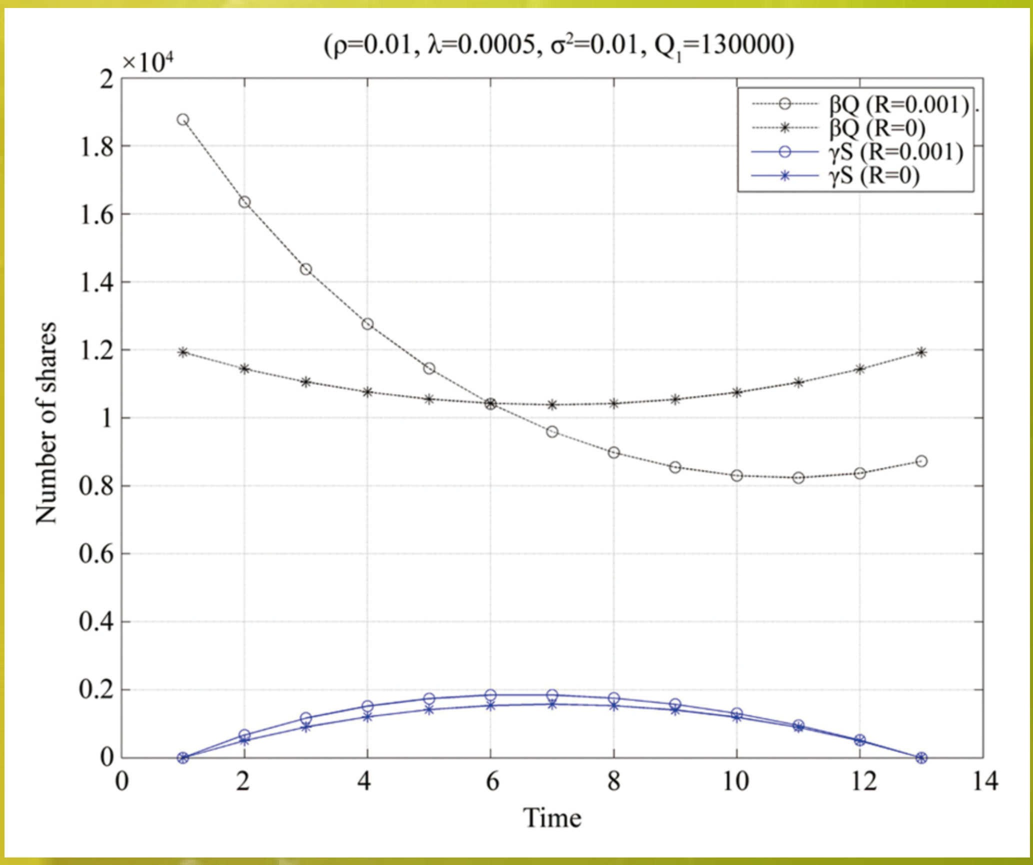
ISSN: 2162-2434 Vol. 5, No. 1, February 2015

Scientific Research Publishing

# Journal of Mathematical Finance

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# Journal of Mathematical Finance (JMF)

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The *Journal of Mathematical Finance* (*JMF*) (Online at Scientific Research Publishing, www.SciRP.org) is published quarterly by Scientific Research Publishing, Inc., USA.

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**Journal of Mathematical Finance, 2015, 5, 1-14**
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51001

# Optimal Execution in Illiquid Market with the Absence of Price Manipulation

**Seiya Kuno[1], Masamitsu Ohnishi[2]**

[1]Center for the Study of Finance and Insurance, Osaka University, Osaka, Japan
[2]Graduate School of Economics, Osaka University, Osaka, Japan
Email: kuno@sigmath.es.osaka-u.ac.jp, ohnishi@econ.osaka-u.ac.jp

Received 26 December 2014; accepted 16 January 2015; published 20 January 2015

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## Abstract

**This article shows the execution performance of the risk-averse institutional trader with constant absolute risk aversion (CARA) type utility by using the condition of no price manipulation defined in the risk neutral sense. From two linear price impact models both satisfying that condition, we have derived the unique explicit optimal execution strategy calculated backwardly with dynamic programming equations. And our study shows that the optimal execution strategy exists in the static class. The derived solution can be decomposed into mainly two components, each giving an explanation of the property of optimal execution volume. Moreover we propose two conditions in order to compare the performance of these two price models, and illustrate that the performances of the two models are surprisingly different under certain conditions.**

## Keywords

**Optimal Execution, Price Manipulation, Algorithmic Trading**

## 1. Introduction

In the competitive market paradigm, it is assumed that security markets are perfectly elastic and all orders can be executed instantaneously. However in real markets, since institutional traders (large traders) usually submit orders of considerable sizes, such traders thus influence the price by their own dealings (called market (price) impact) and create the execution time lag for their orders. Thus the large trader often divides her holdings (orders) into small pieces considering the tradeoff between market impact risk due to her fast execution and volatility risk due to her slow execution. In [1], such a price change (price impact) occurring at each trading period can be divided into three components. Firstly a temporary impact which represents the temporary cost of demanding liquidity and only affects an individual trade, and secondly a transient impact which represents gradual incorpo-

**How to cite this paper:** Kuno, S. and Ohnishi, M. (2015) Optimal Execution in Illiquid Market with the Absence of Price Manipulation. *Journal of Mathematical Finance*, **5**, 1-14. http://dx.doi.org/10.4236/jmf.2015.51001

ration of trade information to the price which derives the gradual price recovery, and finally a permanent impact which affects the prices of all subsequent trades of an agent. These price changes may enable the large trader to manipulate the market. The act of manipulating the market intentionally and through managed actions to make profits actively spoils market public welfare, and is forbidden in many trading venues. With the appearance of electronic trading, this problem got more concerns in financial literature. In optimal execution literature many studies are often conducted as the following way. Firstly, the price process model that considers such a price change under the condition of no price manipulation is built; then, the optimization problem with such a price model in the static or dynamic way in the discrete or continuous time setting is solved.

In this paper, under no price manipulation condition, we consider mainly two types of price model depending on how the price is reverted to its previous price level for the buy trade. Let's call one of them the permanent (impact) price model (as in e.g. [2] and [3]) and the other the transient (impact) price model (as in e.g. [4] and [5]). In the permanent price model, the execution price that lifted up by the large trader's order immediately reverts to a permanent level which is usually higher than the price at the previous trading time. On the other hand, the transient price model considers the price that reverts to a permanent level gradually in time. That is, one of the differences between the two models is whether the temporary impact decays instantly (in the permanent price model) or gradually (in the transient price model). A large number of empirical studies have been reported for the basis of the transient price model in various trading venue, refer to e.g. [6] and references therein. Although many empirical studies also show the non-linearity of the price impact function, we use the linear one for simplicity of calculation.

The main goal of this paper is to derive the optimal execution strategies for these two price models. Then in the equidistance discrete trading time grid setting, we show that the optimal execution strategy of the risk-averse large trader with each price model exists in the static class by deriving backwardly the explicit solution with the dynamic programming equation. This result is similar to the one found in [7] which derives the optimal execution strategy dynamically with the continuous time permanent price model, but our approach with the discrete time transient price model can decompose the optimal solution into various components and then gives the intuitive interpretation about the existence of price manipulation. Moreover, since we found that there exist the optimal execution strategies for two price models in the static class, it can be easy to compare the cost performance by simulations and parameter settings between the price models.

The rest of the paper is organized as follows. In Section 2, we present two price dynamics and two definitions of the price manipulation. In Section 3, we describe the optimization problem and derive explicit solutions for the two price models. Furthermore, we show the property of the optimal execution strategy and illustrate it using the comparative statics. In Section 4, we consider the relationship between two price models. The transient price model is more realistic but a little bit complicated therefore it takes much time when we simulate the execution performance, on the other hand the permanent price model is unrealistic but simple enough to be able to make high-speed trading decision in algorithmic trading system. For that reason, we suggest how to incorporate the intrinsic parameter of the transient price model into the permanent price model. More concretely, we propose two conditions that exist between those two price models under the TWAP (Time Weighted Average Price) strategy, when we attempt to compare the performance of those two price model in the same market. Section 5 contains a conclusion. Calculations and proofs are complicated but can be proved in a straightforward way.

## 2. Market Models and Price Manipulation

In this section, we explain two existing price models in the discrete time setting. One is the permanent impact (price) model proposed by [3], which extends to that of [8] and another is the transient impact (price) model proposed by [4], which is a generalization of that of [5]. A risk-averse institutional trader (after that we call her a large trader in the sense that she submits large order volumes) and many noise traders also called liquidity providers are considered as economic agents. The superscript of each variable denoting $i = pe$ or $tr$ represents the use of the permanent price model or transient price model respectively. Through this paper, we set the exponential decay of the temporary impact in the transient price model, because it satisfies the no price manipulation according to Definition 2 stated later in this section.

### 2.1. Two Price Models

Suppose that $p_t^i$ is the price of a single risky asset at time $t$, $q_t$ is the large trader's execution volume. If

$q_t > 0$, it is the buy trade, on the other hand if $q_t < 0$, it is the sell trade. $Q_t$ is the number of shares which the large trader remains to purchase, if $Q_t > 0$ (or liquidate, if $Q_t < 0$). That is,

$$Q_{t+1} = Q_t - q_t \tag{1}$$

Moreover, $w_t^i$ is the investment capital (wealth). For simplicity, we assume in the following that the large trader plans to purchase the asset. If at time *t*, the large trader submits large amount of her market order $q_t$ just after she has recognized the price at that time $p_t^i$, the order is executed immediately. However, the execution price may not be equal to $p_t^i$. The execution price will be instantly lifted upward from $p_t^i$ to $\hat{p}_t^i$ because of the temporary imbalance of supply and demand. Assume that $\lambda_t$ denotes the price change per share (called price impact), the dynamics of $w_t^i$ and $\hat{p}_t^i$ are,

$$w_{t+1}^i = w_t^i - \hat{p}_t^i q_t, \tag{2}$$

$$\hat{p}_t^i = p_t^i + \lambda_t q_t. \tag{3}$$

The lifted price by the large order reverts to the previous price level to a certain extent.

In the permanent price model, the execution price diminishes instantly to the permanent impact level and the expected price is maintained until the next trading time. That is,

$$p_{t+1}^{pe} = \alpha_t p_t^{pe} + \left(1-\alpha_t\right)\hat{p}_t^{pe} + \varepsilon_{t+1}. \tag{4}$$

Using Equation (3) and (4),

$$p_{t+1}^{pe} = p_t^{pe} + \left(1-\alpha_t\right)\lambda_t q_t + \varepsilon_{t+1}, \tag{5}$$

where $\alpha_t$ represents the deterministic reversion rate of price and $0 \le \alpha_t \le 1$. $\varepsilon_{t+1}$ represents the public news effect on the fundamental price between time *t* and *t* + 1 and is recognized by the large trader at time *t* + 1. Further, $\{\varepsilon_t\}_{t\in[2,T]}$ is an i.i.d.stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and follows

$$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right) \tag{6}$$

All information available to the large trader before her trading at time *t* are

$$\mathcal{F}_t := \sigma\left\{\varepsilon_{s+1} : s = 1, \cdots, t-1\right\}. \tag{7}$$

In the permanent price model, the price impact, the temporary impact and the permanent impact are represented respectively by $\lambda_t$, $\left(1-\alpha_t\right)\lambda_t$, and $\alpha_t\lambda_t$.

The transient price model, on the other hand, is the same as the permanent price model until the submitted order is executed. However the price reversion toa permanent level is not immediate but gradual. We set the time independent rate $\rho$ as the resilience speed. Then we have

$$p_t^{tr} = p_t^0 + \sum_{k=1}^{t-1} \lambda_k \mathrm{e}^{-\rho(t-k)} q_k, \tag{8}$$

where $p^0$ denotes the fundamental price and $p_{t+1}^0 - p_t^0 =: \varepsilon_{t+1}$, defined in (6) and (7). Furthermore, by Equation (8) we get

$$p_{t+1}^{tr} - p_t^{tr} = \varepsilon_{t+1} + \lambda_t \mathrm{e}^{-\rho} q_t - S_t. \tag{9}$$

Here, we define *S* as

$$S_t := \mathrm{e}^{-\rho t}\left(1-\mathrm{e}^{-\rho}\right)\sum_{k=1}^{t-1} \lambda_k \mathrm{e}^{\rho k} q_k = l_{t-1} q_{t-1} + \mathrm{e}^{-\rho} S_{t-1}, \tag{10}$$

where

$$l_t := \lambda_t \left(1-\mathrm{e}^{-\rho}\right)\mathrm{e}^{-\rho}. \tag{11}$$

In this transient price model, the price impact and the transient impact are $\lambda_t$ and $\lambda_t \mathrm{e}^{-\rho(t-k)}$. On the other hand, the temporary and the permanent impact are both 0.

**Remark 1**: *The economic interpretation of* $S_t$ *is the difference between the cumulative transient impact traded from time* 1 *to t* – 1 *viewed at the time t and the one viewed at the time t* + 1. *Since the price reverts to the* *permanent level over and over* (*in the case price is down*), *then* $S_t \geq 0$.

The reason why we use these specific two price models is its viability, as it will explained in the next subsection. The main difference between these two models is whether the effect of the present execution is completely incorporated in the price immediately or not. In the transient price model, since the price after the present execution fall down gradually to the permanent level (in this case 0), the effect of the present execution is partially incorporated in the price at the following trading time, and is completely incorporated after a certain period.

### 2.2. Absence of Price Manipulation

In this subsection, we introduce the concept of price manipulation from the perspective of the feasibility of the price model. This is because the market can easily crash with the price manipulation of the large traders in the current market environment where the high-frequency trading is becoming a main stream. So the construction of the feasible price model is essential to limit such a price manipulation. In the following we introduce two concepts of price manipulation.

**Definition 1** ((*Pure*) *Price manipulation* [9]): *A round trip trade is an execution strategy* $\{q_t\}_{t\in[1,T]}$ *such that* $\sum_{t=1}^{T} q_t = 0$. *A pure price manipulation strategy is a round trip trade such that*

$$E\left[\sum_{t=1}^{T} \hat{p}_t q_t\right] < 0. \tag{12}$$

It is shown in [9] that if the permanent impact is linear in terms of execution volume, then the pure price manipulation is absent from the market in the risk neutral sense. Within the time-homogeneous reversion rate framework, our permanent price model satisfies this condition.

**Definition 2** (*Transaction-triggered price manipulation* [1]): *If the expected execution costs of a buy program can be decreased by intermediate sell trade, the price model admits transaction-triggered price manipulation. That is, there exists* $Q_1$, $T > 0$, *and a corresponding execution strategy* $\tilde{q}$ *for which under a monotone execution strategy q*,

$$E\left[C_T\left(\tilde{q}\right)\right] < \min\left\{E\left[C_T\left(q\right)\right]\right\}. \tag{13}$$

Definition 2 states a stronger condition of the price manipulation than the one given by Definition 1. That is to say, even if the price model satisfies the absence of pure price manipulation, it may not satisfy the absence of the transaction-triggered price manipulation, such as buy and sell oscillation trades.

In this paper, we use an exponential resilience for the transient price model. This does not admit transaction-triggered price manipulation. As shown below in Remark 2, our control for the risk-averse large trader describes that when we apply the round trip trade. 0 trade is always optimal. So, both price models satisfy the condition of the absence of pure price manipulation.

## 3. Optimal Execution

In this section, we show that the optimal execution strategy exists in the static class by deriving the explicit solution with a dynamic programming equation. Suppose that a risk-averse large trader with CARA (Constant Absolute Risk Aversion) type utility of which the risk aversion coefficient is *R* submits large amount of market orders in equally time intervals over the maturity *T*. We consider the problem of the dynamic execution strategy that maximizes the large trader's expected utility from her terminal wealth. Here, we show the optimal execution strategy based mainly on the transient price model. For the permanent price model, we only provide the result since it requires simpler calculation.

### 3.1. Execution Strategy for a Risk-Averse Large Trader

In this case, we define the large trader's expected utility under the trading strategy $\pi$ at time *t* as

$$V_t^{\pi} := E_t^{\pi}\left[-\exp\left\{-Rw_{T+1}^{tr}\right\}\cdot 1_{\{Q_{T+1}=0\}} + \left(-\infty\right)\cdot 1_{\{Q_{T+1}\neq 0\}}\right], \tag{14}$$

where $1_{\{\bullet\}}$ is the indicator function and the right hand side of the Equation (14) represents that it is optimal for the large trader to execute her whole holding orders at maturity *T*. Moreover we define the optimal value function

$$V_t := \operatorname*{ess\,sup}_{\pi} V_t^{\pi} \tag{15}$$

where the subscript *t* of the expectation represents the condition where all the information up to time *t* is available to the large trader.

Because of the Markov property of the dynamics and path independency of the large trader's utility at the final period, $V_t$ is a function of $(w_t, p_t, Q_t, S_t)$, and by principle of optimality, the optimality equation (Bellman equation) becomes as

$$V_t\left(w_t^{tr}, p_t^{tr}, Q_t, S_t\right) = \sup_{q_t \in \mathbb{R}} E\left[V_{t+1}\left(w_{t+1}^{tr}, p_{t+1}^{tr}, Q_{t+1}, S_{t+1}\right) \middle| w_t^{tr}, p_t^{tr}, Q_t, S_t, q_t\right]. \tag{16}$$

We derive the sequence of the optimal execution volumes which attains $V_1$ from the final period *T* by backward induction in *t*.

**Theorem** (*Optimal Execution Strategy with the Transient Price Model*): *When we use the transient price model, the optimal execution volume of a large trader at time t denoted* $q_t^*$ *is represented as the function of the remaining execution volume* $Q_t$ *and the cumulative effect of past executions* $S_t$ *at that time. Then at time t, the optimal execution volume and the corresponding optimal value function are respectively*

$$q_t^* = \frac{D_t Q_t - L_t S_t}{2C_t} = \beta_t Q_t - \gamma_t S_t, \tag{17}$$

*and*

$$V_t\left(w_t^{tr}, p_t^{tr}, Q_t, S_t\right) = -\exp\left\{-R\left(w_t^{tr} - p_t^{tr} Q_t - A_t Q_t^2 - B_t S_t Q_t + K_t S_t^2\right)\right\}, \tag{18}$$

*where we set*

$$\begin{cases} C_t := l_t e^{\rho} + \dfrac{R\sigma_\varepsilon^2}{2} + A_{t+1} - B_{t+1} l_t - K_{t+1} l_t^2 \\ D_t := -\lambda_t e^{-\rho} + R\sigma_\varepsilon^2 + 2A_{t+1} - B_{t+1} l_t \\ L_t := 1 - B_{t+1} e^{-\rho} - 2K_{t+1} l_t e^{-\rho} \end{cases}, \quad \begin{cases} A_t := A_{t+1} + \dfrac{R\sigma_\varepsilon^2}{2} - \dfrac{D_t^2}{4C_t} \\ B_t := B_{t+1} e^{-\rho} - 1 + \dfrac{D_t L_t}{2C_t} \\ K_t := K_{t+1} e^{-2\rho} + \dfrac{L_t^2}{4C_t} \end{cases}, \text{ and } \begin{cases} \beta_t := \dfrac{D_t}{2C_t} \\ \gamma_t := \dfrac{L_t}{2C_t}. \end{cases} \tag{19}$$

*Then a deterministic execution strategy becomes optimal.*

Secondary, we provide the optimal execution strategy for the permanent price model as following corollary.

**Corollary** (*Optimal Execution Strategy with permanent Price Model*):

*When we use the permanent price model, the optimal execution volume of a large trader at time t denoted* $q'^*$ *is represented as the affine function of the remaining execution volume* $Q_t$ *at that time. Then at time t, the optimal execution volume and the optimal value function are*

$$q_t'^* = \frac{D_t' Q_t}{2C_t'} = \beta_t' Q_t, \tag{20}$$

*and*

$$V_t\left(w_t^{pe}, p_t^{pe}, Q_t\right) = -\exp\left\{-R\left(w_t^{pe} - p_t^{pe} Q_t - A_t' Q_t^2\right)\right\}, \tag{21}$$

*where*

$$\begin{cases} C_t' := \alpha_t \lambda_t + \dfrac{R\sigma_\varepsilon^2}{2} + A_{t+1}' \\ D_t' := -\left(1 - \alpha_t\right)\lambda_t + R\sigma_\varepsilon^2 + 2A_{t+1}' \end{cases}, \quad A_t' := A_{t+1}' + \frac{R\sigma_\varepsilon^2}{2} - \frac{D_t'^2}{4C_t'} \text{ and } \beta_t' := \frac{D_t'}{2C_t'}. \tag{22}$$

We provide a short proof of this Theorem in the appendix. For the proof of the Corollary, refer to [10]. The optimal solution for the transient price model consists of two components, $\beta$ and $\gamma$. $\beta$ contributes directly to the optimal solution while $\gamma$ contributes secondarily. If the external factor is added in the permanent price model, $\gamma'$ is also added. Since the terms $\beta_t$, $\gamma_t$, and $S_t$ are deterministic at time *t*, the optimal execution strategy exists in the static class which is supported by the next remark.

**Remark 2**: *For both price models, $Q_t$ can be expressed in $\beta$, $\gamma$, $S$ and $Q_1$. Therefore, by Equation* (10), *$Q_t$ can be controlled determinately andfor $t \geq 2$, we have the expressions below. For the transient price model*

$$Q_t = \prod_{i=1}^{t-1}(1-\beta_i)\cdot Q_1 + \sum_{k=2}^{t-1}\left[\prod_{i=k}^{t-1}(1-\beta_i)\right]\gamma_{k-1}S_{k-1} + \gamma_{t-1}S_{t-1}$$
$$= \prod_{i=1}^{t-1}(1-\beta_i)\left\{Q_1 + \sum_{k=1}^{t-1}\frac{\gamma_k S_k}{\prod_{i=1}^{k}(1-\beta_i)}\right\}, \tag{23}$$

*and for the permanent price model*

$$Q_t = \prod_{i=1}^{t-1}(1-\beta'_i)\cdot Q_1. \tag{24}$$

### 3.2. Properties of the Optimal Execution Strategy under Time-Homogeneous Parameter

The purpose of this subsection is to give an intuitive and intelligible analysis of the optimal strategies mainly for the permanent price model as it is difficult to give an analytical proof for the optimal execution strategy using transient price model. However we can show this intuition and confirm it using some numerical examples. To this end, we set some time-homogeneity assumptions for the impact $\lambda$, the reversion rate $\alpha$ and the resilience $\rho$. That is, $\lambda_t = \lambda$, $\alpha_t = \alpha$, and $\rho_t = \rho$. Here, in particular, we give a proof about comparative statics in risk aversion *R*, and for the other proofs of the properties, please refer to [8] [10], and [1]. For the detailed proofs of following Lemma 1 and propositions, refer to appendix.

