of the related log-normal random variables. Feynman and Kleinert showed that by the method of the path integral, the problem for geometric average can be solved via the effective classical potential [4]. In 1990, Kemna and Vorst discussed the pricing of arithmetic Asian options with MCMC method and proposed Turnbull and Wakeman formula for pricing Arithmetic average option under continuous case. Furthermore, they derived an analytic solution for a geometric average option by changing the diffusion term [5]. In 1995, Rogers and Shi solved the pricing problem with a PDE approach [6].

On the other hand, the introduction of fractional Brownian motion (FBM) should date back to the development of the option pricing theory. In 1900, Bachelier, the father of option pricing theory, first developed arithmetic Brownian motion to model the dynamics of underlying asset [7]. In 1973, F. Black and M. Scholes introduced the Black-Scholes-Merton (BSM) model, which assumed that the stock process followed a geometric Brownian motion. And they derived the well-known BS formula [8]. But empirical studies indicate that the log-returns are usually not normal and show the dependence structure. Additionally, it also reveals that the stock price usually has some properties such as “fat tail”, “self-similarity”, “long-range dependence” [9]. This reveals the disparity between the model and the market.

In 1940, Kolmogorov first introduced the fractional Brownian motion within a Hilbert space, where it was called Wiener Helix. Yaglom studied this process and discussed its moment properties [10]. Further studies have discovered that fractional Brownian motion can be used to model such situations as

- 1) The widths of consecutive annual rings of a tree;
- 2) The temperature at a specific place as a function of time;
- 3) The level of water in a river as a function of time;
- 4) The characters of solar activity as a function of time;
- 5) The values of the log returns of a stock;
- 6) Financial turbulence, *i.e.*, the empirical volatility of a stock, and other turbulence phenomena;
- 7) The prices of electricity in a liberated electricity market.

In 1968, Mandelbrot and Van Ness provided a stochastic integral representation of this process: If $0 < H < 1$, the fractional Brownian motion with Hurst index H is a continuous Gaussian process,

$$\{B_H(t), t \in R\} \quad \text{and} \quad B_H(0) = 0$$

with mean zero and covariance:

$$Cov_H(t, s) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Obviously, $B_H(t)$ coincides with $B(t)$, the standard Brownian motion, when $H = 1/2$. In 2008, Ciprian Necula obtained an explicit fractional BS formula by using Fourier transform [11]. Mandelbrot and Taylor proposed that the stock market should take on the character of fractional Brownian motion [12]. Then Peters introduced fractional Brownian motion to model the dynamics of stock price [13]. After this, many scholars have made outstanding contributions on this topic. In 2000, Duncan *et al.*, Hu and Oksendal developed the fractional

BS formula by using Wick-product [14]. And the systematic option pricing theory under fractional Brownian motion framework has been presented by Hu *et al.* in [15].

The paper is organized as follows. In Section 2, we further give a simple description of fractional Brownian motion framework and review some existing results of power options. Next, the pricing model of Geometric Asian power options has been presented in Section 3. Moreover, closed form solution and call-put parity are also derived. The conclusion remarks and some open problems are discussed in the final part.

2. Preliminary

In general, one can rely on the numerical methods for pricing arithmetic Asian options in Levy models [16]. While geometric averaging options within Levy models can be priced analytically. Since its statistical property can be obtained from the assumption that the stock process follows a log-normal distribution. Kemna and Vorst [5] derived an analytic formula for Asian options of this kind as below.