**Lemma 1** (*Monotone Decrease Property*): *If $\lambda_t = \lambda$ and $\alpha_t = \alpha$, then for the permanent price model, the optimal execution volume decreases monotonously in time. That is,*

$$q_1'^* \geq q_2'^* \geq \cdots \geq q_T'^*. \tag{25}$$

For the proof of Lemma1, refer to [7]. From Lemma 1 the strategy for the permanent price model also satisfies the absence of transaction triggered price manipulation. Therefore,

$$0 \leq \beta_t \leq 1. \tag{26}$$

**Proposition 1**(*Risk Aversion Effect*): *Suppose $R_a$ and $R_b$ are the risk aversion coefficients of the large trader "a" and "b" then the more risk averse the large trader is, the earlier she executes. That is, for all t, if $R_a \geq R_b$, then for the permanent price model,*

$$Q_t(R_a,\lambda,\alpha) \leq Q_t(R_b,\lambda,\alpha). \tag{27}$$

If $R \to \infty$, it is optimal to submit the full volume at the initial time. That is, if the large trader is risk averse enough, she regards the volatility risk as important above all.

**Proposition 2** (*Risk Neutral Trader*): *Suppose $\lambda \neq 0$. If $R \downarrow 0$, then for the permanent price model, the optimal execution strategy is the naïve strategy* (*executing equally at each time*). *That is,*

$$q_t^* = \frac{Q_t}{T-t+1}, \quad \left(\beta_t = \frac{1}{T-t+1}\right) \tag{29}$$

*Moreover, for the transient price model, the optimal execution strategy is time symmetric. Then we form the following property,*

$$q_t^* = q_{T-t+1}^*. \tag{30}$$

**Remark 3**: *The optimal execution strategy for the transient price model does not have the monotone decrease property* (*Lemma* 1). *However from the numerical experiment shown in* **Figure 1**, *the convexity of the optimal execution volume in time can be confirmed for both price models. Moreover, we will also find that, $Q_t(R_a,\lambda,\rho) \leq Q_t(R_b,\lambda,\rho)$*

However, there is analytical difficulty for the proof of this property because the terms of $\beta$ and $\gamma$ depend

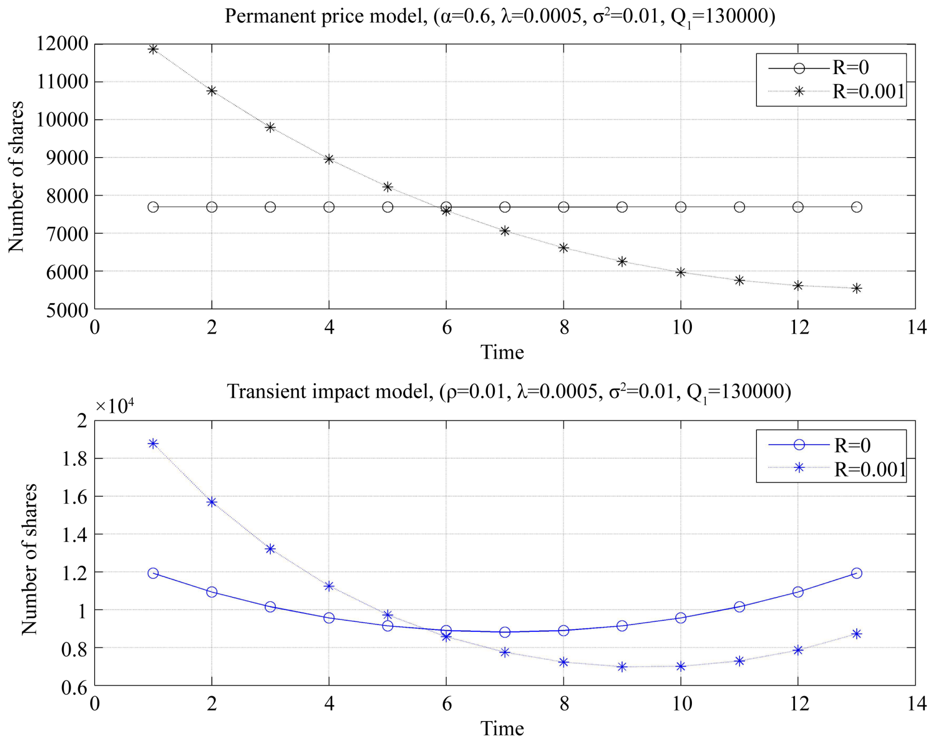


**Figure 1.** Optimal execution strategies for the permanent price model (upper half) and transient price model (lower half).

mutually on each other over time. In fact, when we express the optimal execution volume at time $t + 1$ with the states at time $t$,

$$q_{t+1}^{*} = \beta_{t+1}Q_{t+1} - \gamma_{t+1}S_{t+1} = \left(\beta_{t+1}\left(1-\beta_{t}\right) - l_{t}\beta_{t}\gamma_{t+1}\right)Q_{t} - \left(\mathrm{e}^{-\rho}\gamma_{t+1} - l_{t}\gamma_{t+1}\gamma_{t} - \beta_{t+1}\gamma_{t}\right)S_{t}. \quad (31)$$

**Figure 2** shows the relationship between $\beta Q$ (mainly the effect of the tradeoff between impact risk and volatility risk) and $\gamma S$ (mainly the effect of the expectations of price reversion over time) for the transient price model, which indicates the convexity property in time and also illustrates Proposition 2 (when $R = 0$). This decomposition of the optimal execution volume reveals the relationship between the existence of transaction-triggered price manipulation and the resilience effect. If the execution price reverts to below the previous price level or the unaffected price process has a possible drift (as in [11]), the optimal execution strategy would admit the transaction-triggered price manipulation. The proof of these properties and more detailed analysis of the dependency of the time grid are our ongoing research topics.

Under the time-homogeneity of $\lambda, \alpha$, and $\rho$, we give a simple numerical example of the optimal execution for the intraday trading strategies and support the previous propositions and remarks. The trading time is based on NYSE (New York Stock Exchange), and we divide the intraday into 13 periods (30 minutes length) to consider the execution time lag. For a more detailed explanation, refer to [12]. Assume that we must purchase 130,000 shares of a risky asset within 13 periods and $\lambda = 0.0005, \sigma^2 = 0.01, \alpha = 0.6,$ and $\rho = 0.01$. **Figure 1** illustrates the dependence of the optimal execution strategy on the risk aversion. In the upper (lower) half of **Figure 1**, the black (blue) line correspond to the risk neutral ($R\downarrow 0$) for the permanent (transient) impact model or the dotted black (blue) line correspond to the slightly risk averse large trader ($R = 0.00001$) for the permanent (transient) impact model. We can confirm that if the large trader is risk neutral ($R\downarrow 0$), Proposition 2 is satisfied. Moreover this figure shows that the more risk averse the large trader is, the earlier she executes. **Figure 2** also indicates the absence of transaction-triggered price manipulation since $\beta Q > \gamma S$.

## 4. Comparison of Two Price Models

So far, we considered two price models, the permanent and the transient with intrinsic parameter $\alpha$ and $\rho$.

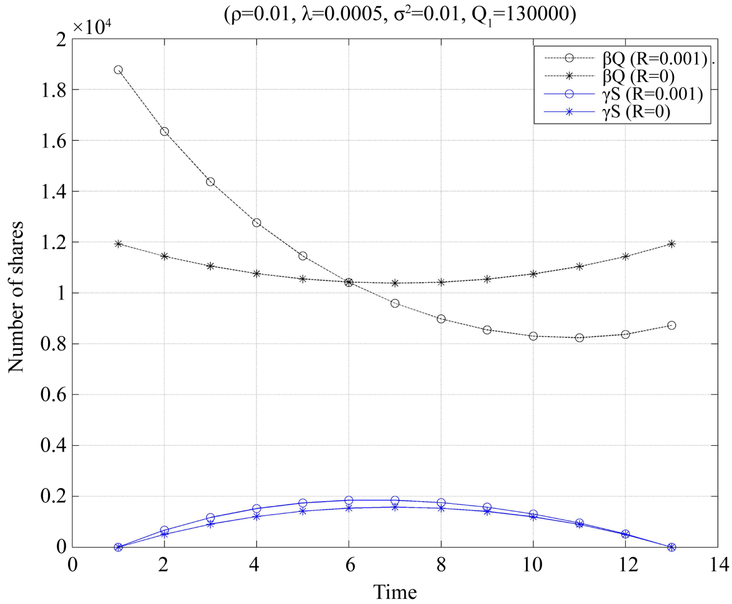


**Figure 2.** The optimal value of two components for the transient price model.

For the two price models describing a real market, if the expected costs derived from these two price models respectively with the same execution volume at the same intervals are different from each other, an arbitrage opportunity may occur between these two models. We should then unify how the information after each trade is incorporated into the price, when we compare the performance of the two price models. So, in order to standardize the market, we should find the relationship between $\alpha$ and $\rho$ so that the two price models are equivalent when the same strategy (TWAP strategy) is used. Here, the TWAP (Time Weighted Average Price) strategy stands for the equally execution over equidistant time interval. One way to do that is to show how to determine the value of parameter $\alpha$ if we can observe the value of $\rho$ however using the permanent price model under unobservable $\alpha$.

Suppose that the expected cost using TWAP strategy over the maturity $T$ with the permanent and the transient price model are respectively $E\left[C_{pe}\right]$ and $E\left[C_{tr}\right]$. Moreover suppose that $\rho$ is fixed. In the following, we define two criteria.

**Definition 3** (*TWAP Cost Equivalent*): *If* $E\left[C_{pe}\right]=E\left[C_{tr}\right]$, *then we say the market is TWAP cost equivalent.*

However, this condition does not satisfy the law of indifference which is a fundamental economic principle. As a stronger condition, we define TWAP equivalent condition as below.

**Definition 4** (*TWAP Equivalent*): *If* $E\left[p_t^{pe}\right]=E\left[p_i^{tr}\right]$, *then we say the market is TWAP equivalent*

We can afterward derive following conditions using Equations (3), (5), (9), (10), and letting $q$ = constant in order to adapt the transient price model according to the permanent price model.

**Condition 1**: *If the market is TWAP cost equivalent, then the following condition holds*:

$$\alpha=1-\frac{2\mathrm{e}^{-\rho}}{(T-1)\left(1-\mathrm{e}^{-\rho}\right)}+\frac{2\left(\mathrm{e}^{-\rho}-\mathrm{e}^{-\rho(T+1)}\right)}{T(T-1)\left(1-\mathrm{e}^{-\rho}\right)^2} \tag{32}$$

**Condition 2**: *If the market is TWAP equivalent, then the following condition holds*:

$$\alpha_t=1-\mathrm{e}^{-\rho t}. \tag{33}$$

The upper (lower) half of **Figure 3** shows that the value of $\alpha$ depending on Condition 1 (Condition 2) when $\rho=0.01$ or $0.5$ or $1$, and $T=13$.

The calculations of these conditions are straightforward. Within Condition 1, the mean of the accumulated transient impact at each time using the transient price model is regarded as the permanent impact, and then is assigned equally to $\alpha$. The upper (lower) half of **Figure 4** illustrates the optimal execution strategies for a risk-

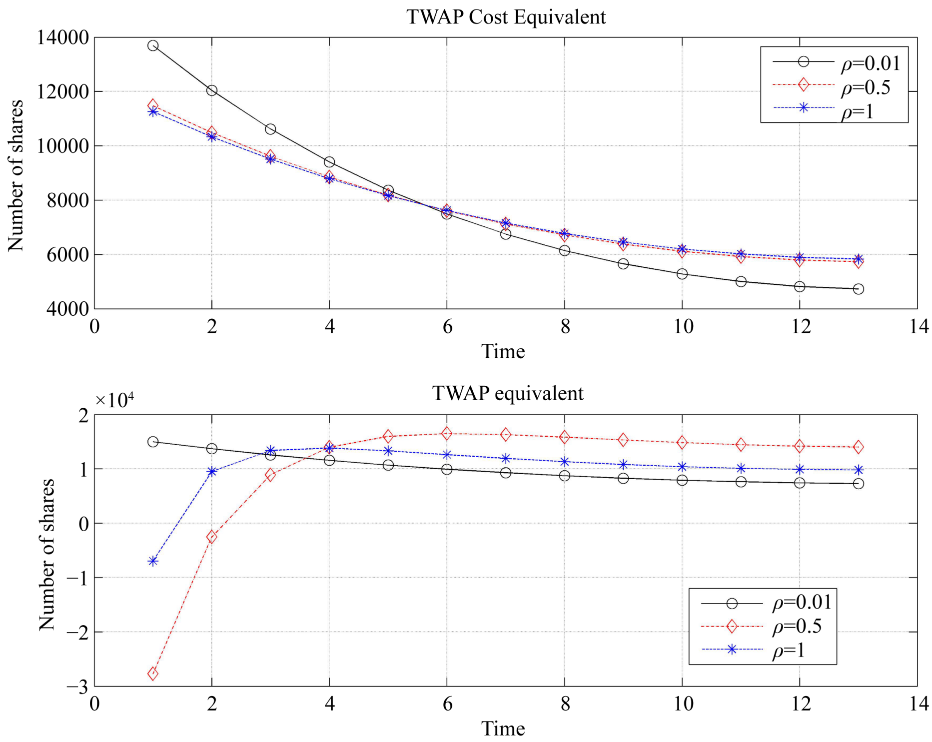


**Figure 3.** The value of α for TWAP cost equivalent (upper half) and TWAP equivalent (lower half).

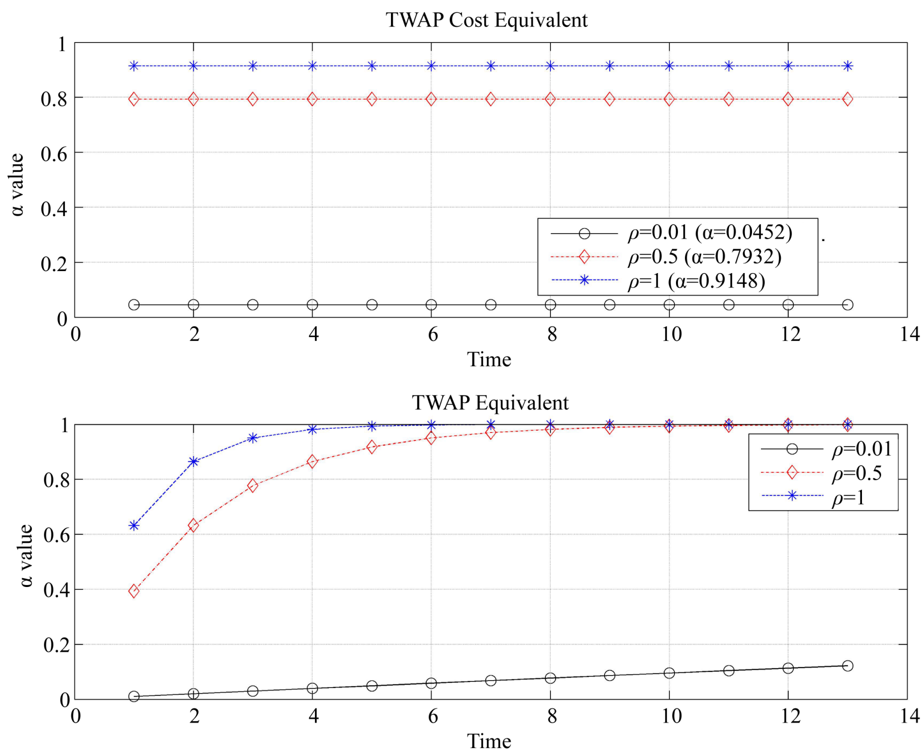


**Figure 4.** Optimal execution for TWAP cost equivalent (upper half) and TWAP equivalent (lower half).

averse large trader corresponding to the value of $\alpha$ in the upper (lower) half of **Figure 3**. We also set $Q_1 = 130000, R = 0.01, \lambda = 0.0005, \sigma^2 = 0.01, T = 13$, $\rho = 0.01$ or $0.5$ or $1$, and $T = 13$. This time, we can confirm that under a certain range of $\rho$, the optimal execution strategy for the permanent price model with Condition 2 does not satisfy the condition of absence of price manipulation stated in Definition 2. Nevertheless the total cost of the permanent price model with TWAP strategy is equal to that of the transient price model with the same TWAP strategy. So, we find that if $\rho$ is time-inhomogeneous then the optimal execution strategy violates the absence of transaction-triggered price manipulation. This fact indicates that although the permanent price model is simple and useful, if one wants to assess the execution performance, the transient price model is more stable in what concerns price manipulation.

**Remark 4**: *When* $\rho \to 0$ *in the transient price model*, *then from Equations* (10), (11), (17), *and* (19),

$$S_t = 0, \ C_t = \frac{R\sigma^2}{2} + A_{t+1} \ \text{ and } \ D_t = -\lambda_t + R\sigma^2 + 2A_{t+1}.$$

*Therefore the optimal execution strategy for the transient price model is the same as the permanent price model one with* $\alpha = 0$.

## 5. Conclusion

In a discrete time setting, we derived an explicit solution for the two price models by solving a dynamic programming equation backwardly from the maturity time. Under the assumptions of a large trader with CARA utility type and public news effects on price modeled as normal random variables, the optimal execution strategy exists in the static class. In particular, since the optimal execution volume for the transient price model consists of two components, that is tradeoff between impact risk and volatility risk, and the expectation of the price reversion, that solution gives consideration to the existence of transaction-triggered price manipulation. From the comparative statics, we also illustrated how the large trader's risk aversion affects the optimal execution strategy. Furthermore, with TWAP strategy we compared the performances of the two price models where the time-homogeneity of the parameters $\alpha$ and $\rho$ plays a significant role in the absence of price manipulation. But it is impossible to capture completely the essence of the price process with parameters using in this study. In recent years, an order driven market becomes mainstream in various trading venues around the world. Therefore, we should specify the shape of limit order book endogenously or exogenously in order to construct the price model. Further research consists on creating more practical models that takes for instance into consideration the intraday liquidity effect among other effects and the nonlinear impact function as empirically stated in [6], [12], and [13].

## Acknowledgements

The authors would like to thank participants of the International Conference on Industrial Engineering Theory, Applications and Practice (IJIE2013) at Pusan National University, the RIMS Workshop on Theory and Application of Mathematical Decision Making under Uncertainty 2013 at Kyoto University, and the Symposium on Stochastic Models 2014 at Tokyo University of Science for their constructive comments.

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## Appendix

*Short proof of Theorem*:

We can derive the optimal execution volume by backward induction from the maturity time $T$. For $t = T$, since the large trader must finish her purchases

$$Q_{T+1} = Q_T - q_T = 0 \tag{34}$$

Then,

$$q_T^* = Q_T \left( = \frac{D_T Q_T - L_T S_T}{2C_T} \right), \tag{35}$$

where we define the maturity condition as

$$\begin{cases} C_T := M \\ D_T := 2M \qquad (M = \text{const}) \\ L_T := 0 \end{cases} \tag{36}$$

and the value function is

$$V_T\left(w_T^{tr}, p_T^{tr}, Q_T, S_T\right) = -\exp\left\{-R\left(w_T^{tr} - p_T^{tr} Q_T - A_T Q_T^2 - B_T S_T Q_T + K_T S_T^2\right)\right\}, \tag{37}$$

and we set

$$\begin{cases} A_T := \lambda_T \\ B_T := 0 \\ K_T := 0 \end{cases} \tag{38}$$

where $A$, $B$ and $K$ are the coefficients of $Q^2$, $Q$, and $S$ respectively.

Next, for $t = T-1$, we first derive her expected utility

$$\begin{aligned} V_{T-1}^{\pi} &= E_{T-1}^{\pi}\left[-\exp\left\{-R\left(w_T^{tr} - p_T^{tr} Q_T - A_T Q_T^2\right)\right\}\right] \\ &= E_{T-1}^{\pi}\left[-\exp\left\{-Rw_{T-1} + R\hat{p}_{T-1} q_{T-1} + R\left(p_{T-1} + \varepsilon_T + \lambda_{T-1} \mathrm{e}^{-\rho} q_{T-1} - \left(1 - \mathrm{e}^{-\rho}\right) \mathrm{e}^{-\rho(T-1)} \sum_{i=1}^{T-2} \lambda_i \mathrm{e}^{\rho i} q_i \right)\right.\right. \\ &\qquad \left.\left. \times \left(Q_{T-1} - q_{T-1}\right) + RA_T\left(Q_{T-1}^2 - 2Q_{T-1} q_{T-1} + q_{T-1}^2\right)\right\}\right] \\ &= -\exp\left\{-Rw_{T-1} + Rp_{T-1} Q_{T-1} + R\left(A_T + \frac{R\sigma^2}{2}\right) Q_{T-1}^2 - RS_{T-1} Q_{T-1} + R\left(\lambda_{T-1} - \lambda_{T-1} \mathrm{e}^{-\rho} + \frac{R\sigma^2}{2} + A_T\right) q_{T-1}^2 \right. \\ &\qquad \left. - R\left[\left(-\lambda_{T-1} \mathrm{e}^{-\rho} + R\sigma^2 + 2A_T\right) Q_{T-1} - S_{T-1}\right] q_{T-1}\right\}, \end{aligned} \tag{39}$$

where we use

$$E_{T-1}\left[\exp\left\{R\left(Q_{T-1} - q_{T-1}\right)\varepsilon_T\right\}\right] = \exp\left\{\frac{R^2}{2}\left(Q_{T-1} - q_{T-1}\right)^2 \sigma^2\right\}.$$

$V_{T-1}^{\pi}$ is a concave function with respect to $q$. Therefore, the maximization of $V_{T-1}^{\pi}$ corresponds to the minimization of the expression in the brace of the exponential appearing in Equation (39). So the problem becomes a quadratic programming problem. Then,

$$q_{T-1}^* = \frac{\left(-\lambda_{T-1} \mathrm{e}^{-\rho} + R\sigma^2 + 2A_T\right) Q_{T-1} - S_{T-1}}{2\left(\lambda_{T-1} - \lambda_{T-1} \mathrm{e}^{-\rho} + \frac{R\sigma^2}{2} + A_T\right)} = \frac{D_{T-1} Q_{T-1} - L_{T-1} S_{T-1}}{2C_{T-1}}, \tag{40}$$

where

$$\begin{cases} C_{T-1} = \lambda_{T-1} - \lambda_{T-1}\mathrm{e}^{-\rho} + \dfrac{R\sigma^2}{2} + A_T = l_{T-1}\mathrm{e}^{\rho} + \dfrac{R\sigma^2}{2} + A_T \\ D_{T-1} = -\lambda_{T-1}\mathrm{e}^{-\rho} + R\sigma^2 + 2A_T \\ L_{T-1} = 1 \end{cases} \tag{41}$$

and the value function is

$$V_{T-1}\left(w_{T-1}^{tr}, p_{T-1}^{tr}, Q_{T-1}, S_{T-1}\right) = -\exp\left\{-R\left(w_{T-1}^{tr} - p_{T-1}^{tr}Q_{T-1} - A_{T-1}Q_{T-1}^2 - B_{T-1}S_{T-1}Q_{T-1} + K_{T-1}S_{T-1}^2\right)\right\},$$

where

$$\begin{cases} A_{T-1} = A_T + \dfrac{R\sigma^2}{2} + \dfrac{D_{T-1}^2}{4C_{T-1}} \\ B_{T-1} = \dfrac{D_{T-1}}{2C_{T-1}} - 1 \\ K_{T-1} = \dfrac{1}{4C_{T-1}} \end{cases} \tag{42}$$

Proceeding similarly for a general time *t*, we obtain the desired results (17), (19) with backward induction.

*Proof of Proposition* 1

From Lemma 1 and Remark 2, we show that if $R_a \geq R_b$, then

$$\beta_t'\left(R_a\right) \geq \beta_t'\left(R_b\right) \tag{43}$$

Denote the terms which does not depend on *R* in $C_t'$ and $D_t'$ as $c_t$ and $d_t$ respectively, then

$$\beta_t'\left(\cdot\right) = \frac{D_t'\left(\cdot\right)}{2C_t'\left(\cdot\right)} = \frac{d_t + R_{(\cdot)}\sigma^2}{c_t + R_{(\cdot)}\sigma^2} = 1 - \frac{c_t - d_t}{c_t + R_{(\cdot)}\sigma^2}. \tag{44}$$

From Remark 2, we have $c_t \geq d_t$. Therefore, $\beta_t'\left(R_a\right) \geq \beta_t'\left(R_b\right)$.

*Proof of Proposition* 2

When $t = T$, the large trader must finish her purchase, therefore $\beta_T = 1$.

Suppose that if $t = k$ then we have

$$\beta_k' = \frac{1}{T-k+1}. \tag{45}$$

We will show for $t = k-1$ that

$$\beta_k' = \frac{1}{T-k+2} \tag{46}$$

So,

$$\beta_{k-1}' = \frac{D_{k-1}'}{2C_{k-1}'} = \frac{-\left(1-\alpha\right)\lambda + 2A_k'}{2\alpha\lambda + 2A_k'} \tag{47}$$

From the assumption of Equation (45) and Equation (22), we get,

$$A_k' = A_{k+1}' - \frac{D_k'}{2}\beta_k' = \frac{T-k}{T-k+1}A_{k+1}' + \frac{\left(1-\alpha\right)\lambda}{\left(T-k+1\right)}.$$

Moreover, from the assumption of Equation (45)

$$\beta_k' = \frac{-\left(1-\alpha\right)\lambda + 2A_{k+1}'}{2\alpha\lambda + 2A_{k+1}'} = \frac{1}{T-k+1}$$

Then, $A_{k+1}$ becomes

$$A'_{k+1} = \frac{\alpha\lambda}{T-k} + \frac{\lambda(1-\alpha)(T-k+1)}{2(T-k)} = \frac{2\alpha\lambda + \lambda(1-\alpha)(T-k+1)}{2(T-k)}.$$

Therefore, from Equation (22), $A'_k$ is represented as,

$$A'_k = \frac{T-k}{T-k+1} \cdot \frac{2\alpha\lambda + \lambda(1-\alpha)(T-k+1)}{2(T-k)} + \frac{(1-\alpha)\lambda}{2(T-k+1)} = \frac{(1+\alpha)\lambda}{2(T-k+1)} + \frac{(1-\alpha)\lambda}{2}.$$

Then, by substituting the above $A'_k$ into Equation (47), we find that

$$\beta'_{k-1} = \frac{-(1-\alpha)\lambda + (1-\alpha)\lambda + \dfrac{(1+\alpha)\lambda}{T-k+1}}{2\alpha\lambda + (1-\alpha)\lambda + \dfrac{(1+\alpha)\lambda}{T-k+1}} = \frac{1}{T-k+2}.$$

That is Equation (46).

**Journal of Mathematical Finance, 2015, 5, 15-25**
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51002

# Modeling Returns and Unconditional Variance in Risk Neutral World for Liquid and Illiquid Market

**Ivivi Joseph Mwaniki**

School of Mathematics, Statistics, Actuarial and Finance Division, University of Nairobi, Nairobi, Kenya
Email: jimwaniki@uonbi.ac.ke

Received 6 January 2015; accepted 25 January 2015; published 28 January 2015

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## Abstract

**This article seeks to model daily asset returns using log-ARCH-Lévy type model which is expected to reproduce most of the stylized features of financial time series data (such as volatility clustering, leptokurtic nature of log returns, joint covariance structure and aggregational Gaussianity) that are empirically found in different types of market. In addition, unconditional variance of daily log returns in risk neutral world of different conditional heteroscedastic models is derived. A key observation is that liquid markets and illiquid market may not have the same underlying dynamics. For instance empirical analysis based on S&P500 index log returns as a liquid market do not have autoregressive part in their first moments while in Nairobi Securities Exchange NSE20 index there is strong presence of autoregressive dynamics of order three, *i.e.* AR(3). Higher moments of both markets are serially correlated.**

## Keywords

**AR-APARCH, Lévy Increments, Generalized Hyperbolic Distribution, Normal Inverse Gaussian, Illiquid Market**

## 1. Introduction

It is well known that the stock price changes are neither independent nor identically distributed. There are linear and nonlinear dependencies between successive price changes. Distributional assumptions concerning risky asset log returns play a key role in option pricing. According to research finding of Mandelbrot [1], evidence indicates that the empirical distributions of daily stock returns differ significantly from the traditional Gaussian model. In search of satisfactory descriptive models for financial data, large number of distributions have been

**How to cite this paper:** Mwaniki, I.J. (2015) Modeling Returns and Unconditional Variance in Risk Neutral World for Liquid and Illiquid Market. *Journal of Mathematical Finance*, **5**, 15-25. http://dx.doi.org/10.4236/jmf.2015.51002

tried (see for example, [2]-[6]).

The deviations from normality become more severe when more frequent data are used to calculate stock returns. Various studies have shown that the normal distribution does not accurately describe observed stock return data. Over the past several decades, some stylized facts have emerged about the statistical behavior of speculative market returns such as aggregational Gaussianity, volatility clustering, etc see [7] [8]. On the same note, most of the literature for example [9]-[12] and references therein, assume that daily log returns, can be modeled by exponential Lévy processes and geometric Lévy process.

There are two important directions in the literature regarding these type of stochastic volatility models. Continuous-time stochastic volatility process represented in general by a bivariate diffusion process, and the discrete time autoregressive conditionally heteroscedastic (ARCH) model of [13] or its generalization (GARCH) as first defined by [14]. Option pricing in GARCH models has been typically done using the local risk neutral valuation relationship (LRNVR) pioneered by [15]. The crucial assumptions in his construction are the conditional, normal distribution of the asset returns under the underlying probability space and the invariance of the conditional volatility to the change of measure. The empirical performance of these normal option pricing models has been studied extensively, for example in [16], [17].

The main focus of this paper is to develop a ARCH type Lévy model which attempts to capture some of the stylized features observed in demeaned log returns from any market data. More so we derive unconditional variance of daily log returns in risk neutral world of different ARCH type models, and an in-depth empirical study in liquid and illiquid market. All parameters are estimated from historical data, *i.e.* for S&P500 index from January 3, 1990 to January 18, 2008 and NSE20 index from March 2, 1998 to July 11, 2007.

The article is organized as follows. Section 2 provides a brief overview of ARCH type models and Lévy increments resulting to parameter estimation of observed salient features. In Section 3 which is our major contribution, unconditional variance of different ARCH type models is presented. Filtered Leptokurtic residuals of Lévy increments are calibrated. Conclusions are drawn in Section 4. Appendix is in the last section.

## 2. ARCH Type Models

ARCH-type models are in general, discrete models used to estimate volatility of financial time series data such stock returns, interest rates and foreign exchange rates. Let

$$r_t = \log\frac{S_t}{S_{t-1}} - \mathbb{E}\log\frac{S_t}{S_{t-1}}$$

where $S_t$ denotes the price of stock at time $t$. Define the following equation

$$r_t = \mu_t + \varepsilon_t \quad \varepsilon_t \sim N\left(0, \sigma_t^2\right), \tag{1}$$

where

$$\sigma_t^2 = \omega + \sum_{i=1}^{p}\alpha_i\varepsilon_{t-i}^2 + \sum_{j=1}^{q}\beta_j\sigma_{t-j}^2, \; t = 1, \cdots, T. \tag{2}$$

where $\sigma_t^2$ is the GARCH(*p*, *q*) volatility process. If $q = 0$ then $\sigma_t$ is ARCH(*p*). [18] and [19] provide a general specifications of volatility dynamic that nest most ARCH type models. In this connection volatility dynamics can be written as

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2 f\left(z_{t-1}\right)$$

where $f\left(z_{t-1}\right)$ is the innovation function. Different GARCH models are mainly characterized by the following specifications of the innovation function $f\left(z_{t-1}\right)$.

$$f\left(z_{t-1}\right) = \begin{cases} z_{t-1}^2, & \text{Simple;} \\ \left(z_{t-1} - \theta\right)^2, & \text{Leverage;} \\ \left\{\left|z_{t-1} - \theta\right| - k\left(z_{t-1} - \theta\right)^2\right\}, & \text{News;} \\ \left(z_{t-1} - \theta\right)^{2\gamma}, & \text{Power;} \\ \left\{\left|z_{t-1} - \theta\right| - \kappa\left(z_{t-1} - \theta\right)^{2\gamma}\right\}, & \text{News and power;} \end{cases} \tag{3}$$

The innovation function is used to model asymmetry and news impact to say the least. These GARCH models can be generalized to allow non-linearity of volatility dynamics by using Box-Cox transformation as follows

$$\sigma_t^{\psi} = \omega + \beta\sigma_{t-1}^{\psi} + \alpha\sigma_{t-1}^{\psi} f\left(z_{t-1}\right), \text{ with } f\left(z_{t-1}\right) = \left(z_{t-1} - \theta\right)^{2\psi} \tag{4}$$

which implies modeling news and power, will nest most of the proposed GARCH models in Literature. Note that the leverage parameter $\theta$ shifts the innovation function, the news parameter $\kappa$ tilts the innovation, and the power parameters $\gamma$ and $\psi$ flatten or steepen the innovation function. Such a model (4) is the Asymmetric Power Autoregressive Conditional Heteroscedastic model *i.e*. APARCH model defined in (5).

The APARCH($m$, $n$) model of can be written as follows

$$X_t = \varepsilon_t,\ \varepsilon_t = \sigma_t z_t,\ z_t \sim i.i.d\left(0,1\right)$$

$$\sigma_t^{\delta} = \omega + \sum_{i=1}^{m} \alpha_i \left(\left|\varepsilon_{t-i}\right| - \gamma_i \varepsilon_{t-i}\right)^{\delta} + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^{\delta} \tag{5}$$

subject to $\omega > 0, \delta \geq 0, \alpha_i \geq 0, -1 < \gamma < 1,$ for $i = 1,\cdots,m$, $\beta_j \geq 0$, for $j = 1,\cdots,n$. and

$$\sum_{i}^{m} k_i + \sum_{j}^{n} \beta_j < 1, \text{ where } k_i = \alpha_i \left(\left|\varepsilon_{t-i}\right| - \gamma_i \varepsilon_{t-i}\right)^{\delta} \tag{6}$$

The model introduces a Box-Cox power transformation on the conditional standard deviation process and on the asymmetric innovations, $\alpha_i \left(\left|\varepsilon_{t-i}\right| - \gamma_i \varepsilon_{t-i}\right)^{\delta}$, adds flexibility of a varying exponent with an asymmetry coefficient to take the leverage effect into account. The properties of APARCH model have been studied, see [20]. The model nests seven other ARCH extensions as special cases.

- ARCH model of [13] when $\delta = 2, \gamma_i$, and $\beta_j = 0$;
- GARCH model of [14] when $\delta = 2$, and $\gamma_i = 0$;
- GJR-GARCH Model of [21] when $\delta = 2$;
- TARCH Model of [22] when $\delta = 1$.

Note that $\mu_t = \mathbb{E}\left(r_t \middle| \mathfrak{F}_{t-1}\right)$ denote the conditional mean given the information set $\mathfrak{F}_{t-1}$ available at time $t-1$. The innovation process for the conditional mean is then given by $\varepsilon_t = r_t - \mu_t$ with corresponding unconditional variance $\sigma^2$ and zero unconditional mean. The conditional variance is defined as $\sigma_t = V\left(r_t \middle| \mathfrak{F}_{t-1}\right)$.

## 2.1. Empirical Data

For simplicity, we focus on daily closing indices $\{S_t\}$ as reported in Nairobi Securities Exchange for NSE20 share index and S&P500 index in New-York Stock Exchange. Daily log-returns $X_t$ of S&P500 index are computed from January 3, 1990 to January 18, 2008 for a total of 4550 daily observations. While for NSE20, share indexes are computed from March 2, 1998 to July 11, 2007 for a total of 2317 daily observations.

All return series exhibit strong conditional heteroscedasticity. The Ljung and Box test rejects the hypothesis of homoscedasticity at all common levels both for returns in S&P500 index and AR(3) residuals of linear regression in NSE20 share index. We estimate GARCH type models assuming conditional normality. With respect to the absolute value of parameter estimates, we find that $\left(0 < \alpha + \beta < 1\right)$ but different for both indices (NSE20 $\left(0 < \alpha + \beta = 0.924238 < 1\right)$, S&P500 $\left(0 < \alpha + \beta = 0.994097 < 1\right)$), indicating the typical higher persistence of shocks in volatility in New York Stock exchange compared to Nairobi Securities Exchange. Model (5) is estimated using Pseudo Maximum Likelihood estimator based on the assumption of conditional normal innovations. The parameter estimates of (8) are reported in **Table 1** and AR-ARCH residual calibrations of GH distribution (9) are presented in **Table 2**. Empirical and kernel densities of fitted distributions for both indices are compared in **Figure 1**.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \varepsilon_t, \varepsilon_t = \sigma_t Z_t, Z_t \sim N\left(0,1\right), \tag{7}$$

$$\sigma_t^{\delta} = \omega + \sum_{i=1}^{m} \alpha_i \left(\left|\varepsilon_{t-i}\right| - \gamma_i \varepsilon_{t-i}\right)^{\delta} + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^{\delta}$$

## 2.2. Lévy Increments

Suppose $\phi\left(u\right)$ is the characteristic function of a distribution. If for every positive integer $n$, $\phi_n\left(u\right)$ is the

**Table 1.** GARCH and GJR model estimates for the indices.

| | NSE20 | | S&P500 | |
|---|---|---|---|---|
| Parameter | GARCH | GJR $(\delta=2)$ | GARCH | GJR $(\delta=2)$ |
| $\phi_1$ | 0.18915 (0.024496) | 0.18136 (0.02424) | | |
| $\phi_2$ | 0.16451 (0.023785) | 0.16245 (0.02352) | | |
| $\phi_3$ | 0.11388 (0.023413) | 0.11516 (0.02308) | | |
| $\omega\times10^4$ | 0.03549 (0.006902) | 0.03458 (0.00647) | 0.006577 (0.001645) | 0.01088 (0.00204) |
| $\alpha$ | 0.15023 (0.017978) | 0.18578 (0.02528) | 0.056461 (0.0067528) | 0.00322 (0.00512) |
| $\beta$ | 0.78763 (0.024753) | 0.79045 (0.02373) | 0.937566 (0.0074845) | 0.93202 (0.0079) |
| GJR $(\gamma)$ | | −0.07332 (0.02592) | | 0.10558 (0.0123) |
| $Q(10)$ | 9.3468 (0.2287) | 8.8337 (0.2648) | 16.5309 (0.08541) | 15.2862 (0.1220) |
| $Q^2(10)$ | 7.1689 (0.5739) | 8.46159 (0.38973) | 6.8918 (0.54835) | 5.9298 (0.6551) |
| lgl | −8363.5 | −8367.7 | −15090.9 | −15090.9 |
| $n$ | 2316 | 2316 | 4549 | 4549 |

Notes: standard errors are in parenthesis. lgl is the log likelihood.

$n^{th}$ power of a characteristic function, we say that the distribution is infinitely divisible. One can define for every such infinitely divisible distribution a stochastic process $X=\{X_t, t\ge 0\}$ called a Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s+t], s,t\ge 0$ has $(\phi(u))^t$ is the characteristic function. For more detailed treatment of Lévy process, see [23].

**Definition 2.1** *The probability density function of the one-dimensional Generalized Hyperbolic distribution is given by the following*:

$$f_{GH}(x;\alpha,\beta,\delta,\mu,\lambda)=\frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\delta\gamma)}\frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\frac{1}{2}-\lambda}}e^{\beta(x-\mu)} \tag{8}$$

where $\gamma^2=\alpha^2-\beta^2$ and $K_\lambda$ is the modified Bessel function of third kind, with the index $\lambda$.

$$K_{\lambda}(\omega)=\frac{1}{2}\int_0^{\infty}\exp\left[-\frac{\omega}{2}\left(v^{-1}+v\right)\right]v^{\lambda-1}dv \tag{9}$$

$\mu$ is the location parameter and can take any real value, $\delta$ is a scale parameter; $\alpha$ and $\beta$ determine the distribution shape and $\lambda$ defines the subclasses of GH and is related to the tail flatness.

The mean and variance of GH distribution are given respectively by the followings

$$E(X)=\mu+\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\frac{K_{\lambda+1}(\zeta)}{K_{\lambda}(\zeta)} \tag{10}$$

and

$$\operatorname{Var}(X)=\delta^2\left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_{\lambda}(\zeta)}+\frac{\beta^2}{\alpha^2-\beta^2}\left[\frac{K_{\lambda+2}(\zeta)}{K_{\lambda}(\zeta)}-\left(\frac{K_{\lambda+1}(\zeta)}{K_{\lambda}(\zeta)}\right)^2\right]\right) \tag{11}$$

where $\zeta=\delta\sqrt{\alpha^2-\beta^2}$. Note that, if $X\sim GH(\lambda,\alpha,\beta,\delta,\nu)$, then

$$X\sim GH\left(-\frac{1}{2},\alpha,\beta,\delta,\mu\right) \text{ has normal-Inverse Gaussian distribution (NIG)}$$

$$X \sim GH(1,\alpha,\beta,\delta,\mu) \text{ hyperbolic distribution (HY)} \quad (12)$$

$$X \sim GH(\lambda,\alpha,\beta,0,\mu) \text{ variance-gamma distribution (VG)} \quad (13)$$

$$f_{GH}(x;\alpha,\beta,\delta,\mu,\lambda) = \frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\frac{1}{2}-\lambda}} e^{\beta(x-\mu)}$$

$$f_{NIG}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1\left(\alpha\delta\sqrt{1+z^2}\right)}{\sqrt{1+z^2}}$$

For more information about GH distribution, see [24].

## 3. Modeling the Underlying

Let $\left(\Omega,\mathcal{F},(\mathfrak{F}_t)_{t\in[0,T]},\mathbb{P}\right)$ be a stochastic basis describing the uncertainty of the economy. We refer to $\mathbb{P}$ as the physical probability measure and $\mathfrak{F}_t$ represent the information flow driven by Brownian motion $B=(B_t)_{t\in[0,T]}$ and Lévy proces $L=(\mathfrak{L}_t)_{t\in[0,T]}$. Let $S_t$ be the price of a stock at time $t$ adapted to the natural filtration $\mathfrak{F}_t$. Define daily log return as $X_t = \log S_t - \log S_{t-1}$. It is well known from our empirical studies that $X_t$ can he represented as $X_t = \mu_t + \varepsilon_t + \xi_t$ where $\mu_t$ is a mean function and $\varepsilon_t, \xi_t$ are the two components of the error term. Moreover, define a $p^{th}$ order autoregressive process $\{X_t\}$ with APARCH(m,n) error as

$$\begin{aligned}
&X_t = \mu_t + \varepsilon_t + \xi_t \\
&\mu_t = \sum_{r=1}^{p}\phi_r X_{t-r} + \mu,\ t\in\mathcal{Z}^+ \\
&\varepsilon_t + \xi_t = \sigma_t\left(Z_t + \sigma\mathfrak{L}_t\right),\ Z_t, \text{and } \mathfrak{L}_t \sim i.i.d(0,1), Z_0 = 0, \mathfrak{L}_0 = 0 \\
&\sigma_t = \text{APARCH}(m,n),\ m,n\in\mathbb{Z}^+
\end{aligned} \quad (14)$$

where $Z_t$ and $\mathfrak{L}_t$ are identically and independently distributed random variables. A general time series model for log returns would be

$$X_t = \mu_t + \sigma_t\left(Z_t + \sigma\mathfrak{L}_t\right),\quad Z_t \sim N(0,1),\ \mathfrak{L}_t \in GH$$

### 3.1. Risk Neutralization

In this section, we construct risk neutral probability measure in the context of [15] and [19]. Duan [15] introduced the GARCH option pricing model by generalizing the traditional risk neutral valuation methodology to the case of conditional heteroscedasticity, the so called Local Risk Neutral Valuation Relationship (LRNVR).

**Definition 3.1** *A pricing measure* $\mathbb{Q}$ *is said to satisfy the locally risk-neutral valuation relationship* (*LRNVR*) *if measure* $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$*, and*

**Table 2.** Calibration of AR-GARCH(1,1) residuals to a class of infinitely divisible distributions.

| NSE20 | GH | HY | NIG | S&P500 | GH | HY | NIG |
|---|---|---|---|---|---|---|---|
| $\lambda$ | −1.79233 | 1.0000 | −0.5000 | $\lambda$ | 2.38336 | 1.0000 | −0.500 |
| $\alpha$ | 0.98225 | 1.15813 | 0.66862 | $\alpha$ | 0.14671 | 1.68640 | 1.33977 |
| $\beta$ | −0.05226 | −0.06604 | −0.05864 | $\beta$ | −0.14279 | −0.14976 | −0.15755 |
| $\delta$ | 1.79373 | 0.45207 | 1.18530 | $\delta$ | 0.04052 | 1.04004 | 1.59588 |
| $\mu$ | 0.12296 | 0.13923 | 0.13014 | $\mu$ | 0.14292 | 0.15130 | 0.16032 |

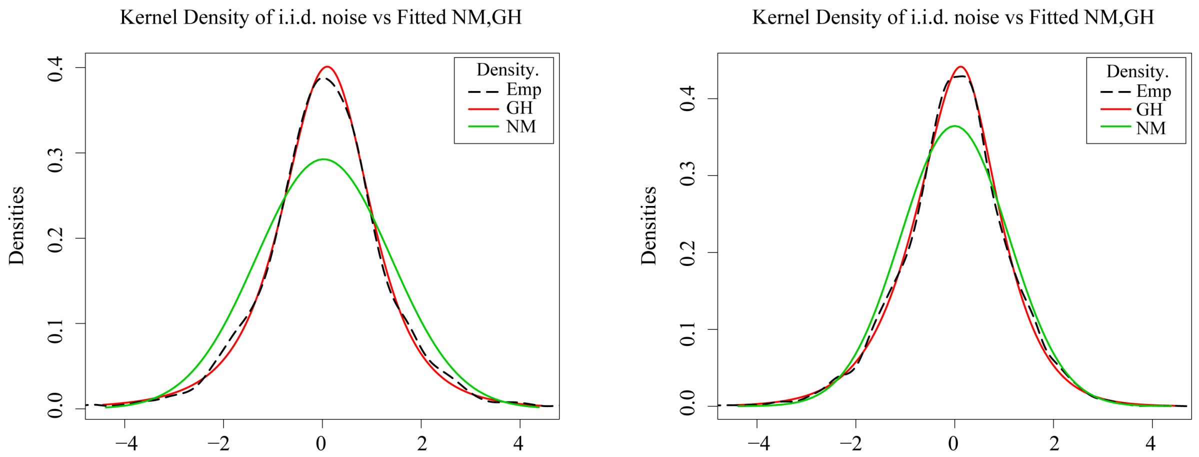


**Figure 1.** Empirical and kernel densities of standardized GARCH filtered Lévy increments of NSE20 index (left) S&P500 index (right) calibrated vs. density of fitted infinitely divisible distributions and normal distributions.

$$E^{\mathbb{Q}}\left[X_t \middle| \mathfrak{F}_{t-1}\right] = r \tag{15}$$

$$\mathrm{Var}^{\mathbb{Q}}\left(X_t \middle| \mathfrak{F}_{t-1}\right) = \mathrm{Var}^{\mathbb{Q}}\left(X_t \middle| \mathfrak{F}_{t-1}\right) \tag{16}$$

almost surely with respect to measure $\mathbb{P}$.