Lemma 2.1 For a geometric average Asian call option, its payoff is given as $(A(T) - K)^+$, where K is the strike price and

$$A(T) = e^{\frac{1}{T-T_0} \int_{T_0}^T \ln S(t) dt}$$

with $[T_0, T]$ being the final time interval over which the average value of the stock is calculated. Then $A(T)$ is log-normally distributed and thus the value of the option at time T_0 is as following:

$$C(S(T_0), T_0) = S(T_0) e^{-\frac{1}{2} \left(r + \frac{1}{6} \sigma^2 \right) (T-T_0)} F_N(d_1) - K e^{-r(T-T_0)} F_N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S(T_0)}{K} + \frac{1}{2} \left(r + \frac{1}{6} \sigma^2 \right) (T-T_0)}{\sigma \sqrt{1/3} \cdot (T-T_0)} \quad \text{and} \quad d_2 = d_1 - \frac{1}{\sqrt{3}} \sigma \sqrt{T-T_0}.$$

In the FBM framework, stock is assumed to be the underlying asset and denoted by $S(t)$ for simplicity. Then the price process can be modeled by the following PDE:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB_H(t), \quad (2)$$

where $S(t)$ is the spot price of the underlying security at time t , μ, σ are both constant, $B_H(t)$ being the standard FBM. Under risk neutral probability, (2) turns into the following form:

$$\frac{dS(t)}{S(t)} = r dt + \sigma d\tilde{B}_H(t), \quad (3)$$

where r is the risk-free rate and $\tilde{B}_H(t)$ is the FBM under the new measure. The corresponding fractional BS formula was developed in [13,17]. For a European call option, its price process $C(S(t), t)$ satisfies the following PDE:

$$HS^2 \sigma^2 t^{2H-1} \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0.$$

Solving the above PDE obtains the pricing formula:

$$C(S(0), 0) = S(0) F_N(d_1) - K F_N(d_2),$$

where $F_N(\cdot)$ denotes the cumulative distribution function of standard normal distribution with

$$d_1 = \frac{\ln \frac{S(0)}{K} + rT + \frac{\sigma^2}{2} T^{2H}}{\sigma T^H}, \quad d_2 = d_1 - \sigma T^H.$$

Furthermore, Biagini and Oksendal etc. extended the model by assuming that the risk-free rate and dividend rate are non-random functions [15]. Then the option pricing formula under FBM framework is given as below:

Lemma 2.2 Suppose the dynamics of asset price follows PDE (3) under the risk neutral probability. If the risk-free rate $r(t)$ and dividend rate $q(t)$ are non-random functions and the payoff function at maturity $f(S(T), T)$ is bounded, then the price of a European option $V(S(t), t)$ satisfies the following:

$$V(S(t), t) = e^{-\int_t^T r(s) ds} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} f\left(S(t) e^{\int_t^T [r(s)-q(s)] ds - \frac{1}{2}\sigma^2(T^{2H}-t^{2H}) + \sigma\sqrt{T^{2H}-t^{2H}} x}\right) dx.$$

Recently, kinds of derivatives have been introduced to satisfy the need of market. Power options, one kind of the newly developed options, have aroused more and more attention. They are used to change the return structure and also possess more flexibility comparing with vanilla options. Power options can be seen as instruments of risk management and getting higher return, thus attracting more and more application. For a European power call, there exist two types in terms of different payoff functions:

$$f(S(T), T) = \begin{cases} (S^n(T) - K)^+ & \text{if } S^n(T) > K \\ (S(T) - K)^+ & \text{if } S(T) > K \end{cases},$$

where n is a positive number.

In 2005, Y. Wang *et al.* [18] derived an explicit formula of European power options. And Y. Xiao *et al.* [19] studied some properties of power options under Brownian motion framework. D. In 2006, Y. Wang *et al.* [20] generalized the model to Geometric Asian power options and obtained the solution under continuous case. In this paper, we further extend the underlying asset follows a fractional Brownian motion. Zhao [21] and S. Zhou [22] considered the European power options under FBM framework and deduced the pricing formula and call-put parity as well.

In this paper, we consider the Asian power options under FBM framework. Correspondingly, for an Asian power call option with fixed strike price K and maturity T , there exist two types of payoff functions:

$$f(A(T), T) = \begin{cases} (A^n(T) - K)^+ & \text{if } A^n(T) > K \\ (A(T) - K)^+ & \text{if } A(T) > K \end{cases}, \quad (4)$$

where $A(T)$ is defined as (1). We discuss the case of (4), since the results for the other type can be obtained by the same procedure.