For some commonly used assumptions concerning utility functions and distributions of change of consumption, [15] shows that a representative agent maximizes his expected utility using the LRNVR measure $\mathbb{Q}$. Risk neutralization should leave the variance unchanged and should transform the conditional expectation so that the discounted expected price of the underlying asset becomes a martingale. It is worth noting that in the case of homoscedasticity process, $(p = 0, q = 0)$, the conditional variances become the same constant and the LRNVR reduces to conventional risk neutral valuation relationship.

Consider the general model of daily log returns under the data generating probability measure $\mathbb{P}$ as

$$X_t = \ln\left(S_t / S_{t-1}\right), \text{ where, } \begin{cases} X_t = \mu_t + \varepsilon_t + \xi_t; \\ \varepsilon_t \middle| \mathfrak{F}_{t-1} \sim N\left(0, \sigma_t\right); \\ \sigma_t^2 = \omega + \alpha\varepsilon_t + \beta\sigma_{t-1}^2 \end{cases} \tag{17}$$

where the parameters $\omega > 0, \alpha > 0$ and $\beta > 0$ and $1 - \beta - \alpha > 0$ and given $\sigma_0$. The sequence $\{\varepsilon_t\}$ and $\{\xi_t\}$ are conditionally independent, while $\mathfrak{F}_{t-1}$ is the past information set. $\mu_t$ represents the conditional expectation of returns.

The pricing measure $\mathbb{Q}$ shifts the error term $\varepsilon_t$ by some measurable function $\lambda_t$, so that the conditional expectation of $X_t$ becomes equal to $r$. In the case of AR(1)APARCH(1,1)-Lévy filter, we follow the [25] argument. Therefore under the equivalent martingale measure $\mathbb{Q}$ the model (16) translates to

$$X_t = \mu_t + \varepsilon_{1t} + \varepsilon_{2t}; \tag{18}$$

$$= \mu_t + \sigma_t\left(Z_t - \lambda_{1t}\right) + \sigma_t\left(\mathfrak{L}_t - \lambda_{2t}\right), \begin{cases} \mu_t = \nu + \phi X_{t-1}; \\ \lambda_{1t} = \left(\mu_t - r\right)/\sigma_t; \\ \lambda_{2t} = \mathbb{E}\mathfrak{L}_t; \\ \sigma_t^2 = f\left(\sigma_s^2, Z_s, \lambda_{1s}; -\infty < s < t\right); \end{cases} \tag{19}$$

The LRNVR implies that under the risk neutral measure $\mathbb{Q}$ the return process evolves as

$$X_t = r + \sigma_t\left(Z_t + \mathfrak{L}_t - \mathbb{E}\mathfrak{L}_t\right), \begin{cases} Z_t \sim N(0,1), \mathfrak{L}_t \sim NIG(\Theta); \\ \Theta = \left(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}\right); \end{cases} \tag{20}$$

$$\sigma_t^2 = \omega + \alpha\left(Z_{t-1} - \lambda_{t-1}\right)^2 \sigma_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{21}$$

$$\lambda_{t-1} = \left(\mu_{t-1} - r\right)/\sigma_{t-1}, \tag{22}$$

$$\mu_{t-1} = v + \phi X_{t-2}, \tag{23}$$

It follows quite easily that

$$\mathbb{E}^{\mathbb{Q}}\left[X_t \middle| \mathfrak{F}_{t-1}\right] = r \text{ and } \mathrm{Var}^{\mathbb{Q}}\left(X_t \middle| \mathfrak{F}_{t-1}\right) = \mathrm{Var}^{\mathbb{Q}}\left(X_t \middle| \mathfrak{F}_{t-1}\right) = \sigma_t^2\left(1 + \mathrm{Var}^{\mathbb{Q}}\mathfrak{L}_t\right) \tag{24}$$

## 3.2. Unconditional Variance

The following propositions provide the unconditional variance for the process $X_t$ under $\mathbb{Q}$

**Proposition 3.1** *Consider AR*(3) *APARCH*(1,1) *Lévy filter, with* $\delta = 2$ *and* $k = 0$ *which implies AR*(3)-*GARCH*(1,1) *Lévy model, the unconditional variance of* $X_t$ *under the LRNVR equivalent measure* $\mathbb{Q}$ *is*

$$\mathrm{Var}^{\mathbb{Q}} X_t = \frac{\left(1+\mathrm{Var}\mathfrak{L}_t\right)\left(\omega + \alpha\left[v - r\left(1 - \sum_{j=1}^{3}\phi_j\right)\right]^2 + 2r\sum_{i\neq j}^{3}\phi_j\phi_i\right)}{1 - \alpha\left[1 + \left(1+\mathrm{Var}\mathfrak{L}_t\right)\left(\sum_{j=1}^{3}\phi_j^2\right)\right] - \beta}$$

Proof: See Appendix. □

**Proposition 3.2** *A special case of AR*(1)*GARCH*(1,1)*Lévy filter the unconditional variance under the LRNVR equivalent measure* $\mathbb{Q}$ *is given by*

$$\mathrm{Var}^{\mathbb{Q}} X_t = \frac{\left(1+\mathrm{Var}\mathfrak{L}_t\right)\left[\omega + \alpha\left(v - r\left(1-\phi\right)\right)^2\right]}{1 - \alpha\left(1 + \phi^2\left(1+\mathrm{Var}\mathfrak{L}_t\right)\right) - \beta}$$

Proof: See Appendix. □

**Example 3.1** *In case of Hyperbolic distribution we substitute mean and variance respectively into* (25). *Where the parameters used maximize the likelihood function of Hyperbolic distribution. i.e. Let* $\zeta_{HP} = \delta_{HP}\sqrt{\alpha_{HP}^2 - \beta_{HP}^2}$ *then,*

$$\mathbb{E}\mathfrak{L}_t = \mu_{HP} + \frac{\beta_{HP}\delta_{HP}}{\sqrt{\alpha_{HP}^2 - \beta_{HP}^2}}\frac{K_2\left(\zeta_{HP}\right)}{K_1\left(\zeta_{HP}\right)}, \text{ and} \tag{25}$$

$$= 0.0073397 \tag{26}$$

$$\mathrm{Var}\mathfrak{L}_t = \delta_{HP}^2\left(\frac{K_2\left(\zeta_{HP}\right)}{\zeta_{HP}K_1\left(\zeta_{HP}\right)} + \frac{\beta_{HP}^2}{\alpha_{HP}^2 - \beta_{HP}^2}\left[\frac{K_3\left(\zeta_{HP}\right)}{K_1\left(\zeta_{HP}\right)} - \left(\frac{K_2\left(\zeta_{HP}\right)}{K_1\left(\zeta_{HP}\right)}\right)^2\right]\right) \tag{27}$$

$$= 1.713026 \tag{28}$$

Consider a discrete time economy, where interest rates and returns are paid after each time interval of equal spaced length. Suppose there is a price for risk, measured in terms of a risk premium that is added to the risk free interest rate *r* to build the expected next period return. As in Duan [15], we adopt and extend the ARCH-M model of [26] with the risk premium being linear functional of the conditional standard deviation, hence the following model under $\mathbb{P}$,

$$X_t = r + \lambda\sigma_t + \varepsilon_t \quad \text{where} \begin{cases} \varepsilon_t \middle| \mathfrak{F}_{t-1} = \sigma_t\left(Z_t + \mathfrak{L}_t\right), & \mathfrak{L}_t \text{ infinitely divisible density;} \\ Z_t \sim N\left(0,1\right), & Z_t \text{ Standard normal;} \\ \sigma_t^2 = \omega + \left(\alpha\sigma_{t-1}Z_{t-1}\right)^2 + \beta\sigma_{t-1}^2, & \mathrm{GARCH}\left(1,1\right); \end{cases} \tag{29}$$

The parameters $\omega$, $\alpha$, and $\beta$ are constant parameters satisfying stationarity and positivity conditions, while the constant parameter $\lambda$ may be interpreted as the unit price for risk. If we change the function $\sigma_t^2$ in (29) to model news impact, we get threshold GARCH model of [21] where

$$g(x) = \omega + \alpha_1 x^2 \mathbb{I}_{x<0} + \alpha_2 x^2 \mathbb{I}_{x\geq 0} \tag{30}$$

hence the resulting TGARCH Lévy filter model

$$X_t = r + \lambda\sigma_t + \varepsilon_t \text{ where } \begin{cases} \varepsilon_t \mid \mathfrak{F}_{t-1} = \sigma_t\left(Z_t + \mathfrak{L}_t\right), & \mathfrak{L}_t \text{ infinitely divisible density;} \\ Z_t \sim N(0,1), & Z_t \text{ Standard normal;} \\ \sigma_t^2 = g\left(\sigma_{t-1}Z_{t-1}\right) + \beta\sigma_{t-1}^2, & \text{TGARCH}(1,1); \end{cases} \tag{31}$$

**Proposition 3.3** *The unconditional variance of the GARCH-M Lévy filter model under the LRNVR equivalent martingale measure* $\mathbb{Q}$ *is*

$$\text{Var}^{\mathbb{Q}} X_t = \frac{\omega\left(1 + \text{Var}\mathfrak{L}_t\right)}{1 - \alpha\left(1 + \lambda^2\right) - \beta} \tag{32}$$

Proof: See Appendix. □

**Proposition 3.4** *The unconditional variance of the TGARCH-M Lévy filter model under equivalent martingale measure* $\mathbb{Q}$ *is*

$$\text{Var}^{\mathbb{Q}} X_t = \frac{\omega\left(1 + \text{Var}\mathfrak{L}_t\right)}{1 - \alpha_1\psi(\lambda) - \alpha_2\left(1 + \lambda^2 - \psi(\lambda)\right) - \beta} \tag{33}$$

where

$$\psi(u) = \frac{u}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right) + \left(1 + u^2\right)\Phi(u) \tag{34}$$

and $\Phi(u)$ denoting the cumulative standard normal distribution function.

Proof: See Appendix. □

## 4. Concluding Remarks

This article develops an log-ARCH-Lévy type risk neutral model. The proposed method delivers predictive distribution of the payoff function for a given econometric model. As a result, the probability distribution could be useful to market participants who wish to compare the model predictions to the potential prices in liquid and illiquid markets.

Any effective option pricing model is expected to be consistent with distributional and time series properties of the underlying asset. The proposed model accommodates most of the observed stylistic fact about financial time series data *i.e*. skewness and leptokurtic nature of demeaned GARCH filtered log returns and perhaps aggregational Gaussianity. In summary,

- developed markets and emerging markets may not have the same underlying dynamics. It would be incorrect to assume that a universal model for the underlying process for all markets.
- The presence of linear autoregressive dynamics AR(3)-GARCH(1,1) effects in NSE20 index affects the unconditional variance in risk neutral world. S&P500 index was found to follow GARCH(1,1) plus leptokurtic residual which was calibrated in one class of generalized hyperbolic distributions,say for example, Normal inverse Gaussian (NIG).
- The presence of autoregressive dynamics, *i.e*. AR(3)-GARCH(1,1) model of NSE20 index as an example of illiquid market would have an impact in pricing options, if the index were to be used as an underlying process.

The log-ARCH-Lévy model is very tractable compared to other jump-diffusion or stochastic volatility models. It attempts to addresses the drawbacks of local volatilities. Further refinements and extensions are left for future research.

## Acknowledgements

Comments from the Editor and the anonymous referee are acknowledged. Financial support from International Science Progam (Sweden)/EAUMP is greatly appreciated.

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## Appendix

Proof of proposition 3.1

Given $X_t = r + \sigma_t\left(Z_t + \mathfrak{L}_t - \mathbb{E}\mathfrak{L}_t\right); \lambda_t = \left(\mu_t - r\right)/\sigma_t; \mu_t = v + \sum_{j=1}^{3}\phi_j y_{t-j}$ We note that $\mathbb{E}^{\mathbb{Q}} X_t = r$ and

$$\mathbb{E}^{\mathbb{Q}}\left[X_t^2\right] = \mathbb{E}^{\mathbb{Q}}\left(r^2 + 2r\sigma_t Z_t\left(\mathfrak{L}_t - \mathbb{E}\mathfrak{L}_t\right) + \sigma_t^2 Z_t^2\left(\mathfrak{L} - \mathbb{E}\mathfrak{L}_t\right)^2\right) = r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_t^2\left(1 + \mathrm{Var}\mathfrak{L}_t\right)$$

$$\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2\right] = \omega + \alpha\mathbb{E}^{\mathbb{Q}}\left(Z_{t-1} - \lambda_{t-1}\right)^2\sigma_{t-1}^2 + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 = \omega + \alpha\left(\mathbb{E}^{\mathbb{Q}}\left[\sigma_{t-1}^2\right] + \mathbb{E}^{\mathbb{Q}}\left(\mu_t - r\right)^2\right) + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2$$

after rearranging and simple algebra

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\mu_{t-1} - r\right]^2 &= v^2 + \left(r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2\left(1 + \mathrm{Var}\mathfrak{L}_t\right)\right)\left(\sum_{j=1}^{3}\phi_j^2\right) - 2vr\left(1 - \sum_{j=1}^{3}\phi_j\right) + r^2\left(1 - \sum_{j=1}^{3}\phi_j\right) + \cdots \\ &= v^2 + \left(r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2\left(1 + \mathrm{Var}\mathfrak{L}_t\right)\right)\left(\sum_{j=1}^{3}\phi_j\right) + r\left(1 - 2v\right)\left(1 - \sum_{j=1}^{3}\phi_j\right) + 2r^2\sum_{j\neq k}^{3}\phi_j\phi_k\end{aligned}$$

Thus under stationarity, the unconditional expectations are independent of $t$

$$\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2\right] = \frac{\omega + r^2\left(\sum_{j=1}^{3}\phi_j\right) + r\left(1 - 2v\right)\left(1 - \sum_{j=1}^{3}\phi_j\right) + 2r^2\sum_{j\neq k}^{3}\phi_j\phi_k}{1 - \alpha\left[1 + \left(1 + \mathrm{Var}\mathfrak{L}_t\right)\sum_{j=1}^{3}\phi_j^2\right] - \beta}$$

Therefore, the unconditional variance of AR(3)GARCH(1,1)Levy filter model under LRNVR equivalent martingale measure is

$$\begin{aligned}\mathrm{Var}^{\mathbb{Q}} X_t &= \frac{\left(1 + \mathrm{Var}\mathfrak{L}_t\right)\left(\omega + \alpha\left[v^2 - 2vr\left(1 - \sum_{j=1}^{3}\phi_j\right) + r^2\left(1 - 2\sum_{j=1}^{3}\phi_j + \sum\phi_j^2\right) + 2r\alpha\sum_{j\neq k}^{3}\phi_j\phi_k\right]\right)}{1 - \alpha\left[1 + \left(1 + \mathrm{Var}\mathfrak{L}_t\right)\left(\sum_{j=1}^{3}\phi_j\right)\right] - \beta} \\ &= \frac{\left(1 + \mathrm{Var}\mathfrak{L}_t\right)\left(\omega + \alpha\left[v - r\left(1 - \sum_{j=1}^{3}\phi\right)\right]^2 + 2\alpha r\sum_{i\neq j}^{3}\phi_j\phi_i\right)}{1 - \alpha\left[1 + \left(1 + \mathrm{Var}\mathfrak{L}_t\right)\left(\sum_{j=1}^{3}\phi_j^2\right)\right] - \beta}\end{aligned}$$

Proof of proposition 3.2

This is a special case of (3.1) with $\phi_1 = \phi$ and $\phi_2 = \phi_3$.

Proof of proposition 3.3

It is a special case of proposition 3.4 when we take $\alpha_1 = \alpha$ and $\alpha_2 = 0$.

Proof of proposition 3.4

Under measure $\mathbb{Q}$.

$$X_t = r + \varepsilon_t = r + \sigma_t\left(\lambda + Z_t + \mathfrak{L}_t - \mathbb{E}\mathfrak{L}_t\right)$$

where $\lambda$ is the risk premium and

$$\sigma_t^2 = \omega + \alpha_1\sigma_{t-1}^2\left(Z_{t-1} - \lambda\right)^2_{\mathbb{I}_{Z_{t-1}^{\prime}<0}} + \alpha_2\sigma_{t-1}^2\left(Z_{t-1} - \lambda\right)^2_{\mathbb{I}_{Z_{t-1}^{\prime}<0}}$$

$$\mathrm{Var}^{\mathbb{Q}} X_t = \mathbb{E}^{\mathbb{Q}} y_t^2 - r^2 = r^2 + \mathbb{E}\sigma_{t-1}^2\left(1 + \mathrm{Var}\mathfrak{L}_t\right) - r^2$$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\sigma_t^2 &= \omega + \alpha_1\psi\left(\lambda\right)\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 + \alpha_2\left[1 + \lambda^2 - \psi\left(\lambda\right)\right]\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 \\ &= \frac{\omega}{1 - \alpha_1\psi\left(\lambda\right) - \alpha_2\left(1 + \lambda^2 - \psi\left(\lambda\right)\right) - \beta}\end{aligned}$$

$$\text{thus } \operatorname{Var}^{\mathbb{Q}}(X_t) = \frac{\omega(1+\operatorname{Var}\mathfrak{L}_t)}{1-\alpha_1\psi(\lambda)-\alpha_2(1+\lambda^2-\psi(\lambda))-\beta}$$

where, $\psi(u) = \frac{u}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)+(1+u^2)\Phi(u)$

and $\phi(u)$ denoting the cumulative standard normal distribution. Note that $Z'_{t-1} \sim N(-\lambda,1)$ and

$$\begin{aligned}
&\mathbb{E}\left[Z_t'^2 \mathbb{I}_{Z_t'<0} \middle| \mathfrak{F}_{t-1}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} Z^2 \exp\left(-(Z+\lambda)^2/2\right)\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda}(u-\lambda)^2 \exp\left(-u^2/2\right)\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} u^2 \exp\left(-u^2/2\right)\mathrm{d}u - \frac{2\lambda}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} u \exp\left(-u^2/2\right)\mathrm{d}u + \frac{\lambda^2}{\sqrt{2\pi}}\int_{-\infty}^{\lambda}\exp\left(-u^2/2\right)\mathrm{d}u \\
&= \frac{-\lambda}{\sqrt{2\pi}}\exp\left(-\lambda^2/2\right)+\Phi(\lambda)+\frac{2\lambda}{\sqrt{2\pi}}\exp\left(-\lambda^2/2\right)+\lambda^2\Phi(\lambda) \\
&= \frac{\lambda}{\sqrt{2\pi}}\exp\left(-\frac{\lambda^2}{2}\right)+(1+\lambda^2)\phi(\lambda) \\
&=: \psi(\lambda)
\end{aligned}$$

Therefore, for positive support

$$\mathbb{E}^{\mathbb{Q}}\left[Z_t^2 \mathbb{I}_{Z_t\geq 0} \middle| \mathfrak{F}_{t-1}\right] = 1+\lambda^2-\psi(\lambda)$$

**Journal of Mathematical Finance, 2015, 5, 26-39**
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51003

# Testing Continuous-Time Interest Rate Model for Chinese Repo Market

**Huimin Zhao*, Fangping Peng**

Sun Yat-Sen Business School, Sun Yat-sen University, Guangzhou, China
Email: *zhaohuim@mail.sysu.edu.cn, pengfp@mail.sysu.edu.cn

Received 8 January 2015; accepted 26 January 2015; published 29 January 2015

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## Abstract

**This paper tests the popular continuous-time interest rate models for Chinese repo market to address what and how the interest rates change with the marketlization in China. Using Bandi [1]'s method, we get the functional nonparametric estimation of drift and diffusion terms and the local time of the process. We find that the interest rates of China during the period from 1993 to 2003 are bimodal distributed and propose a two-regime model which can fit the data better. We also study the probabilities that the process will stay the two regimes respectively and its transition probability that the process transfers from one regime to another regime.**

## Keywords

**Chinese Repo Market, Interest Rate, Nonparametric Estimation**

## 1. Introduction

The short rate is fundamental to the pricing of fixed-income securities. Large literature devotes itself to the estimation of the short term interest rate process using different models and methods. In continuous time finance, the dynamic evolution of the spot interest rate process is usually driven by a Markov stochastic differential equation. Diffusion processes have become the standard tool for modelling prices in financial markets for derivative pricing and risk management purposes. Although such continuous time processes offer analytic tractability, the parameters of the process are often difficult to estimate from the data because sample data are available only at discrete time points.

Literature has documented different parametric models for short rate dynamics, each attempting to capture particular features of observed interest rate movements. However, empirical tests of these models have yielded mixed results. Therefore, nonparametric techniques are well-used to remove some distributional restrictions im-

*Corresponding author.

**How to cite this paper:** Zhao, H.M. and Peng, F.P. (2015) Testing Continuous-Time Interest Rate Model for Chinese Repo Market. *Journal of Mathematical Finance*, **5**, 26-39. http://dx.doi.org/10.4236/jmf.2015.51003

posed by parametric models.

Ait-Sahalia [2] compares their implied parametric density to the same density estimated nonparametrically and finds strong evidence that CEV diffusions with linear drifts do not fit the data well. Stanton [3] employs the first-order nonparametric method to estimate drift and diffusion of the short rate, whose results also indicate that there is substantial evidence of nonlinearity in the drift. Jiang and Knight [4] investigate the finite sample properties of various estimators using the Monte Carlo simulation. They observe that while all the parametric diffusion estimators perform well, the parametric drift estimators perform poorly. Moreover, both the nonparametric diffusion and drift estimators perform reasonably well.

An assumption commonly made in nonparametric methods is the stationarity of the process. Notwithstanding the advantages of assuming stationarity, it would be helpful to allow for martingale and other possible forms of non-stationary behavior in the process. Motta and Hafner [5] study locally stationary factor models by the nonparametric estimation. Florens and Simoni [6] investigate the nonparametric estimation of an instrumental regression. Restrepo-Tobn and Kumbhakar [7] apply nonparametric estimation to study US banks. Kristensen [8] tests a diffusion model by nonparametric estimation.

Bandi and Philips [9] construct a nonparametric method for scalar diffusion models without imposing the stationary assumption. They assume recurrence which is less restrictive than stationarity. Bandi and Neuyen [10] derive the properties of local time. They also develop a procedure for estimating functions non-parametrically from data observed only at discrete time intervals based on US short rate data. Johannes [11] applies the same method on US 3-month Treasury bill data even though his results reflects negatively on one-factor diffusion model.

There is no large literature investigating Chinese short interest rate market. Interest rates can be regarded as a benchmark to distribute rare capital by interest rate mechanism in the financial market. It is meaningful to study whether the interest rate is decided by the mechanism of market competition or not. Hong and Lin [12] test the discrete-time model for the Chinese spot interest rate. Most of literature focuses on the term structure model or monetary policy of China, such as Duffee and Stanton [13], Siegel [14] and He and Wang [15].

In this paper, we study the interest rate behavior of China based on the observed 7 days repo rate for Shanghai market. The repo rate provides the benchmark for the interest rate of marketability and pricing of national debt futures. With the interest rate marketlization of China, the movement of interest rate reflects the principle of the supply and demand tightly. We follow Bandi and Philips [9]'s method to assume recurrence only and examine how well it could fit China data under non-parametric model without stability. We find that the interest rates behaved very differently during the two subperiods, so we assume the density of the process is bimodal. Based on the evidence of local time of sub-sample data, we estimate the parameters for a two regime model with the year 1999 as the change point.

The paper is organized as follows. Section 2 introduces the data and method. Section 3 gives the empirical results. Section 4 discusses the two-regime model and its properties implied by the empirical results. Conclusions are given in Section 5.

## 2. Data and Method

### 2.1. Data

We use 7-day repo rate of Shanghai market of China as the proxy of Chinese repo market. The data are retrieved from database of China Center for Economic Research (CCER) of Peking University. On the pre-holiday days such as the one-week holiday on the labor day, National day and Chinese new year, the interest rates are abnormally high since they are not real interest rates for 7 days, so I removed these from my observations. The final data set is composed of 2052 daily observations from January 4, 1995 to December 31, 2003. The short rate is continuously compounded yield to maturity. **Figure 1** gives the changing of time series of the sample data.

From **Figure 1**, it is clear that the data has a different feature before and after 1999. Before 1999, interest rates stayed at a higher level, but they dropped dramatically after 1999. This is consistent with the change of term structure in Chinese money market. **Figure 1** also shows the daily change (difference between the two successive days) of the spot rates. It also shows similar pattern with the daily data: interest rates become less volatile since 1999.

We study the behavior of short interest rates by two sub-samples 1995.01-1998.12 and 1999.01-2003.12. The results of preliminary analysis of the whole sample and sub-samples are shown in **Table 1**. Panel A presents

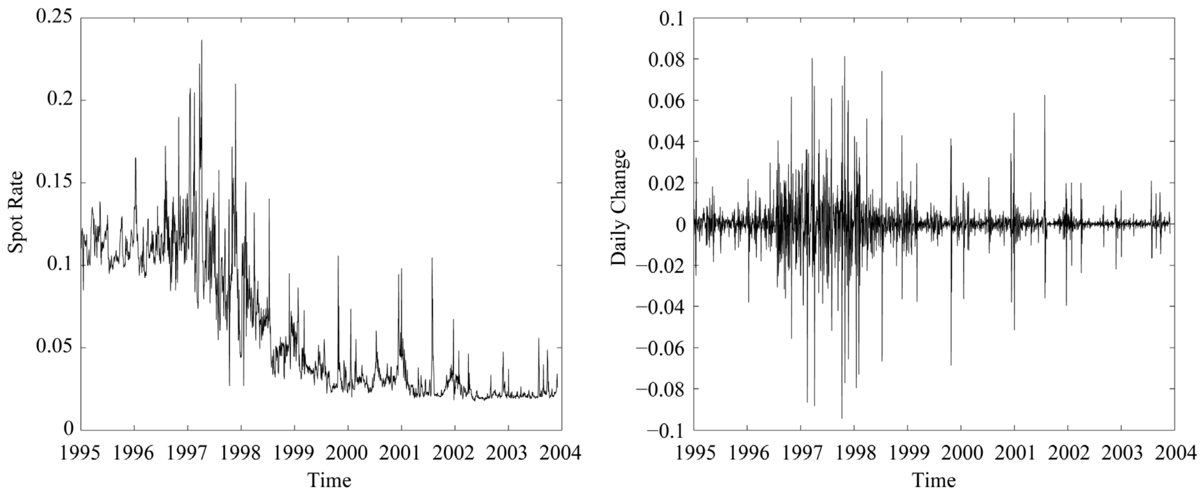


**Figure 1.** The figures shows the daily time-series and daily changes of 7-day repo rate for Shanghai market respectively. The sample period is 1995.01-2003.12 (2052 observations).

**Table 1.** Descriptive statistics of repo rate and hypothesis test.

| A. Summary statistics of daily repo rate $r_t$ | | | | | |
|---|---|---|---|---|---|
| Sample period | Mean | Std | Skewness | Kurtosis | First Autocorr |
| 1995.01-2003.12 | 0.0606 | 0.0416 | 0.7804 | −0.4660 | 0.9626 |
| 1995.01-1998.12 | 0.0989 | 0.0310 | 0.1340 | 0.8521 | 0.8789 |
| 1999.01-2003.12 | 0.0283 | 0.0097 | 2.6744 | 13.0090 | 0.7757 |
| B. Summary statistics of daily change of repo rate $r_t - r_{t-1}$ | | | | | |
| Sample period | Mean | Std | Skewness | Kurtosis | First Autocorr |
| 1995.01-2003.12 | −4.38E−05 | 0.0114 | −0.4823 | 18.1432 | −0.1928 |
| 1995.01-1998.12 | −7.46E−05 | 0.0152 | −0.3962 | 9.8552 | −0.1620 |
| 1999.01-2003.12 | −1.79E−05 | 0.0065 | −0.4006 | 35.0510 | −0.3350 |

C. Hypothesis test for daily rate of two subperiods

$H_0: \sigma_1^2 = \sigma_2^2$ F value: 8.6169 F critical value at 1%: 1.152

result: $H_0$ is rejected at 1% level

$H_0: \mu_1 = \mu_2$ U value: 67.07 U critical value at 1%: 2.58

result: $H_0$ is rejected at 1% level

D. Hypothesis test for daily change of two subperiods

$H_0: \sigma_1^2 = \sigma_2^2$ F value: 4.6177 F critical value at 1%: 1.152

result: $H_0$ is rejected at 1% level

$H_0: \mu_1 = \mu_2$ U value: 0.1065 U critical value at 10%: 1.65

result: $H_0$ cannot be rejected at 10% level

E. Wilcoxon Rank Sum Test for daily repo rate $r_t$

$H_0: F_1(r) = F_2(r)$ U value: 71.98 critical value at 1%: 4.9

result: $H_0$ is rejected at 1% level

F. Wilcoxon Rank Sum Test for daily change rate of two subperiods

$H_0: F_1(r) = F_2(r)$ U value: 14.15 critical value at 1%: 4.9

result: $H_0$ is rejected at 1% level

This table presents the mean, standard deviation, skewness, kurtosis, and the first autocorrelation of the daily data and daily change of entire sample period and two subperiods. It also gives the hypothesis test about mean and variance of daily rate and daily change rate of two subperiods respectively. Panel E and F give the Wilcoxon Rank Sum test to test whether two subperiods have the same distribution.

the statistics of continuously compounded annualized daily repo rate $r_t$. The first autocorrelation of whole sample is close to 1, and two subperiods have significant different means, standard deviations and skewness. After 1998, interest rates have a higher mean, higher positive skewness and lower volatility. With the marketlization of interest rates, the distribution of them may become more asymmetry because of the stochastic market. Panel B gives the summary statistics of daily change of repo rate $r_t - r_{t-1}$. The first autocorrelation of the daily change is lower and negative with a negative and positive kurtosis.

Panel C shows the result of hypothesis test for daily rate. It shows that the two subperiods have significant different means and variances. But this may induce that the stationarity of the whole data cannot be guaranteed. Using the same hypothesis test with the Panel C, I tested for daily change of the two subperiods in Panel D. The null hypothesis that the two subperiod samples have the volatility was rejected at 1% level.

Furthermore, from Panel A and Panel B of **Table 1**, the skewness and kurtosis of repo rate and daily change rate are not consistent based on the Wilcoxon Rank Sum test in Panel E and F. This test is a nonparametric alternative to test whether the two samples have the same distribution when their distribution are not known. We find that the two subperiod data follow the different distributions with the different mean and variance. This means that the stationarity of the whole data process may not be guaranteed. **Table 2** shows the result for the linear stationary test. The null hypothesis of a unit root was rejected at 5% level based on the augmented Dickey-Fuller test (ADF, see Harvey [16]).

**Figure 2** gives the frequency histogram of the whole data. The height presents the times that the repo rate appears in a small vicinity of a point. It is clear that there are two peaks in the figure at about 3% and 11%.

Based on the above analysis using the repo rate data sample, we add a state variable into our model for our empirical study.

## 2.2. The Model

We assume that the short rate follows a stochastic differential equation as follows:

$$\mathrm{d}r_t = \mu(r_t, s_t)\mathrm{d}t + \sigma(r_t, s_t)\mathrm{d}z_t \tag{1}$$

where $z_t$ is a standard Brownian motion, $\mu$ and $\sigma$ are the drift and diffusion of interest rate process respectively which depend on the values of the short rate $r_t$ and a state variable $s_t$ which has two states 1 and 2. Models such as interest rate models of Cox, Ingersoll and Ross (CIR) [17], Vasicek [18], Hull and White [19] are special cases of this model.

But parametric interest rate models may not fit historical data well. Ait-Sahalia [2] reject "$\cdots$every parametric model of the spot rate [previously] proposed in the literature". Jiang and Knight [4] also think that the

**Table 2.** Unit root test for repo rate.

| A. Estimates of parameters | | | | | | |
|---|---|---|---|---|---|---|
| Parameters | $\mu$ | $\phi$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ |
| Estimation | 0.0012 | −0.0212 | −0.2394 | −0.1666 | −0.1597 | −0.1324 |
| Standard error | 0.0004 | 0.0059 | 0.0222 | 0.0224 | −0.0219 | −0.02062 |
| T value | 2.8044 | −3.5900 | −10.7800 | −7.4100 | −7.1300 | −6.0390 |

B. Augmented Dickey-Fuller T Test

$H_0$: there exits a unit root

T statistics: $\hat{\tau}_\mu = \frac{\hat{\phi}}{\left(\mathrm{avar}\left(\hat{\phi}\right)\right)^{1/2}} = -3.59$ Critical value at 5%: −0.86

Result: the null hypothesis of a unit root is rejected at 5%.

This table presents the statistics of Augmented Dickey Fuller *T* test for the daily annualized yield on repo rate for Shanghai market. The model used in the test is: $\Delta r_t = \mu + \phi r_{t-1} + \sum_{i=1}^{4} \phi_i \Delta r_{t-i} + u_t$. Panel A reports the estimates of parameters with standard error and *t*-statistics. Panel B reports the test of unit root. The sample period is 1995.01-2003.12.

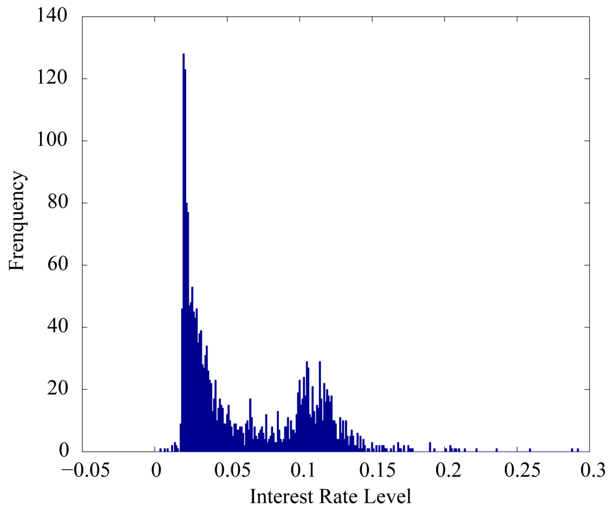


**Figure 2.** This figure shows that frequency histogram of time-series of 7-day repo rate for Shanghai market. The sample period is 1995.01-2003.12.

parametric drift estimator performed very poorly. Therefore, we follow the nonparametric estimation techniques which is popular in recent literature related.

The basis for our Monte Carlo simulation is a time-discretization of (1) over a daily interval ( $\Delta = 1$ day)

$$r_{t+\Delta} = r_t + \mu(r_t, s_t)\Delta + \sigma(r_t, s_t)\epsilon_t^{\Delta} \tag{2}$$

where $\epsilon_t^{\Delta}$ is a standard normal process with zero mean and $\Delta$ variance.

After the drift and diffusion estimates are obtained, the next short rate will be simulated according to this data-generating process. After repeating this process a large number, *G*, sample paths from the true continuous-time model are produced, then the Mento Carlo confidence bands can be determined.

## 2.3. Nonparametric Estimation Method

As Johannes [11] mentioned, nonparametric estimation method firstly requires little prior information relating to the functional form of the conditional expectations, so it doesn't need to estimate the type of the function as parametric estimation. Second, nonparametric estimators focus on local effects. This implies that the abnormal or very volatile sub-sample will not change any of the conclusion. The final advantage of nonparametric estimation method is that the estimators are feasible and easy to evaluate.

Based on the nonparametric model of Stanton [3] and econometric estimation, which is wildly used by Jiang [20], Bandi [1] and Johannes [11], we suppose that the short rate process follows one factor model, not considering the state variable $s_t$ with $n$ observations of interest rates $r_t$ at $t = t_1, t_2, \cdots, t_n$, *i.e.*, $r_{\Delta}, r_{2\Delta}, \cdots, r_{n\Delta}$. The model and data-generating process are the following:

$$\mathrm{d}r_t = \mu(r_t)\mathrm{d}t + \sigma(r_t)\mathrm{d}z_t \tag{3}$$

$$r_{t+\Delta} = r_t + \mu(r_t)\Delta + \sigma(r_t)\epsilon_t^{\Delta} \tag{4}$$

where the parameters in Equations (3) and (4) are the same as in Equation (2).

The estimators of drift and diffusion terms are:

$$\hat{\mu}(r) = \frac{\sum_{i=1}^{n-1} K\left(\frac{r_{i\Delta} - r}{h}\right)\left(\frac{r_{(i+1)\Delta} - r_{i\Delta}}{\Delta}\right)}{\sum_{i=1}^{n-1} K\left(\frac{r_{i\Delta} - r}{h}\right)} \tag{5}$$

$$\hat{\sigma}^2(r) = \frac{\sum_{i=1}^{n-1} K\left(\frac{r_{i\Delta} - r}{h}\right)\left(\frac{\left(r_{(i+1)\Delta} - r_{i\Delta}\right)^2}{\Delta}\right)}{\sum_{i=1}^{n-1} K\left(\frac{r_{i\Delta} - r}{h}\right)} \tag{6}$$

where $K(.)$ is a Gaussian Kernel,

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$h$ is the window width depending on the size and disperse of observations. Scott [21] suggest the window width

$$h = \hat{\sigma} T^{-\frac{1}{m+4}}$$

where $\hat{\sigma}$ is standard deviation of observations, $T$ is the number of observations and *m* is the dimension. The approximations converge to the true functions at a rate $\Delta^k$, where $\Delta$ is the time between successive observations and $k$ is an arbitrary positive integer.

This nonparametric method has been developed but they either rely on the existence of a time-invariant marginal density for the underlying process (Jiang [20], Jiang and Knight [4]), or stationarity which is assumed despite robustness to deviation from it (Stanton [3]). So Bandi [1] proposes local time to describe the data. Based on our previous analysis, stationarity of the short rate process cannot be guaranteed, so we also use local time to grasp more information of data.

## 2.4. Local Time

Bandi [1] uses new fully functional methods to exploit the spatial properties, embodied in the local time (classical references are Chung and Williams [22]; karatzas and Shreve [23]; Revuz and Yor [24]) of interest rate which is robust against deviations from stationarity. Spatial densities and their functionals can be regarded as new descriptive tools for the series that are non-stationary or stationarity cannot be guaranteed, as in Bandi [1] which assume recurrence, a weaker assumption than the stationary condition.

**Definition 1** *If* $X_t$ *is a continuous semi-martingale, then exists a nondecreasing stochastic process* (*nondecreasing in* $t$, *that is*) $L_X(t,a)$, *called the chronological local time of* $X$ *at a. This process is defined, almost surely, as*

$$\bar{L}_X(t,a) = \frac{1}{\sigma^2(a)} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{[a,a+\epsilon[}(X_s) \mathrm{d}[X]_s \tag{7}$$

This formula gives the amount of time in real time units that the process $X_t$ spends in the spatial neighborhood of a point $a$. This spatial density assumes importance particularly when the underlying process is non-stationary, as they furnish the possibility of characterizing some of the features of the data, *i.e.*, the location of the process. In fact, in the presence of non-stationarity, conventional descriptive statistics fail to provide reliable information given the tendency of the data to drift away from a particular point. So spatial densities can be regarded as new descriptive tools for series that are non-stationary or stationary cannot be guaranteed.

Recurrence requires the continuous trajectory of the process to visit any set in its range an infinite number of times over time almost surely. It makes economic sense because interest rates are expected to return to the values in their range over and over again. It is meaningful to estimate the drift and diffusion functions at each point in the range of the sample interest rate process. The density of the observations plays a role in the operation of the asymptotic. This information is contained in the estimated local time of the spot interest rate process.

In order to show precise inference on the drift of process of a point (*i.e.*, to achieve statistically consistent estimates), we require the estimated local time of the process at that point to be large. Its properties and estimation are shown in the following section.

## 3. Empirical Results

### 3.1. Nonparametric Estimation

According to the previous analysis, we derive the estimation of drift and diffusion from the above estimators in Equations (5) and (6) and obtain the 1000 simulated interest rate paths using the Monte Carlo simulation method. Then we estimate the drift and diffusion for every path.

Drift and diffusion estimates for the single-factor model in Equation (1) and their Monte Carlo confidence bands are given in the **Figure 3**. We report estimates from Equations (3) and (4) for $r_t \in [0, 0.18]$, which cover the 99.6% of the data.

The simulation results indicate that the estimates are unbiased. Because there are few observations are high rates, the confidence intervals are relatively wide. Especially the diffusion estimation fits well based on **Figure 3**. At lower interest rate levels, it has a lower variance. As interest rates go up, variance increases accordingly.

### 3.2. Local Time Estimation

Local time gives the amount of time that the process spends in the vicinity of one point. Bandi [1] also derive the estimator of local time:

$$\hat{\bar{L}}_r(r) = \frac{\Delta}{h} \sum_{i=1}^{n} K\left(\frac{r_{i\Delta} - r}{h}\right) \tag{8}$$

By virtue of recurrence, interest rates may visit every level over time which opens up the possibility of recovering the true function by using a single trajectory of the process over a long time, through a combination of infill and long span asymptotic. Bandi [1] suggest that the asymptotic 95% confidence interval for $\hat{\bar{L}}_r(r)$ is given by

$$\hat{\bar{L}}_r(r) \pm 1.96\left(8k \frac{h}{\hat{\sigma}^2} \hat{\bar{L}}_r(r)\right) \tag{9}$$

where the parameters in Equations (8) and (9) are consistent in the whole paper. These asymptotic confidence bands resemble conventional intervals for probability densities.

**Figure 4** gives the plot of local time of the short rate of the entire data sample (2167 daily observations). The modes show up at around 3% and 11%. Given the features of the estimation procedure in Bandi [1], we expect to be able to identify the functions of interest rate at points that are visited frequently. After a quick look at the

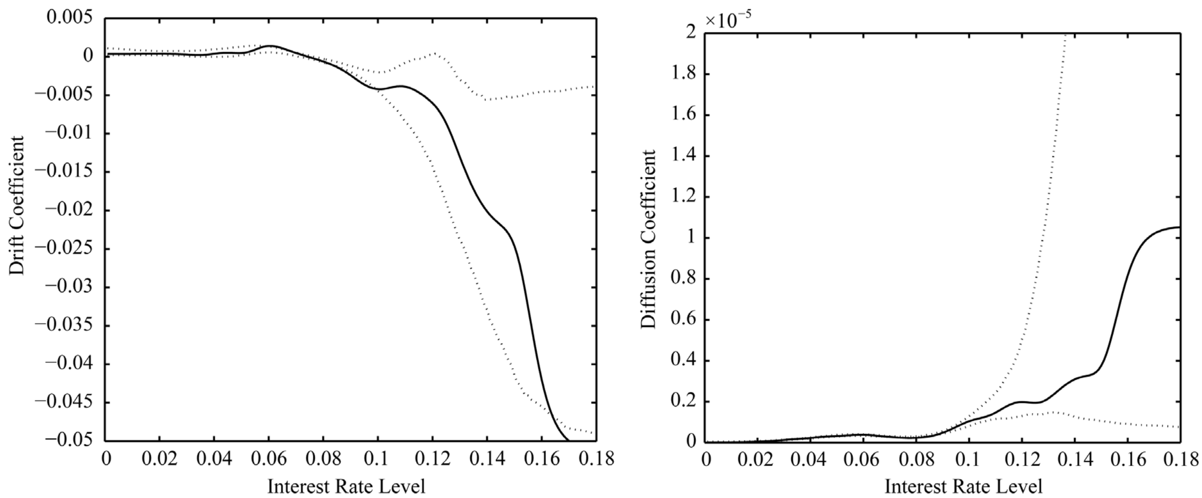


**Figure 3.** The figure shows that the result of nonparametric estimates of drift and diffusion terms the single-factor diffusion model respectively. The sample period is from January 1995 to December 2003 (2052 daily observations). The solid line is the drift function estimated from repo rate data and the dot lines are 95% Monte Carlo confidence bands.

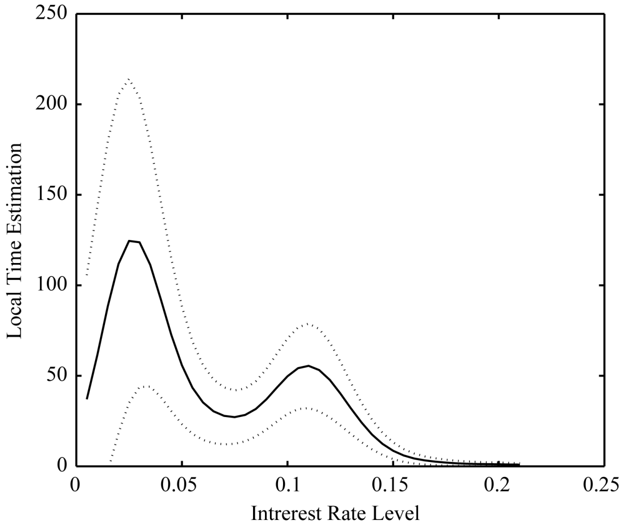


**Figure 4.** The figure shows that the estimates of local time process of the repo rate series examined in this study. The sample period is January, 1995 to December, 2003 (2052 annualized daily observations). The straight line is the pointwise nonparametric estimates of the local time process and the dot lines are the corresponding 95% asymptotic confidence bands.

graph of the estimated local time, we anticipate that problems would arise in the 17% - 21% range, as the time spent by the sample process in this range is quite small. The density in the figure is bimodal, the spatial density of the process appears to be bimodal. Compared with the frequency histogram of the repo rate in **Figure 3**, we can find that they are very similar. Therefore, the local time can be the approximation of density of the one path for the underlying process.

From the feature of the data, the interest rates had a higher level before 1999,, but after 1999, interest rates went down and kept a lower level until 2003. Therefore, two different time horizon can be considered: 1995.01-1998.12 and 1999.01-2003.12. **Figure 5** presents their local time estimation respectively.

We find that the two peaks in **Figure 4** appear in **Figure 5** separately. For the time horizon 1995.01-1998.12, the interest rates below 5% have a very low frequency. For the time horizon 1999.01-2003.12, because 98% of data is below 5%, local times for interest rates above 5% are close to zeros. These features provide the evidence to consider the effect of a state variable.