3. Pricing Model

3.1. General Hypotheses

The general assumptions in this model are as below:

A1 The dynamics of underlying asset follows fractional Brownian motion.

A2 The risk-free rate $r(t)$ is non-random function.

A3 There are no transaction costs or taxes in buying or selling stocks, options, *i.e.*, the market is frictionless.

A4 Dividends are paid on the underlying stock during the option life with the rate of $q(t)$.

A5 The option can only be exercised at maturity.

A6 The market does not admit arbitrage.

3.2. Pricing Framework with Constant Risk Free Rate and Dividend Rate

Before coming to the case of non-random risk-free rate, we first consider the case that the risk-free rate and dividend rate are both constant. Let $t = 0$ for simplicity. Under above assumptions, the dynamics of stock price process (3) takes the following form under risk neutral measure:

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sigma d\tilde{B}_H(t),$$

where r, q are the risk-free rate and dividend rate, respectively, $\tilde{B}_H(t)$ being the FBM under risk-neutral probability. Then the stock price process can be derived

$$S(t) = S(0) \exp\left((r - q)t - \frac{1}{2}\sigma^2 t^{2H} + \sigma \tilde{B}_H(t)\right).$$

It implies that $S(t)$ is log-normally distributed with

$$\ln S(t) \sim N\left(\ln S(0) + (r - q)t - \frac{1}{2}\sigma^2 t^{2H}, \sigma^2 t^{2H}\right)$$

under risk neutral measure.

First, we derive the closed form solution for Geometric Asian options with fixed strike price. Since its payoff at maturity is $(A(T) - K)^+$, the following theorem is derived:

Theorem 3.1 Suppose the dynamics of underlying asset follows PDE (3) under risk neutral probability. If the risk free rate $r(t)$ and dividend rate $q(t)$ are constant, then a Geometric Asian call option at time $t = 0$ is priced as below:

$$C(S(0), T) = S(0) e^{-\frac{1}{2}(r+q)T - \frac{1}{4(2H+1)(H+1)}\sigma^2 T^{2H}} F_N(d_1) - Ke^{-rT} F_N(d_2),$$

where

$$d_2 = \frac{\ln \frac{S(0)}{K} + \frac{1}{2}(r - q)T - \frac{1}{2(2H + 1)}\sigma^2 T^{2H}}{\sigma T^H / \sqrt{2(H + 1)}} \quad \text{and} \quad d_1 = d_2 + \frac{\sigma T^H}{\sqrt{2(H + 1)}}. \tag{6}$$

Proof of Theorem 3.1

Define

$$G(T) = \frac{1}{T} \int_0^T \ln S(t) dt \quad \text{and} \quad A(T) = \exp(G(T)).$$

From the definition, $G(T)$ is normally distributed with mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$, which are calculated explicitly as below.

$$\begin{aligned} \tilde{\mu} &= \tilde{E}[G(T)] = \frac{1}{T} \int_0^T \tilde{E}[\ln S(t)] dt \\ &= \ln S(0) + \frac{1}{T} \int_0^T (r - q)t dt - \frac{1}{2T} \int_0^T \sigma^2 t^{2H} dt \\ &= \ln S(0) + \frac{1}{2}(r - q)T - \frac{1}{2(2H + 1)}\sigma^2 T^{2H} \end{aligned}$$

and

$$\begin{aligned} \tilde{\sigma}^2 &= \text{Var}[G(T)] = \tilde{E}[G(T) - \tilde{\mu}]^2 \\ &= \frac{1}{T^2} \int_0^T \int_0^T \sigma^2 \tilde{E}[B_H(t) B_H(\tau)] dt d\tau \\ &= \frac{1}{2T^2} \int_0^T \int_0^T \sigma^2 (|t|^{2H} + |s|^{2H} - |t - s|^{2H}) dt d\tau \\ &= \frac{1}{2(H + 1)}\sigma^2 T^{2H} \end{aligned}$$

where $\tilde{E}[\cdot]$ denotes the expectation under risk neutral probability.