**Figure 6** shows that the drift estimation using the non-parametric estimation method for the two subperiods 1995.01-1998.12 and 1999.01-2003.12 respectively. It can be seen that the drifts are very close to zeros for two subperiods, but other parts below 4% and above 14% for subperiod 1995.01-1998.12 are mean-reversion. It is surprising that for subperiod 1999.01-2003.12, mean-reversion speed is very low for 1% - 5%, especially from 2.5% - 5%, the drifts behave like a martingale. At the same time, from the corresponding local time figures, they have higher local time and cover more than 98% of subperiod data respectively.

This pattern appears again for their diffusion estimation in **Figure 7**. The corresponding variances over two subperiods are low and relative stable. The Monte Carlo 95% confidence bands are very close. This means that the data have a big change after 1999. Considering their different states, we use two-regime model to fit the data in the follows.

## 4. Discussion

### 4.1. Two Regime Model

From the previous analysis, we consider the effect from the state variable. The model is the following:

$$\mathrm{d}r_t = \left[\mu_1(r_t) + \left(\mu_2(r_t) - \mu_1(r_t)\right)(s_t - 1)\right]\mathrm{d}t + \left[\sigma_1(r_t) + \left(\sigma_2(r_t) - \sigma_1(r_t)\right)(s_t - 1)\right]\mathrm{d}z_t \tag{10}$$

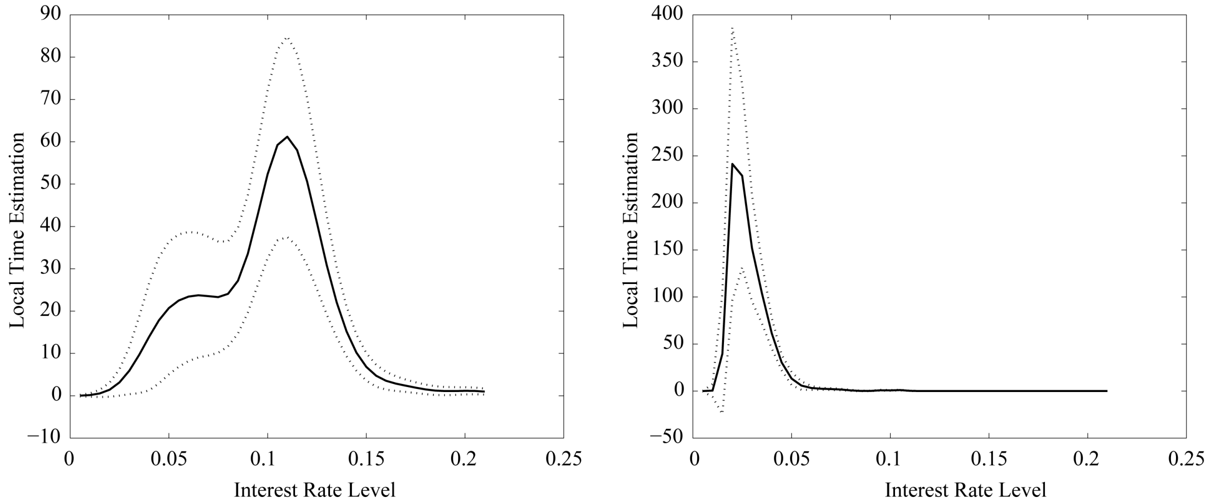


**Figure 5.** Figures show the local time process of the repo rate series for two sample periods respectively. The first sample period is from January, 1995 to December, 1998 (939 annualized daily observations). The second sample period is from January, 1999 to December, 2003 (1113 annualized daily observations). The straight line is the pointwise nonparametric estimates of the local time process and the dot lines are the corresponding 95% asymptotic confidence bands.

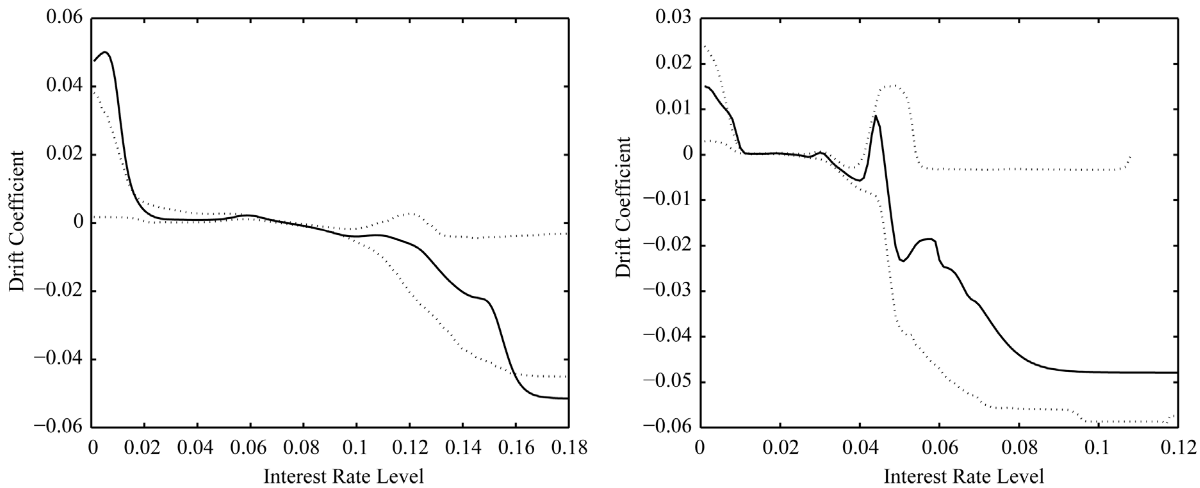


**Figure 6.** Figures show that the result of estimates of drift term for the single-factor diffusion model for two subperiods respectively. The sample periods are from January 1995 to December 1998 (939 daily observations) and from January 1995 to December 1998 (939 daily observations). The solid line is the drift function estimated from repo rate data and the dot lines are 95% Monte Carlo confidence bands.

where $s_t$ is a stochastic state variable which satisfies:

$$s_t = \begin{cases} 1, & p = P\left(s_t = 1 \middle| r_t\right) \\ 2, & q = P\left(s_t = 2 \middle| r_t\right) = 1 - p \end{cases}$$

When process $s_t$ equals to 1 at time $t$, the interest rates stay at the state 1 with probability $P\left(s_t = 1 \middle| r_t\right)$ and the process follows the following model with probability $P\left(s_t = 1 \middle| r_t\right)$:

$$\mathrm{d}r_t = \mu_1\left(r_t\right)\mathrm{d}t + \sigma_1\left(r_t\right)\mathrm{d}z_t \tag{11}$$

When $s_t$ equals to 2, the interest rates stay at the state 2 with probability $1 - P\left(s_t = 1 \middle| r_t\right)$ and the process follows the following model with probability $1 - P\left(s_t = 1 \middle| r_t\right)$:

$$\mathrm{d}r_t = \mu_2\left(r_t\right)\mathrm{d}t + \sigma_2\left(r_t\right)\mathrm{d}z_t \tag{12}$$

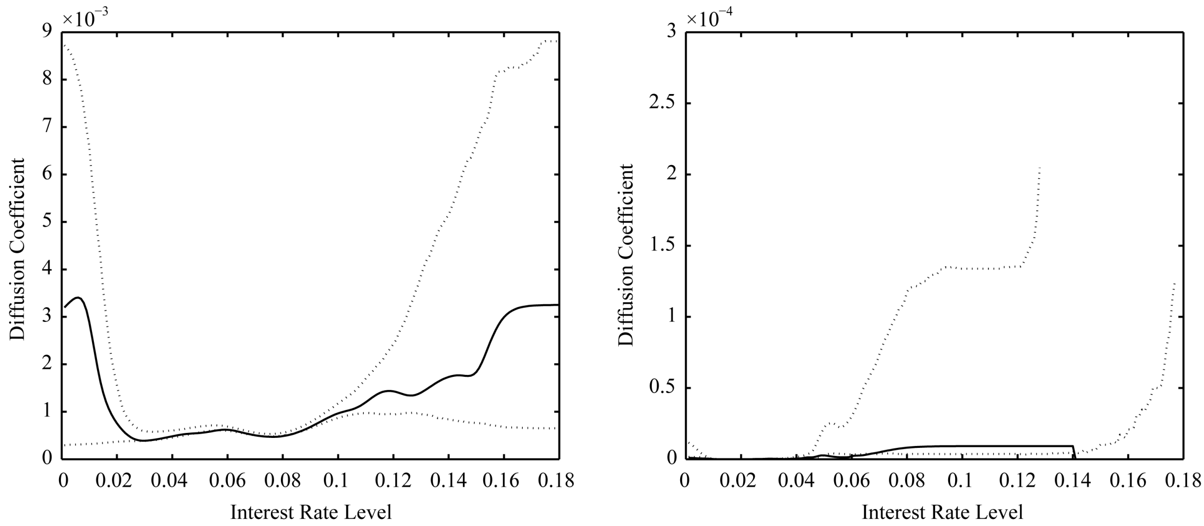


**Figure 7.** Figures show that the result of estimates of diffusion term for the single-factor diffusion model for two subperiods respectively. The sample periods are from January 1995 to December 1998 (939 daily observations) and from January 1995 to December 1998 (939 daily observations). The solid line is the drift function estimated from repo rate data and the dot lines are 95% Monte Carlo confidence bands.

We then obtain the estimation of the conditional probability $P\left(s_t \middle| r_t\right)$ based on our sample data. For example, $P\left(s_t = 2 \middle| r_t\right)$ is estimated as $P\left(s_t = 2 \middle| r_t, \hat{\theta}\right)$ from our discrete data in the below section. The transition probabilities $P\left(s_t = 2 \middle| s_{t-1} = 1\right)$, $P\left(s_t = 1 \middle| s_{t-1} = 2\right)$, $P\left(s_t = 1 \middle| s_{t-1} = 1\right)$ and $P\left(s_t = 2 \middle| s_{t-1} = 2\right)$ are the the transition probabilities $p_{12}$, $p_{21}$, $p_{11}$ and $p_{22}$ respectively in the below.

Then from the Equation (10), we consider the model relying on the state variable. The probability $P\left(s_t \middle| r_t\right)$ will be estimated and plotted in the following sections based on our sample data and parameters estimated by $P\left(s_t = 1 \middle| r_t, r_{t-1}; \hat{\theta}\right)$. Then we assume the short rates follow one of interest rate models (Vasicek model here) with a probability relying on the short rate at time $t$ and $t-1$ and parameters.

Based on the previous analysis for drift and diffusion terms, we assume that $\mu_j\left(r_t\right)$ and $\sigma_j\left(r_t\right)$ $\left(j = 1,2\right)$ have the same form as the Vasicek model:

$$\mu_j\left(r_t\right) = \alpha_j + \beta_j r_t, \quad \text{for} \quad j = 1,2$$

$$\sigma_j\left(r_t\right) = \sigma_j, \quad \text{for} \quad j = 1,2$$

where $j$ = 1 or 2 and $\alpha_1$, $\beta_1$ and $\sigma_1$ mean that the process is in regime 1, which is also $s_t = 1$ and $\alpha_2$, $\beta_2$ and $\sigma_2$ mean regime 2 which is $s_t = 2$ ($\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\sigma_1 \neq \sigma_2$). The change in regimes is itself a random variable and unobservable. A complete time series model would therefore include a description of the probability law governing the change from $\alpha_1$, $\beta_1$ and $\sigma_1$ and $\alpha_2$, $\beta_2$ and $\sigma_2$.

Given the discrete data, the data generating process is:

$$r_t = \theta_{s_t} + \phi_{s_t} r_{t-1} + \tilde{\sigma}_{s_t} \epsilon_t \tag{13}$$

where $\epsilon_t \sim N\left(0,1\right)$. So there is a relationship:

$$\frac{\alpha_{s_t}}{1-\beta_{s_t}} = \theta_{s_t}, \quad \frac{\sigma_{s_t}}{1-\beta_{s_t}} = \tilde{\sigma}_{s_t}, \quad \frac{1}{1-\beta_{s_t}} = \phi_{s_t} \tag{14}$$

In our model, we only have two states and $s_t$ equals to 1 or 2. With the daily data, we then test the model in Equation (13). We assume that $r_t$ follows a normal distribution with mean $\theta_j + \phi_j r_{t-1}$ and variance $\tilde{\sigma}_j^2$ for $j = 1,2$. Such a process is described as a two-state Markov Chain with transition probabilities $\left\{p_{ij}\right\}_{i,j=1,2}$ which are:

$$P\left\{s_t = j \middle| s_{t-1} = i\right\} = p_{ij}, \quad \text{for} \quad i, j = 1,2$$

So for a two-state Markov chain, the transition matrix is

$$Q = \begin{pmatrix} p_{11} & 1-p_{22} \\ 1-p_{11} & p_{22} \end{pmatrix}$$

No loss of generality, we assume that this two-state Markov chain is ergodic provided that $p_{11} < 1$, $p_{22} < 1$ and $p_{11} + p_{22} > 0$.

The unconditional probability that the process will be in regime 1 at any given date should be the follows:

$$P\{s_t = 1\} = \frac{1-p_{22}}{2-p_{11}-p_{22}}$$

It is obvious that

$$P\{s_t = 2\} = 1 - P\{s_t = 1\} = \frac{1-p_{11}}{2-p_{11}-p_{22}}$$

The matrix of *m*-period-ahead transition probabilities for an ergodic two-state Markov chain is given by:

$$P^m = \begin{pmatrix} \dfrac{(1-p_{22}) + \gamma^m (1-p_{11})}{2-p_{11}-p_{22}} & \dfrac{(1-p_{22}) - \gamma^m (1-p_{22})}{2-p_{11}-p_{22}} \\ \dfrac{(1-p_{11}) - \gamma^m (1-p_{11})}{2-P_{11}-p_{22}} & \dfrac{(1-p_{11}) + \gamma^m (1-p_{22})}{2-p_{11}-p_{22}} \end{pmatrix}$$

Thus, if the process is currently in state 1, the probability in state 2 after m periods later is given by

$$P\{s_{t+m} = 2 | s_t = 1\} = \frac{1-p_{11} - \lambda^m (1-p_{11})}{2-p_{11}-p_{22}}$$

where $\gamma = -1 + p_{11} + p_{22}$.

Given a short rate, whether it stay in regime 1 or 2 is unknown, but we can estimate the probability for any states.

### 4.2. Estimation of Two Regime Model

From Hamilton [25], there is the maximum likelihood estimation from the observed data $r_t$ as the following:

$$\hat{p}_{ij} = \frac{\sum_{t=2}^{T} P\{s_t = j, s_{t-1} = i | r_T\}}{\sum_{t=2}^{T} P\{s_{t-1} = i | r_T\}}, \quad \text{for} \quad i, j = 1, 2 \tag{15}$$

where let $r_T = (r_T', r_{T-1}', \cdots, r_1')$ be a vector containing all observations obtained through date *T*. Our first probability $P(s_{t-1})$ and $P(s_t = j, s_{t-1} = i | r_T)$ begin from $P(s_1)$ and $P(s_2 = j, s_1 = i | r_T)$.

We suppose virtually certainty from observations from regime *j*, so that $P\{s_{t-1} = i | r_T\}$ equals to unity for those observations that came from regime *j* and equals to zero for those observations that came from other regimes.

Following the method of Hamilton [26], the EM algorithm is:

$$\sum_{t=2}^{T} \left(r_t - \theta_j^{(l+1)} - \phi_j^{(l+1)} r_{t-1}\right) p(s_t = j | r_T; \lambda_l) = 0, \quad j = 1, 2,$$

$$\sum_{t=2}^{T} r_{t-1} \left(r_t - \theta_j^{(l+1)} - \phi_j^{(l+1)} r_{t-1}\right) p(s_t = j | r_T; \lambda_l) = 0, \quad j = 1, 2,$$

$$\tilde{\sigma}_j^{2(l+1)} = \frac{\sum_{t=2}^{T} \left(r_t - \theta_j^{(l+1)} - \phi_j^{(l+1)} r_{t-1}\right)^2 p(s_t = j | r_T; \lambda_l)}{\sum_{t=2}^{T} p(s_t = j | r_T; \lambda_l)}, \quad j = 1, 2,$$

$$p_{11}^{(l+1)} = \frac{\sum_{t=2}^{T} p(s_t = 1, s_{t-1} = 1 | r_T; \lambda_l)}{\sum_{t=2}^{T} p(s_{t-1} = 1 | r_T; \lambda_l)},$$

$$p_{22}^{(l+1)} = \frac{\sum_{t=2}^{T} p(s_t = 2, s_{t-1} = 2 | r_T; \lambda_l)}{\sum_{t=2}^{T} p(s_{t-1} = 2 | r_T; \lambda_l)},$$

$$\rho^{(l+1)} = p(s_1 = 1 | r_1; \lambda_l).$$

where $\lambda_l = \left(p_{11}^{(l)}, p_{22}^{(l)}, \theta_1^{(l)}, \theta_2^{(l)}, \phi_1^{(l)}, \phi_2^{(l)}, \tilde{\sigma}_1^{2(l)}, \tilde{\sigma}_2^{2(l)}, \rho^{(l)}\right)$.

Because we know

$$p\left(r_t \mid s_t, r_{t-1}, \theta_1, \theta_2, \phi_1, \phi_2, \tilde{\sigma}_1^2, \tilde{\sigma}_2^2\right) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_{s_t}^2}} \exp\left[-\left(r_t - \theta_{s_t} - \phi_{s_t} r_{t-1}\right)^2 \Big/ \left(2\tilde{\sigma}_{s_t}^2\right)\right]$$

and

$$p\left(s_t = 1\right) = p\left(s_t = 1, s_{t-1} = 1\right) + p\left(s_t = 1, s_{t-1} = 2\right)$$

Using the whole data sample (2052 observations), we calculate the smoothed probabilities $p\left(s_t = j, s_{t-1} = i \mid r_T, \lambda_l\right)$. These smoothed probabilities are used in equations of EM algorithm to calculate the parameters. The estimated results are reported in **Table 3**.

It is known that $p_{11}$ and $p_{22}$ are 87.83% and 92.24%. From Equation (13), we find that our two-regime model is the following Vasicek model:

$$\mathrm{d}r_t = \left(0.01 - 0.1204 r_t\right)\mathrm{d}t + 0.0204\mathrm{d}z_t \tag{16}$$

$$\mathrm{d}r_t = \left(0.00003 + 0.0009 r_t\right)\mathrm{d}t + 0.0022\mathrm{d}z_t \tag{17}$$

This means that once the process enters a regime, it will remain in that state with a high transition probability. Furthermore, in regime 1, mean-reversion parameter is larger, but it is different for regime 2 in which the drift coefficient is very close to zero. These are very reasonable, because the interest rates are lower and not so volatile as regime 1. Both average change rates of two regimes are very close to 0, but their variances differ.

The inference about the value of $s_t$ for a single date is obtained. A probabilistic inference in the form of $P\left\{s_t = 2 \mid r_t; \hat{\theta}\right\}$ can be calculated for each date $t$ in the sample. The resulting series is plotted as a function of $t$ in **Figure 8**.

It is obvious that after 1999 probability was very high and close to 1 most of the time. In reality, it is known that when Chinese interest rates remain at a lower level, high economic growth rate gives pressure towards lower rates. Interest-rate liberalization in China is necessary.

## 5. Conclusions

In this paper, I study the interest rate behavior of China based on the observed 7 days repo rate of Shanghai market. The repo rate provides the benchmark for the interest rate of marketability and pricing of national debt futures.

Following Bandi and Philips [9]'s method, we assume recurrence only and examine how well it can fit China data under the non-parametric model. Because we find that interest rates behave very differently during the two subperiods which is against the stationarity of the short rate process, we assume that the drift and diffusion terms in the interest rate model rely not only on the short rate, but also on a state variable.

We find that the density of the process is bimodal. Two regime model could be better to capture the interest

**Table 3.** Two regime model for repo rate.

| Parameters | $\theta_1$ | $\theta_2$ | $\phi_1$ | $\phi_2$ |
|---|---|---|---|---|
| Estimation | 0.0089 | 0.000034 | 0.8925 | 1.0009 |
| Standard error | 0.008 | 0.00018 | 0.1 | 0.002 |
| Parameters | $\tilde{\sigma}_1^2$ | $\tilde{\sigma}_2^2$ | $p_{11}$ | $p_{22}$ |
| Estimation | 0.00033 | 0.0000048 | 0.8783 | 0.9224 |
| Standard error | 0.00001 | 0.000000003 | 0.0254 | 0.029 |

This table presents the result of two regime model and the data generating process is: $r_t = \theta_{s_t} + \phi_{s_t} r_{t-1} + \tilde{\sigma}_{s_t}\epsilon_t$. Where $\epsilon_t \sim N(0,1)$ and the state, $s_t$, follows a two-state Markov chain model with $P\left(s_t = 1 \mid s_{t-1} = 1\right) = p_{11}$ and $P\left(s_t = 2 \mid s_{t-1} = 2\right) = p_{22}$. The model is estimated using maximum likelihood approach. The sample is daily annualized yield on repo rate for Shanghai market and the sample period is 1995.01-2003.12.

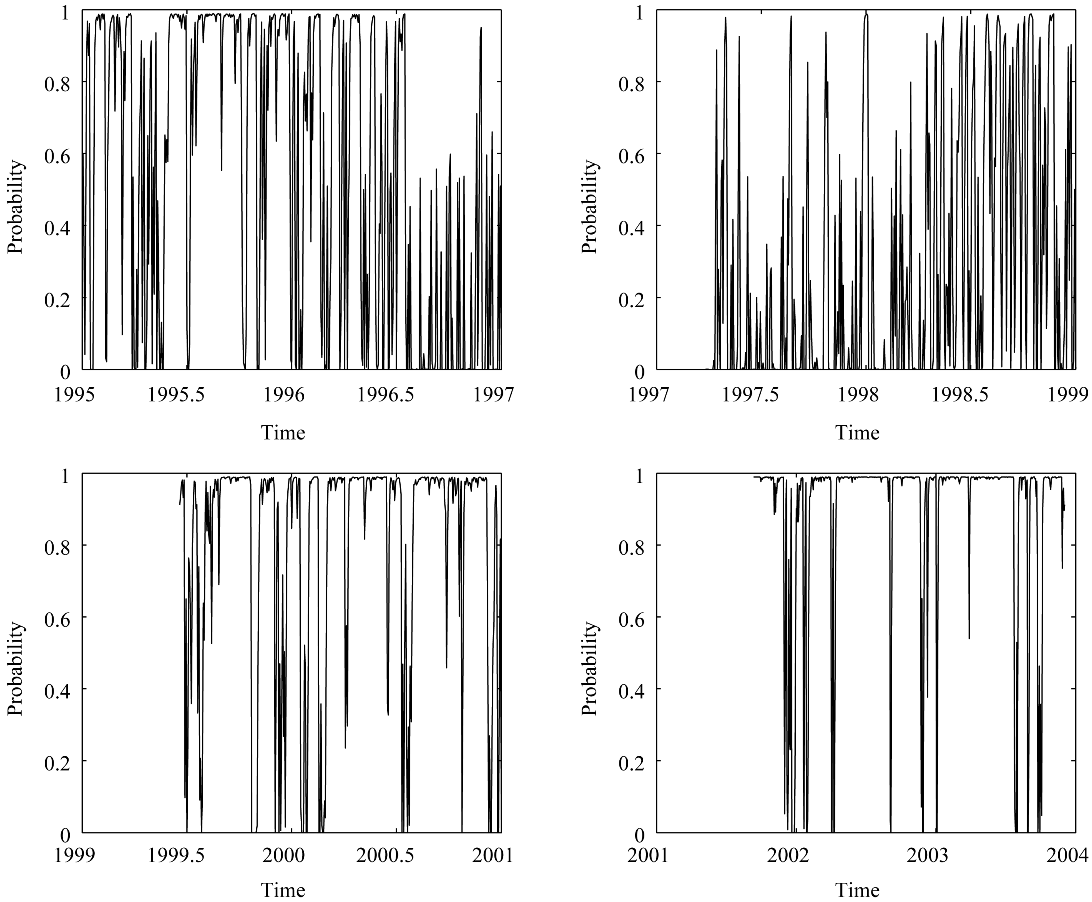


**Figure 8.** This figure shows that result of probability that the short rate is in state 2 which is at a low level or $P\left\{s_t = 2 \middle| r_t, r_{t-1}; \hat{\theta}\right\}$ plotted as a function of *t*. The sample period is from January 1995 to December 2003 (2052 daily observations.

rates of China. Based on the evidence of local time of sub-sample data, we estimate the parameters and examine the properties of two-regime model. Using functional nonparametric method, we test the Vasicek model at different states. The short rates behave like a martingale in regime 2. We also calculate the probabilities that the process will stay in regime 1 and regime 2, and the probability that process will transfer from one state to another and the inference probability for a single date.