Consequently, we can conclude that $A(T)$ is log-normally distributed, i.e., $\ln A(T) = G(T) \sim N(\tilde{\mu}, \tilde{\sigma}^2)$. For a geometric Asian option, its payoff at maturity is $(A(T) - K)^+ = (\exp(G(T)) - K)^+$. Hence under the risk neutral measure, the value of a call option is:

$$C(S(0), T) = e^{-rT} \tilde{E}\left[(A(T) - K)^+\right] = e^{-rT} \int_D (e^x - K) \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} e^{-\frac{(x - \tilde{\mu})^2}{2\tilde{\sigma}^2}} dx,$$

where $D = \{x : A(T) > K\} = \{x : e^x > K\}$. And simple computations lead to that:

$$\begin{aligned}
C(S(0), T) &= e^{-rT} \int_D (e^{\tilde{\mu} + \tilde{\sigma}y} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = e^{-rT + \tilde{\mu} + \frac{\tilde{\sigma}^2}{2}} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\tilde{\sigma})^2}{2}} dy - Ke^{-rT} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= e^{-rT + \tilde{\mu} + \frac{\tilde{\sigma}^2}{2}} \int_{-d_2 - \tilde{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - Ke^{-rT} F_N(d_2) \\
&= S(0) e^{-\frac{1}{2}(r+q)T - \frac{1}{2(2H+1)}\sigma^2 T^{2H} + \frac{1}{4(H+1)}\sigma^2 T^{2H}} F_N(d_1) - Ke^{-rT} F_N(d_2) \\
&= S(0) e^{-\frac{1}{2}(r+q)T - \frac{1}{4(2H+1)(H+1)}\sigma^2 T^{2H}} F_N(d_1) - Ke^{-rT} F_N(d_2)
\end{aligned}$$

where

$$\begin{aligned}
D &= \{x : A(T) > K\} = \{y : e^{\tilde{\mu} + \tilde{\sigma}y} > K\} = \{y : \tilde{\mu} + \tilde{\sigma}y > \ln K\} \\
&= \left\{ y : y > -\frac{\ln \frac{S(0)}{K} + \frac{1}{2}(r-q)T - \frac{\sigma^2}{2(2H+1)}T^{2H}}{\sigma T^H / \sqrt{2(H+1)}} \right\}
\end{aligned}$$

and $d_i, i=1, 2$ are defined as (6).

The Theorem is proved.

Moreover, by the same procedure, we can conclude that under FBM, the price of a Geometric Asian put option is valued at:

$$P(S(0), T) = Ke^{-rT} F_N(-d_2) - S(0) e^{-\frac{1}{2}(r+q)T - \frac{1}{4(2H+1)(H+1)}\sigma^2 T^{2H}} F_N(-d_1),$$

where $d_i, i=1, 2$ are defined same as Theorem 3.1.

Next, we extend the options to Geometric Asian power ones and obtain the following results:

Theorem 3.2 Suppose the dynamics of underlying asset follows PDE (3) under risk neutral probability. If the risk free rate $r(t)$ and dividend rate $q(t)$ are constant and the payoff function at maturity is given as (4). Then the price of a Geometric Asian power call, $C(S(0), T)$, is obtained:

$$C(S(0), T) = S^n(0) e^{-rT + \frac{n}{2}(r-q)T - \frac{n}{2(2H+1)}\sigma^2 T^{2H} + \frac{n^2}{4(H+1)}\sigma^2 T^{2H}} F_N(D_1) - Ke^{-rT} F_N(D_2)$$

where

$$D_2 = \frac{\ln \frac{S(0)}{\sqrt[n]{K}} + \frac{1}{2}(r-q)T - \frac{1}{2(2H+1)}\sigma^2 T^{2H}}{\sigma T^H / \sqrt{2(H+1)}} \quad \text{and} \quad D_1 = D_2 + \frac{n\sigma T^H}{\sqrt{2(H+1)}}. \quad (7)$$