From our results, China's recent interest rate stays in regime 2 in which the interest rate keeps at a low level with a high probability. Interest rate marketization of China will enable market forces to play a greater role in determining the allocation of credit, and economy will be more responsive to changes in rates. The liberalization of rates is a landmark change, and it represents another major milestone in China's transformation to a market economy.

We acknowledge helpful comments from Chu Zhang, Jiang Luo, Jin Zhang and seminar participants at the Hong Kong University of Science and Technology. The research of this paper has been partially supported by grants from the Fundamental Research Funds for the Central Universities (Project No. 1209022), National Natural Science Foundation of China (Project No. 71303265, No. 71272201 and No. 71231008).

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**Journal of Mathematical Finance, 2015, 5, 40-48**
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51004

# Entrepreneurship Dynamics under Time Inconsistent Preferences

**Yang Liu, Jinqiang Yang**

School of Finance, Shanghai University of Finance and Economics, Shanghai, China
Email: breadyoung@outlook.com, yang.jinqiang@mail.sufe.edu.cn

Received 21 January 2015; accepted 9 February 2015; published 11 February 2015

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## Abstract

**We investigate the implications of time inconsistent preferences on the entrepreneurial decision making. We use a time varying preference model to capture the optimal liquidation choice and investment allocation for the averse risk agent in the incomplete market. Compared to standard entrepreneurial dynamic framework, our model shows that inconsistent preferences may lead to under investment when the entrepreneur faces liquidity constraint and over investment when his liquid wealth is far away from the liquidation boundary. Moreover, the possibility of changing to the future stage has ambiguous influences on the exit decision and optimal investment.**

## Keywords

**Time Inconsistency, Liquidity Constraints, Precautionary Saving, Optimal Liquidation, Investment Strategy**

## 1. Introduction

Entrepreneurs face high uncertainty and liquidity constraints, both of which have significant influences on the business decision making process. These uncertainty and constraints are important determinants for capital accumulation, exit decision and asset allocation. Since [1], the real option approach has become an essential part of optimization problem for the entrepreneurs. [2] develop an incomplete-market q-theoretic model to study entrepreneurship dynamics and find that the illiquidity, idiosyncratic risks and borrowing constrains result in business decisions, consumption and asset allocation decisions different with the standard complete markets profit-maximizing analysis for entrepreneurial firms.

In the standard optimization model framework, it is assumed that agents have a constant rate of time preference. However, virtually every experimental research on time preference indicates that this assumption is unrealistic. The most relevant effect of time inconsistent is the preference change with time. Usually, an agent

**How to cite this paper:** Liu, Y. and Yang, J.Q. (2015) Entrepreneurship Dynamics under Time Inconsistent Preferences. *Journal of Mathematical Finance*, **5**, 40-48. http://dx.doi.org/10.4236/jmf.2015.51004

has different preferences at different stages. In this article, we focus on the entrepreneurs' decision making under non-constant preference.

In our paper, we try to focus on the following interesting questions: What is the impact of time inconsistent preference on the entrepreneurial valuation? How could the time varying discount rate affects the liquidation choices? How would the entrepreneur allocate the wealth between investment, consumption and public equity? We extend the entrepreneurial optimization model in [2] by incorporating the time inconsistence preferences.

Our model gives three main contributions. First, the time inconsistent agent will under invest when his liquid wealth is close to liquidation boundary and over invest when the agent is far away from the exit threshold. Second, time inconsistent preferences weaken the effects of risk aversion on the liquidation decisions. Third, increasing the possibility of the birth of the future stage with utility discount has ambiguous influences on the exit decision and investment strategy and finally the decisions approach the steady result in the future stage.

Our research relates to the literature about time inconsistency model and its application. [3] models time varying impatience with quasi hyperbolic discount functions and explains why consumers have asset-specific marginal propensities. [4] describe the equilibrium of a discrete-time exchange economy in which consumers with arbitrary subjective discount factors and homothetic period utility functions follow linear Markov consumption and portfolio strategies. [5] consider two types of goods: goods with immediate costs and delayed benefits, and goods with immediate benefits and delayed costs. With time inconsistency model, they explain how to design optimal contract respond to consumer biases. [6] extend the real option framwork to model the investment timing decisions of entrepreneurs with time inconsistent preferences.

The remainder of the paper proceeds as follows: Section 2 presents the model; Section 3 derives the solutions; Section 4 provides the quantitative results and Section 5 concludes.

## 2. Model

In this section, we set out the framework for the basic model of entrepreneurial optimization problem. The liquidation option is described and followed by a discussion of the nature of time inconsistent preferences.

Time is continuous and horizon is infinite. There is a single perishable consumption good. The agent derives utility from a consumption process $C$ according to

$$J_t = \mathbb{E}_t\left[\int_t^{\infty} D(t,s)U(C_s)\mathrm{d}s\right] \tag{1}$$

where $U(C)$ is a concave function. For tractability, we choose $U(C)=\frac{C^{1-\gamma}}{1-\gamma}$, where $\gamma>0$ is the coefficient of relative risk aversion. $D(t,s)$ denotes the agent's intertemporal discount function: the agent's value at time $t$ of \$1 received at the future time $s$. We thus have

$$D(t,s)=\begin{cases}\mathrm{e}^{\zeta(s-t)}, & \text{if } s\in[t,\tau);\\ \beta\mathrm{e}^{\zeta(s-t)}, & \text{if } s\in[\tau,\infty).\end{cases} \tag{2}$$

for $s>t$. As in [7], the present stage could last for a random duration of time. For simplicity, we assume that the lifespan of present agent $\tau-t$ is exponentially distributed with parameter $\lambda$. Stated in another way, the birth of future agent is modeled as a Poisson process with intensity $\lambda$. We define $\beta=\tilde{\beta}^{1-\gamma}$ where $\tilde{\beta}\in[0,1]$ measures the degree of the agent's utility discount in future stage. We assume the agent is in the present stage, and thus the value function is:

$$J_t = E_t\left[\int_t^{\tau}\mathrm{e}^{\zeta(s-t)}U(C_s)\mathrm{d}s+\int_{\tau}^{\infty}\beta\mathrm{e}^{\zeta(s-t)}U(C_s)\mathrm{d}s\right] \tag{3}$$

Consider the setting for a standard entrepreneurial problem as in [2]. The agent possesses a firm and our production specification features the widely used "AK" technology augmented with capital adjustment costs. Let $I$ denote the gross investment. The change of capital stock $\mathrm{d}K$ is given by: $\mathrm{d}K_t=(I_t-\delta K_t)\mathrm{d}t$, where $\delta\geq 0$ is the rate of depreciation. The firm's productivity shock $\mathrm{d}A_t$ over the period $(t,t+\mathrm{d}t)$ is independently and identically distributed (iid), and is given by: $\mathrm{d}A_t=\mu_A\mathrm{d}t+\sigma_A\mathrm{d}Z_t$, where $Z$ is a standard Brownian motion, $\mu_A$ and $\sigma_A$ are the mean and volatility of the productivity shock respectively. The firm's operating revenue over period $(t,t+\mathrm{d}t)$ is proportional to $K_t$ and is given by $K_t\mathrm{d}A_t$. The firm's operating profit $\mathrm{d}Y_t$ over the same period is given by:

$$\mathrm{d}Y_t = K_t \mathrm{d}A_t - I_t \mathrm{d}t - G(I_t, K_t)\mathrm{d}t \tag{4}$$

where $G(I,K)$ is the adjustment cost. We assume that the firm's adjustment cost $G(I,K)$ is homogeneous of degree one in $I$ and $K$, and write $G(I,K)$ in the homogeneous form as: $G(I,K) = g(i)K$, where $i = I/K$ is the firm's investment-capital ratio. For simplicity, we assume $g(i) = \frac{\theta i^2}{2}$, where the parameter $\theta$ measures the degree of the adjustment cost.

The entrepreneur has an option to liquidate capital at any moment. Liquidation is irreversible and gives a terminal value $lK$, where $l > 0$ is a constant. Let $T^l$ denote the optimal liquidation time. When the entrepreneur is not well-diversified, liquidation provides an important channel to manage the downside risk exposure.

The agent can invest in risk free asset and public equity. These two financial asset represent the standard investment opportunities in the classical [8] model. The risk free asset accumulates with a constant interest rate $r$. Incremental return of public equity, $\mathrm{d}R_t$, over time period $\mathrm{d}t$ is iid: $\mathrm{d}R_t = \mu_R \mathrm{d}t + \sigma_R \mathrm{d}B_t$, where $B_t$ is a standard Brownian motion, $\mu_R$ and $\sigma_R$ are the constant expected mean and volatility, respectively. The sharp ratio for the public equity is: $\eta = \frac{\mu_R - r}{\sigma_R}$. Let $\rho$ denote the correlation coefficient between the public equity and the entrepreneurial business. The non diversifiable risks $\epsilon = \sigma_A \sqrt{1-\rho^2}$ play a role in the decision making process.

Let $X_t$ be the amount allocated to the risky public equity at time $t$, and $W_t$ denote the liquid financial wealth process. Before $T^l$, the agent holds the firm and acts as an entrepreneur. The liquid financial wealth process $W_t$ evolves as follows:

$$\mathrm{d}W_t = \left[r(W_t - X_t) + \mu_R X_t - C_t\right]\mathrm{d}t + \mathrm{d}Y_t + \sigma_R X_t \mathrm{d}B, \quad 0 < t < T^l \tag{5}$$

After exiting from the business, the agent retires and $W_t$ accumulates in the following form:

$$\mathrm{d}W_t = \left[r(W_t - X_t) + \mu_R X_t - C_t\right]\mathrm{d}t + \sigma_R X_t \mathrm{d}B_t, \quad t > T^l \tag{6}$$

The agent is allowed to borrow against capital at all times in our model. To make sure the debt is risk free, we set the liquidation value of the capital $lK$ greater than outstanding debt:

$$W_t + lK_t \ge 0, \quad 0 < t < T^l \tag{7}$$

The optimization problem of the agent involves the maximization of the utility defined as (3). First, before liquidation $(t < T^l)$, the entrepreneurial objective is to choose a consumption process $C_t$, a portfolio allocation rule $X_t$, the investment process $I_t$ and an optimal liquidation timing strategy $T^l$ to maximize the utility subject to the wealth dynamics (5) and borrowing constrain (7). After the liquidation option has been exercised, the entrepreneur collects the liquidation proceeds and retires. And then the agent chooses optimal allocation between the risk free asset, public equity and consumption.

## 3. Solution

### 3.1. Benchmark: Time Consistent Preference

As a benchmark, we consider the case in which the entrepreneurial preference is time consistent. The constant preference case reduces to [2], and the solution to this problem is summarized in Proposition 1.

**Proposition 1.** *The entrepreneur operates the business if and only if* $w = \frac{W}{K} \ge \underline{w}_1$. *Before liquidation the entrepreneurial value function* $J_C(K,W)$ *is given by* $J_C(K,W) = \frac{(bF(K,W))^{1-\gamma}}{1-\gamma}$, *where* $b = \zeta\left[1 + \frac{1-\gamma^{-1}}{\zeta}\left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\right]^{\gamma/(\gamma-1)}$. *The scaled certainty equivalent (CE) wealth* $f(w) = F(K,W)/K$ *solves* *the following ordinary differential equation* (*ODE*):

$$0=\frac{\gamma mf(w)(f'(w))^{1-\gamma^{-1}}-\zeta f(w)}{1-\gamma}-\delta f(w)+(r+\delta)wf'(w)+(\mu_A-\rho\eta\sigma_A)f'(w)$$
$$+\frac{(f(w)-(w+1)f'(w))^2}{2\theta f'(w)}+\frac{\eta^2 f(w)f'(w)}{2h(w)}-\frac{\epsilon^2 h(w)f'(w)}{2f(w)}, \quad (8)$$

where $m=\zeta+(1-\gamma^{-1})\left(r-\zeta+\frac{\eta^2}{2\gamma}\right)$ and $h(w)=\gamma f'(w)-\frac{f(w)f''(w)}{f'(w)}$. When $w$ approaches $\infty$, $f(w)$ approaches the complete-markets solution given by $\lim_{w\to\infty} f(w)=f^{FB}(w)=w+q^{FB}$, where $q^{FB}$ is the scaled average q in complete-market:

$$q^{FB}=1+\theta\left[(r+\delta)+\sqrt{(r+\delta)^2-\frac{2}{\theta}(\mu_A-\rho\eta\sigma_A-(r+\delta))}\right] \quad (9)$$

The ODE (8) satisfies the following boundary conditions at endogenous liquidation choice $\underline{w}_1$: $f(\underline{w}_1)=\underline{w}_1+l$ and $f'(\underline{w}_1)=1$. The optimal consumption $c=C/K$, investment $i=I/K$ and the public equity allocation-capital ratio $x=X/K$ are given by: $c(w)=mf(w)(f'(w))^{-\gamma^{-1}}$, $i(w)=\frac{1}{\theta}\left(\frac{f(w)}{f'(w)}-w-1\right)$ and $x(w)=-\frac{\rho\sigma_A}{\sigma_R}+\frac{\mu_R-r}{\sigma_R^2}\frac{f(w)}{h(w)}$. After exiting entrepreneurship, the agent's value function takes the following homothetic form: $V_1(W)=\frac{(bW)^{1-\gamma}}{1-\gamma}$

## 3.2. Time Inconsistent Case

Consider the case of an entrepreneur who makes decisions under the belief that future selves act in the interest of the current self. This assumption has been analyzed in [9]. In addition, this assumption is also consistent with empirical evidence on 401 (K) investment (see [10]), and health club attendance (see [11]).

Assume the entrepreneur is in the present stage where the value function takes the form as (3). The standard dynamic programming argument implies that the agent's optimal consumption, investment and public equity allocation solve the following Hamilton-Jaobi-Bellman (HJB) equation of value function $J(K,W)$:

$$\zeta J=\max_{C,I,X}\frac{\zeta C^{1-\gamma}}{1-\gamma}+(I-\delta K)J_K+(rW+(\mu_R-r)X+\mu_A K-I-G(I,K)-C)J_W$$
$$+\left(\frac{\sigma_A^2K^2+2\rho\sigma_A\sigma_R KX+\sigma_R^2X^2}{2}\right)J_{WW}+\lambda(\beta J_C-J), \quad (10)$$

where $J_C$ is the value function in time consistence case. It is easy to verify that $J(K,W)$ takes the form:

$$J(K,W)=\frac{(bP(K,W))^{1-\gamma}}{1-\gamma} \quad (11)$$

Let $\underline{W}_2$ denote the entrepreneurial endogenous liquidation boundary and $\underline{w}_2=\underline{W}_2/K$. The following proposition summarizes the solution for the optimal decisions making and scaled CE wealth $p(w)=P(K,W)/K$.

**Proposition 2.** *The scaled CE wealth* $p(w)$ *solves the following ODE*:

$$0=\frac{m\gamma p(w)(p'(w))^{1-\gamma^{-1}}-\zeta p(w)}{1-\gamma}-\left(\delta+\frac{\lambda}{1-\gamma}\right)p(w)+\frac{\lambda\beta}{1-\gamma}f(w)\left(\frac{p(w)}{f(w)}\right)^{\gamma}+(r+\delta)wp'(w)$$
$$+(\mu_A-\rho\eta\sigma_A)p'(w)+\frac{(p(w)-(w+1)p'(w))^2}{2\theta p'(w)}+\frac{\eta^2 p(w)p'(w)}{2h(w)}-\frac{\epsilon^2 h(w)p'(w)}{2p(w)}, \quad (12)$$

When $w$ approaches $\infty$, $p(w)$ approaches the complete-market solution given by $\lim_{w\to\infty} p(w) = p^{FB}(w) = \alpha\left(w + q^{FB}\right)$, where $\alpha$ measures the influence of time inconsistence preference on the valuation in complete-market. $\alpha$ is the solution to the following function: $\frac{m\gamma}{1-\gamma}\alpha^{1-\gamma^{-1}} + \frac{\lambda\beta}{1-\gamma}\alpha^{\gamma-1} + r + \frac{\eta^2}{2\gamma} + \frac{\zeta+\lambda}{\gamma-1} = 0$. The ODE (12) satisfies the following conditions at the endogenous optimal liquidation choice $\underline{w}_2$: $p(\underline{w}_2) = \alpha(\underline{w}_2 + l)$ and $p'(\underline{w}_2) = \alpha$.

The consumption, investment and public equity allocation are given by $c(w) = mp(w)p'(w)^{-\gamma^{-1}}$, $i(w) = \frac{1}{\theta}\left(\frac{p(w)}{p'(w)} - w - 1\right)$ and $x(w) = -\frac{\rho\sigma_A}{\sigma_R} + \frac{\mu_A - r}{\sigma_R^w}\frac{p(w)}{h(w)}$. After the liquidation, the agent's value function takes the form: $V_2(W) = \frac{(b\alpha W)^{1-\gamma}}{1-\gamma}$.

## 4. Quantitative Results

*Parameter choices*. Where possible, we borrow the parameters from [2]. We set $r = \zeta = 4.6\%$. For public equity, $\mu_R = 10.6\%$ and $\sigma_R = 20\%$. Adjusted cost $\theta = 2$ and depreciation $\delta = 12.5\%$. For production shock, $\mu_A = 20\%$ and $\sigma_A = 10\%$. Capital liquidation price $l = 0.9$.

### 4.1. Optimal Liquidation Boundary

**Figure 1** plots the effects of risk aversion and correlation on liquidation boundaries. Panel A presents $\underline{w}_1 - \underline{w}_2$ in different levels of $\gamma$. [2] study the effects of risk aversion in time consistent case and find that a higher $\gamma$ entrepreneur will exit earlier. We find the trend stays the same in time inconsistent case while the liquidation boundary $\underline{w}_2$ is always lower than $\underline{w}_1$. In addition, the results show the difference $\underline{w}_2 - \underline{w}_1$ is larger with a higher risk aversion. Actually, the time inconsistent preference weakens the effects of risk aversion. That is, higher risk averse agent will delay the exit compared to the time consistent case. The entrepreneur would like to maintain the firm operation longer considering the existence of utility discount in future stage. Panel B indicates that the influences of correlation between the entrepreneurial business and public equity on the liquidation decision is ambiguous in both cases. $\rho$ measures the systematic risks involved in the firm operation, and thus increasing $\rho$ brings more systematic risks which encourages the agent exit sooner. On the other hand, correlation between the firm and the public equity offers a way for the entrepreneur to hedge the risks and thus delay the liquidation. Panel B shows the non monotonic result and we find that the hedge effect is a little more

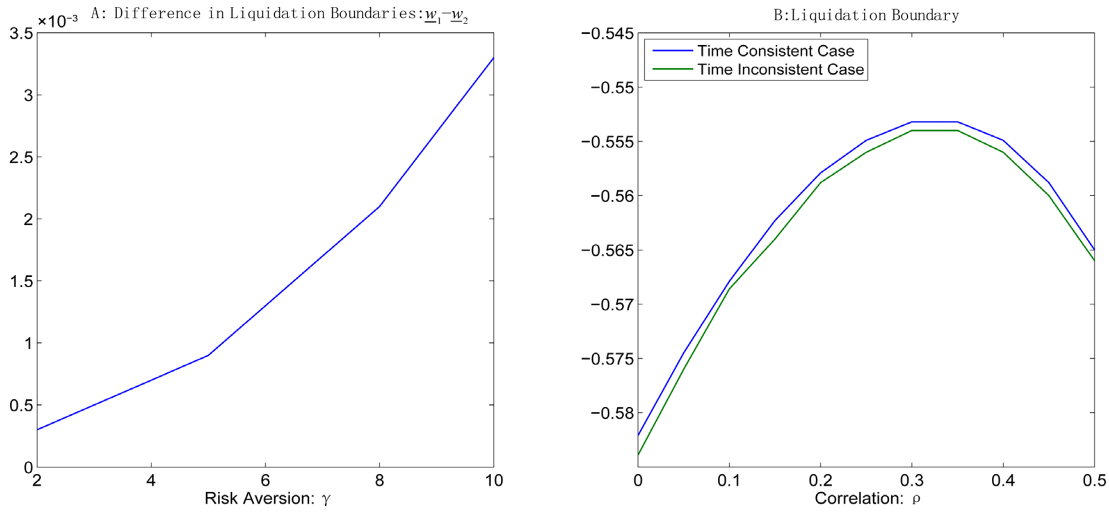


**Figure 1.** Effects of risk aversion and correlation on liquidation choices.

significant in time inconsistent case. Therefore, $\underline{w}_2$ is always lower than $\underline{w}_1$.

The magnitude of utility discounting parameter $\tilde{\beta}$ and Poisson process intensity $\lambda$ determine the degree of the entrepreneurial time inconsistency. **Figure 2** presents the effects of $\tilde{\beta}$ and $\lambda$ in our model. Panel A plots the CE wealth in complete-market $p^{FB}$ when $W = K$ (*i.e.* $w = 1$) at various levels of $\lambda$. $p^{FB}$ decrease sharply when $\lambda$ is around 0. With $\lambda$ increasing, $p^{FB}$ turns to be stable and approaches the future state ($\lambda = 1$ case) steady result. Panel B plots the liquidation boundary at different $\lambda$. The setting of inconsistent preferences requires the entrepreneur to maintain the firm operation longer as to offset the utility discount in future stage while panel B indicates that the liquidation decision is non monotonic with $\lambda$. When the agent's preference just changes to inconsistency from consistent setting (near $\lambda = 0$), the desire to maintain the firm operation dominates and the liquidation boundary decreases with $\lambda$. But the typical entrepreneurial dynamics turns to dominate and the exit choice approaches the future stage steady result in a higher level of $\lambda$. Panel C and D plot the influences of utility discount $\tilde{\beta}$ on $p^{FB}$ and $\underline{w}_2$. Decreasing $\tilde{\beta}$ leads to a lower $p^{FB}$ and delays the liquidation. Panel D indicates that the agent lacks sensitivity to $\tilde{\beta}$ when the discount factor is close to 1 and the liquidation boundary is reduced sharply at a lower level of $\tilde{\beta}$.