Proof of Theorem 3.2

For the power option discussed in Theorem 3.2, its payoff at the maturity is $(A^n(T) - K)^+ = (\exp(nG(T)) - K)^+$. Based on Theorem 3.1, the closed form solution is deduced as below:

$$\begin{aligned}
C(S(0), T) &= e^{-rT} \int_{D'} (e^{n(\tilde{\mu} + \tilde{\sigma}y)} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = e^{-rT + n\tilde{\mu}} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{n\tilde{\sigma}y - \frac{y^2}{2}} dy - Ke^{-rT} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= e^{-rT + n\tilde{\mu} + \frac{n^2\tilde{\sigma}^2}{2}} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-n\tilde{\sigma})^2}{2}} dy - Ke^{-rT} F_N(D_2) \\
&= e^{-rT + n\tilde{\mu} + \frac{n^2\tilde{\sigma}^2}{2}} F_N(D_1) - Ke^{-rT} F_N(D_2) \\
&= S^n(0) e^{-rT + \frac{n}{2}(r-q)T - \frac{n}{2(2H+1)}\sigma^2 T^{2H} + \frac{n^2}{4(H+1)}\sigma^2 T^{2H}} F_N(D_1) - Ke^{-rT} F_N(D_2)
\end{aligned}$$

where

$$D' = \left\{ x : A(T)^n > K \right\} = \left\{ y : e^{n(\tilde{\mu} + \tilde{\sigma}y)} > K \right\} = \left\{ y : \tilde{\mu} + \tilde{\sigma}y > \ln \sqrt[n]{K} \right\}$$

$$= \left\{ y : y > - \frac{\ln \frac{S(0)}{\sqrt[n]{K}} + \frac{1}{2}(r-q)T - \frac{\sigma^2}{2(2H+1)}T^{2H}}{\sigma T^H / \sqrt{2(H+1)}} \right\}$$

and $D_i, i = 1, 2$ are defined as (7).

The proof is complete.

Furthermore, similar procedure obtains the price of the corresponding put option:

$$P(S(0), T) = Ke^{-rT} N(-D_2) - S^n(0) e^{-rT + \frac{n}{2}(r-q)T - \frac{n}{2(2H+1)}\sigma^2 T^{2H} + \frac{n^2}{4(H+1)}\sigma^2 T^{2H}} N(-D_1),$$

where $D_i, i = 1, 2$ are defined same as Theorem 3.2.

3.3. Generalized Pricing Framework

Finally we consider the case that the risk-free rate and dividend rate are non-random functions. Under the risk neutral measure, the dynamics of stock price is obtained

$$S(t) = S(0) \exp \left(\int_0^t (r(s) - q(s)) ds - \frac{1}{2} \sigma^2 t^{2H} + \sigma \tilde{B}_H(t) \right).$$

Thus we obtain that $S(t)$ is log-normally distributed and

$$\ln S(t) \sim N \left(\ln S(0) + \int_0^t (r(s) - q(s)) ds - \frac{1}{2} \sigma^2 t^{2H}, \sigma^2 t^{2H} \right).$$

Consider that $A(T)$ is defined as (1), and we denote $\ln A(T) \sim N(\tilde{\mu}, \tilde{\sigma}^2)$. Simple computations lead to

$$\tilde{\mu} = \ln S(0) - \frac{1}{2(2H+1)} \sigma^2 T^{2H} + \frac{1}{T} \int_0^T \int_0^t (r(s) - q(s)) ds dt,$$

$$\tilde{\sigma}^2 = \frac{1}{2(H+1)} \sigma^2 T^{2H}.$$

Furthermore, the desired formula of option price can be derived as below:

$$\begin{aligned} C(S(0), T) &= e^{-\int_0^T r(s) ds} \int_{D'} \left(e^{n(\tilde{\mu} + \tilde{\sigma}y)} - K \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= e^{-\int_0^T r(s) ds + n\tilde{\mu}} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{n\tilde{\sigma}y - y^2}{2}} dy - Ke^{-\int_0^T r(s) ds} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= e^{-\int_0^T r(s) ds + n\tilde{\mu} + \frac{n^2\tilde{\sigma}^2}{2}} \int_{-D_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-n\tilde{\sigma})^2}{2}} dy - Ke^{-\int_0^T r(s) ds} F_N(d_2) \\ &= e^{-\int_0^T r(s) ds + n\tilde{\mu} + \frac{n^2\tilde{\sigma}^2}{2}} F_N(D_1) - Ke^{-\int_0^T r(s) ds} F_N(D_2) \\ &= S^n(0) e^{-\int_0^T r(s) ds + \frac{n}{T} \int_0^T \int_0^t (r(s) - q(s)) ds dt - \frac{n\sigma^2 T^{2H}}{2(2H+1)} + \frac{n^2\sigma^2 T^{2H}}{4(H+1)}} F_N(D_1) - Ke^{-\int_0^T r(s) ds} F_N(D_2), \end{aligned}$$

where

$$D_2 = \frac{\ln \frac{S(0)}{\sqrt[n]{K}} - \frac{1}{2(2H+1)} \sigma^2 T^{2H} + \frac{1}{T} \int_0^T \int_0^t (r(s) - q(s)) ds dt}{\sigma T^H / \sqrt{2(H+1)}} \quad \text{and} \quad D_1 = D_2 + \frac{n\sigma T^H}{\sqrt{2(H+1)}}. \tag{8}$$

As for the geometric Asian power put option, its price takes the following form:

$$P(S(0), T) = Ke^{-\int_0^T r(s) ds} F_N(-D_2) - S^n(0) e^{-\int_0^T r(s) ds + \frac{n}{T} \int_0^T \int_0^t (r(s) - q(s)) ds dt - \frac{n\sigma^2 T^{2H}}{2(2H+1)} + \frac{n^2 \sigma^2 T^{2H}}{4(H+1)}} F_N(-D_1),$$

where $D_i, i = 1, 2$ are defined as (8).

Remark: Theorem 3.2 is just a particular case of the conclusions in [5]. And the Geometric Asian options are just the case of $n = 1$. Finally, the call-put parity can also be obtained for Geometric Asian power options.

Theorem 3.3 Suppose the dynamics of underlying asset follows PDE (3) under risk neutral probability. If the risk free rate $r(t)$ and dividend rate $q(t)$ are constant and the payoff function at maturity is given as (4). Then its call-put parity is as below:

$$C(S(0), T) - P(S(0), T) = S(0) e^{-rT + \frac{n}{2}(r-q)T - \frac{n}{2(2H+1)}\sigma^2 T^{2H} + \frac{n^2}{4(H+1)}\sigma^2 T^{2H}} - Ke^{-rT}.$$

Moreover, for the case of non-random functions of $r(t)$ and $q(t)$, the following holds:

$$C(S(0), T) - P(S(0), T) = S^n(0) e^{-\int_0^T r(s) ds + \frac{n}{T} \int_0^T \int_0^t (r(s) - q(s)) ds dt - \frac{n\sigma^2 T^{2H}}{2(2H+1)} + \frac{n^2 \sigma^2 T^{2H}}{4(H+1)}} - Ke^{-\int_0^T r(s) ds}.$$

Here we omit the proof and leave it for readers.

4. Conclusions

In this paper, we first consider the geometric Asian options with constant risk-free rate and dividend rate under FBM, and derive the call-put parity. Furthermore, we discuss the geometric Asian power options. Finally, for the general case of non-random risk-free rate and dividend rate, we obtain the corresponding pricing formulas and call-put parity. If $H = 1/2$, FBM is Brownian motion and all the conclusions coincide with that of classical exponential Brownian motion framework [8]. If $n = 1$, the solutions are the same as the results in BSM model [5].

For the Asian option with floating strike price, its pricing is of great interest in our future study. For the arithmetic Asian option, the statistical property is not as well as that of geometric kind. Therefore, our future focus will be on how to use numerical method to study its properties. Moreover, another interest is to calibrate the model with market data.

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