### 4.2. CE Wealth and Investment Decision

The time inconsistency affects not only the liquidation choice but also the wealth and operation strategies. **Figure 3** plots the effects of inconsistency on CE wealth, Tobin's *q*, entrepreneurial investment and consumption. We define private enterprise value $Q(K,W)$ for the firm as follows: $Q(K,W) = P(K,W) - W$. The entrepreneurial average q is given by the ratio between private enterprise value $Q(K,W)$ and capital: $q(w) = \dfrac{Q(K,W)}{K} = p(w) - w$. Panel A and B show that the CE wealth and average $q$ is significantly lower in time inconsistent case. Panel C plots the investment decision and indicates that there exist both over- and under-investment in time inconsistent case compared to the constant preference. The entrepreneur will invest less when his liquid wealth approaches the liquidation threshold. In a higher level of liquid wealth, on the other hand, the inconsistent preference agent will invest more than standard model. Panel D presents the consumption in two preference settings. The inconsistent agent will consume less and this reflects their precautionary saving considering the utility discount in the future stage.

**Figure 4** plots the effects of $\lambda$ and $\tilde{\beta}$ on the entrepreneurial wealth and investment decisions when the

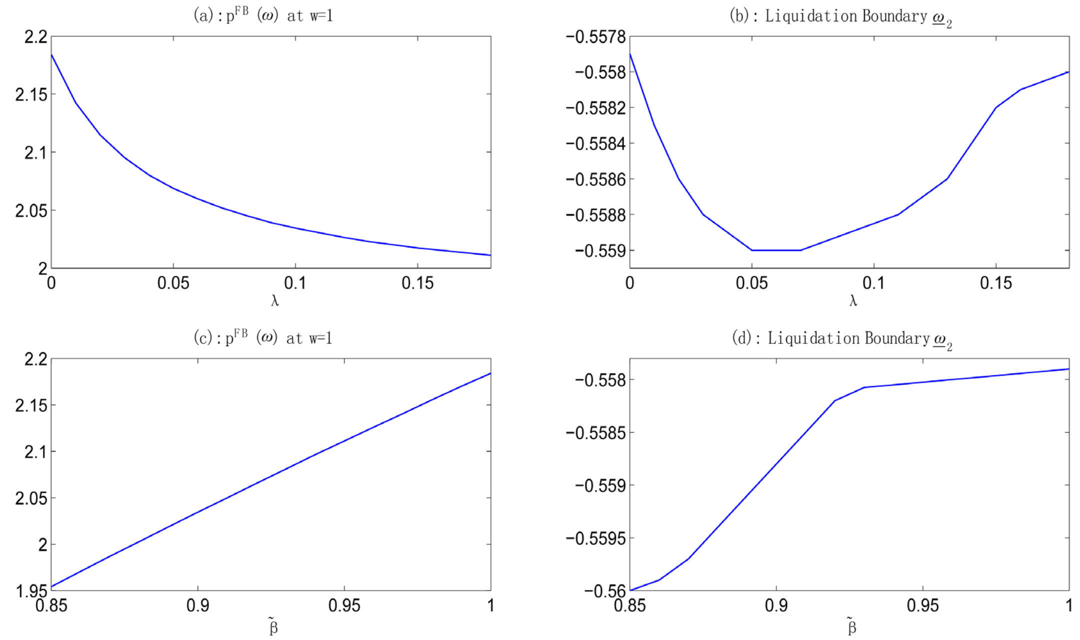


**Figure 2.** Effects of $\lambda$ and $\tilde{\beta}$ on compete-market CE wealth and liquidation choices.

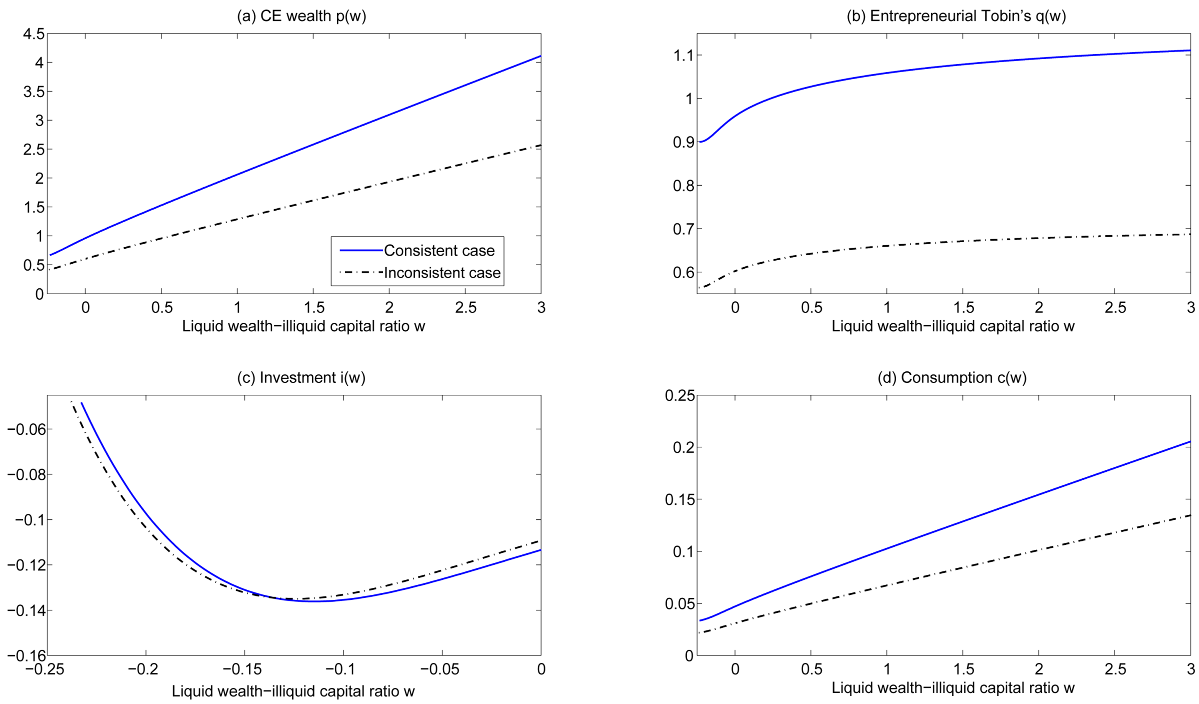


**Figure 3.** Effects of time inconsistence preferences on CE wealth and entrepreneurial investment and consumption decisions.

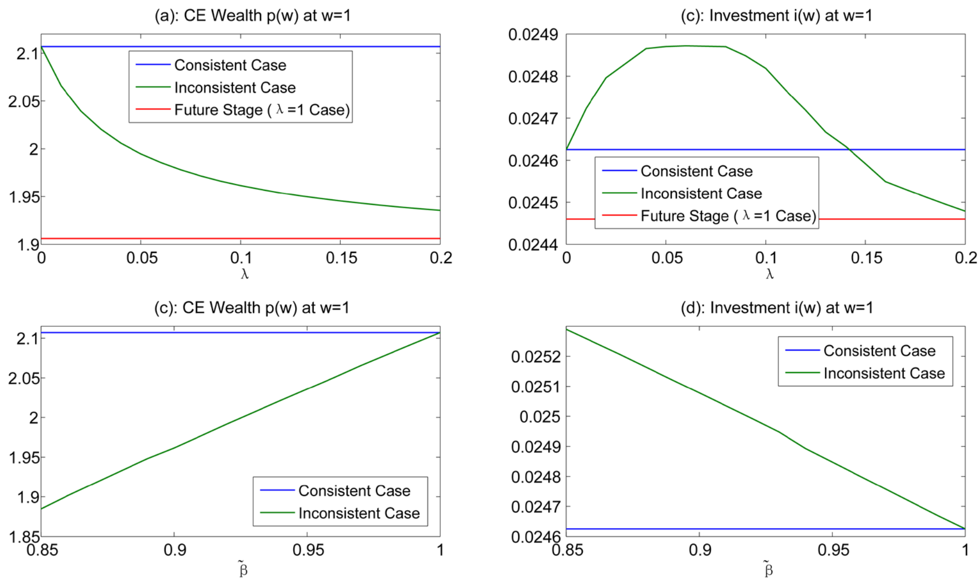


**Figure 4.** Effects of poison intension $\lambda$ and utility discount $\tilde{\beta}$ on CE wealth and investment.

agent's liquid wealth equals the capital $(w=1)$. CE wealth decreases sharply when $\lambda$ is close to 0. With $\lambda$ increasing, $p(w)$ turns to be stable and approaches the future stage value ($\lambda=1$ case). Panel B shows that the investment decision is non monotonic with $\lambda$. As we have stated above, the inconsistent preference encourages the entrepreneur maintain the firm operation longer. In the investment decision process, it is intuitive that the inconsistent agent would invest more than consistent case as a compensation for the utility discount in future stage. On the other hand, higher $\lambda$ means it would be more possible for the birth of the future stage. Therefore the entrepreneur will invest less and make more precautionary saving. Panel B exhibits this ambiguous effects. At a low level of $\lambda$, the entrepreneur will invest more to offset the utility discount in the future stage. With $\lambda$

increasing, the precautionary saving starts to dominate and the investment turns to approach the lower future stage investment strategy. Panel C and D plot the CE wealth and investment at various $\tilde{\beta}$. $\tilde{\beta}$ captures the degree of the utility discount in the future stage and thus it is intuitive that the CE wealth will be reduced when $\tilde{\beta} \neq 0$. Panel C shows that $p(w)$ decrease sharply when reducing $\tilde{\beta}$. Panel D indicates that investment is higher than the consistent case and our result presents the negative correlation between $\tilde{\beta}$ and $i(w)$ exists all the time with inconsistency setting. With $\tilde{\beta}$ decreasing, the precautionary saving become meaningless given the huge utility discount in future compared to present age and thus the agent would like to invest more in the firm operation to obtain risky but high return in contemporary stage.

## 5. Conclutions

This paper extends the entrepreneurial dynamics model to account for time inconsistent preferences. Entrepreneurs need to formulate the investment decisions taking into account the possibility of future stage with utility discount. This sets up a conflict between two opposing forces. First, the agent desires to take advantage of the option to exit, and also has an incentive to invest more and longer to offset the utility discount in future stage. Second, the time inconsistent preference lead to motivation of precautionary saving and thus reducing the investment. We extend the model of [2] to consider the decision making process for an industry made up of time inconsistent entrepreneurs.

We find that time inconsistency leads to under investment when the entrepreneurial liquid wealth is close to the liquidation boundary and over investment when the liquid wealth is far away from the exit threshold. For further analysis, we study the effects of some key factors in our model. Inconsistency weakens the effects of risk aversion which accelerates the liquidation. The effects of the correlation between the firm and the public equity are ambiguous and non monotonic in both cases, but the inconsistent setting delays the exit decision compared to consistent model. The magnitude of Poisson process intensity $\lambda$ and the utility discount $\tilde{\beta}$ determine the degree of the entrepreneurial time inconsistency. Increasing $\lambda$ promotes the possibility for the birth of the future stage in which there exists utility discount. We find that the entrepreneurial CE wealth is negatively correlated with $\lambda$ and will approach the future stage steady value with $\lambda$ increasing. The effects of $\lambda$ on the liquidation decisions and the investment strategy are ambiguous. At a low level, increasing $\lambda$ delays the liquidation and promotes the investment. But a higher $\lambda$ accelerates the exit and reduces the investment. $\tilde{\beta}$ measures the utility discount in future stage and thus the entrepreneurial CE wealth decreases sharply when reducing $\tilde{\beta}$. The investment is negatively correlated to $\tilde{\beta}$. The liquidation boundary becomes stable when $\tilde{\beta}$ is close to 1 but decreases significantly when $\tilde{\beta}$ is lower.

## Acknowledgements

Jinqiang Yang acknowledges support by Natural Science Foundation of China (#71202007 #71472117), New Century Excellent Talents in University (#NCET-13-0895) and Chen Guang Project of Shanghai Municipal Education Commission and Shanghai Education Development Foundation (#12CG44).

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Journal of Mathematical Finance, 2015, 5, 49-57
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51005

# A Regime Switching Model for the Term Structure of Credit Risk Spreads

**Seungmook Choi[1], Michael D. Marcozzi[2]**

[1]Department of Finance, University of Nevada Las Vegas, Las Vegas, NV, USA
[2]Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV, USA
Email: seungmook.choi@unlv.edu, marcozzi@unlv.nevada.edu

Received 22 January 2015; accepted 10 February 2015; published 13 February 2015

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## Abstract

**We consider a rating-based model for the term structure of credit risk spreads wherein the creditworthiness of the issuer is represented as a finite-state continuous time Markov process. This approach entails a progressive drift in credit quality towards default. A model of the economy is presented featuring stochastic transition probabilities; credit instruments are valued via an ultra parabolic Hamilton-Jacobi system of equations discretized utilizing the method-of-lines finite difference method. Computations for a callable bond are presented demonstrating the efficiency of the method.**

## Keywords

**Optimal Stopping, Failure Rate, Regime Switching, Credit Risk Spreads**

## 1. Introduction

When pricing of credit instruments subject to default risk, market participants typically assume that default is unpredictable, using dynamics derived from rating information in order to take advantage of credit events (cf. [1]). Generally, they fall into a loose hierarchy known as reduced-form models. The most ubiquitous approach involving *hazard rate* models wherein default risk via unexpected events is modeled by a jump process. In this framework, credit-risky securities are priced as discounted expectation under the risk neutral probability measure with modified discount rate (cf. [2], [3]). Although conceptually simple and easy to implement, these models are limited by the appropriate calibration of the hazard rate process. More generally, *spread modeling* represents spreads directly and eliminates the need to make assumptions on recovery (cf. [4], [5]). Finally, *rating based* models consider the creditworthiness of the issuer to be a key state variable used to calibrate the risk-neutral hazard rate (cf. [6]-[8]). A progressive drift in credit quality toward default (an absorbing state) is now allowed as opposed to a single jump to bankruptcy, as in many hazard rate models. Rating based models are particularly useful for the pricing of securities whose payoffs depend on the rating of the issuer.

In this paper, we consider a rating based regime switching model for the term-structure of credit risk spreads in continuous time (cf. [9], [10]). A unique feature of our model is the inclusion of stochastic transition probabilities. Credit instruments are then characterized as the solution to a ultraparabolic Hamilton-Jacobi system of equations for which we develop a methods-of-lines finite difference method. Computations are presented for a rating based callable bond which validates the applicability and efficiency of the method.

## 2. Model of the Economy

In this section, we introduce the dynamics of the risk-less and risky term structures of interest rates as well as the bankruptcy process. To this end, we assume the existence of a unique equivalent martingale measure such that all risk-less and risky zero-coupon bond prices are martingales after normalization by the money market account (cf. [11], [12]). Without loss of generality, we suppose a single risky zero-coupon bond price and continuous trading over a finite time interval $\left[0,\tilde{T}\right]$. We let $\mathcal{E}(t)$ $\left(0<t<\tilde{T}\right)$ denote a continuous time Markov process on the regime (or *états*) space $\mathbb{I}_m=\{0,1,2,\cdots,m\}$ with associated transition probabilities $P_{ij}(t)=Pr\left\{\mathcal{E}(t+\Delta t)=j\,\middle|\,\mathcal{E}(t)=i\right\}$, for all $\Delta t>0$; it follows that

$$0\le P_{ij}\quad\text{and}\quad\sum_{j=0}^{m}P_{ij}=1,\tag{2.1}$$

for $i\in\mathbb{I}_m$. Let $\boldsymbol{P}_i(t)=\left(P_{i0},P_{i1},\cdots,P_{im}\right)$ represent the $i^{th}$-state transition distribution.

We define the transition probabilities as follows. The $0^{th}$-state we associate with default, in which case $\boldsymbol{P}_0(s)=(1,0,\cdots,0)$. For $i=1,2,\cdots,m$, we define the $i^{th}$-state transition dynamics consistent with the non-negativity constraint in (2.1) such that $(j=1,2,\cdots,m-1)$

$$\mathrm{d}P_{ij}(s)=\alpha_{ij}\left(\overline{p}_{ij}-P_{ij}\right)\mathrm{d}s+\sigma_{ij}\beta_j\left(\boldsymbol{P}_i\right)\mathrm{d}W_{ij}(s),\tag{2.2a}$$

$$P_{ij}(t)=p_{ij}\in(0,1),\tag{2.2b}$$

for $0<t<s<\tilde{T}$, where

$$\beta_j^2\left(\boldsymbol{P}_i\right)=\begin{cases}P_{ij} & \text{if } j<m-1\\ P_{ij}\left(1-\sum_{\varepsilon=1}^{m-1}P_{i\varepsilon}\right) & \text{if } j=m-1\end{cases}$$

and $0<\overline{p}_{ij}$ is the mean transition level satisfying $\sum_{\varepsilon=1}^{m-1}\overline{p}_{i\varepsilon}\le 1$, $0\le\alpha_{ij}$ is the rate of reversion to the mean, $0\le\sigma_{ij}$ and $\mathrm{d}W_{ij}$ is a Wiener process. From (2.1), it follows that $P_{im}=1-P_{i0}-P_{i1}-\cdots-P_{im-1}$ and so

$$\mathrm{d}P_{im}(s)=-\sum_{j=1}^{m-1}\alpha_{ij}\left(\overline{p}_{ij}-P_{ij}\right)\mathrm{d}s-\sum_{j=1}^{m-1}\sigma_{ij}\beta_j\left(\boldsymbol{P}_i\right)\mathrm{d}W_{ij}(s),\tag{2.2c}$$

$$P_{im}(t)=p_{im}=1-\sum_{j=1}^{m-1}p_{ij}\in(0,1).\tag{2.2d}$$

We relate the *transition* matrix $\Pi=\left(P_{ij}\right)$ to the regime dynamics via the infinitesimal generator $\Lambda$,

$$\Lambda=\lim_{h\to 0^+}\frac{\Pi(h)-\boldsymbol{I}}{h},$$

such that

$$\frac{\mathrm{d}\boldsymbol{P}(s)}{\mathrm{d}s}=\boldsymbol{P}(s)\Lambda,$$

for $0<t<s<\tilde{T}$, and

$$\boldsymbol{P}(t)=\boldsymbol{P},$$

where $\boldsymbol{P}=(P_0,P_1,\cdots,P_m)$ is the vector of probabilities $P_j(s)=Pr\{E(s)=j\}$. Without loss of generality, we associate $E(s)$ with the vector $\boldsymbol{\mathcal{E}}(s)\in\{\boldsymbol{e}_0,\boldsymbol{e}_1,\cdots,\boldsymbol{e}_m\}$, $\boldsymbol{e}_0=(1,0,\cdots,0)$, $\boldsymbol{e}_1=(0,1,\cdots,0)$, $\cdots$, $\boldsymbol{e}_m=(0,0,\cdots,1)$, subject to the dynamics

$$\mathrm{d}\boldsymbol{\mathcal{E}}(s)=\boldsymbol{\mathcal{E}}(s)\Lambda\mathrm{d}s+\mathrm{d}\boldsymbol{M}(s), \tag{2.3a}$$

$$\boldsymbol{\mathcal{E}}(t)=\boldsymbol{e}, \tag{2.3b}$$

for $0<t<s<\tilde{T}$, where $\boldsymbol{M}(s)$ is a martingale with respect to the filtration generated by $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{P}(s)=\mathbb{E}[\boldsymbol{\mathcal{E}}(s)]$ ([13], Chap 4.8; [14], Part III, App. B; [15], Chap 8). In particular, the state of the system is known at inception such that $\boldsymbol{e}_i=\mathbb{E}[\boldsymbol{\mathcal{E}}(t)]=\boldsymbol{P}(t)=\boldsymbol{P}$, for some $i\in\mathbb{I}_m$.

We suppose that the *risky* interest rate $R$ follows a state specific Cox-Ingersall-Ross dynamic given by

$$\mathrm{d}R(s;\boldsymbol{\mathcal{E}})=\alpha(\boldsymbol{\mathcal{E}})\cdot(\overline{r}(\boldsymbol{\mathcal{E}})-R)\mathrm{d}s+\sigma(\boldsymbol{\mathcal{E}})\sqrt{R}\cdot\mathrm{d}W(s) \tag{2.4a}$$

for $0<t<s<\tilde{T}$, with mean reversion level $\overline{r}(\boldsymbol{\mathcal{E}})$ and rate of reversion to the mean $\alpha(\boldsymbol{\mathcal{E}})$, such that

$$R(t;\boldsymbol{e})=r(\boldsymbol{e}), \tag{2.4b}$$

where $\mathrm{d}W$ is a Wiener process. In default $\alpha(\boldsymbol{e}_0)=\sigma(\boldsymbol{e}_0)=0$, otherwise $\alpha(\boldsymbol{e}_i)=\alpha_i$ and $\sigma(\boldsymbol{e}_i)=\sigma_i$. The risky bond price $B$ associated with a maturity $T$ satisfies

$$\frac{\mathrm{d}B(s)}{\mathrm{d}s}=R(s;\boldsymbol{e})B(s), \tag{2.5a}$$

$$B(t)=b. \tag{2.5b}$$

We consider the risk-less interest rate $\varrho$ to satisfy

$$\mathrm{d}\varrho(s;\boldsymbol{\mathcal{E}})\mathrm{d}s=0,$$

$$\varrho(t;\boldsymbol{\mathcal{E}})=\rho(\boldsymbol{\mathcal{E}}),$$

where in default $\rho(\boldsymbol{e}_0)=0$ for convenience, and $\rho(\boldsymbol{e}_i)=\rho$ otherwise.

For a given contract $\psi$, we define the *value* function associated with the joint Markov ultradiffusion process (2.2)-(2.5) such that

$$v(t,b,r,\boldsymbol{\pi},\boldsymbol{e}_i)=\mathbb{E}\left\{\exp\left[-\varrho(\boldsymbol{\mathcal{E}})\cdot(T-t)\right]\cdot\psi(T,B(T),R(T),\Pi(T),\boldsymbol{\mathcal{E}})\right\}, \tag{2.6}$$

for $0<t<T<\tilde{T}$, where $\boldsymbol{\pi}=(p_{ij})$.

In particular, for a non-coupon paying bond $\psi(T,\boldsymbol{e}_0)=\delta$ and $\psi(T,\boldsymbol{e}_i)=1$ otherwise, where $\delta$ is the default recovery rate, whereas for a callable bond $\psi(T,b,\boldsymbol{e}_0)=0$ and $\psi(T,b,\boldsymbol{e}_i)=\max\{b-E(\boldsymbol{e}_i),0\}$ otherwise, for some rating based exercise price $E(\boldsymbol{e}_i)$. Generalization of (2.6) and the subsequent analysis to include early exercise features follows routinely and will not be considered here.

## 3. Characterization

Letting $v_i(t,b,r,\boldsymbol{\pi})=v(t,b,r,\boldsymbol{\pi},\boldsymbol{e}_i)$ and

$$\boldsymbol{v}(t,\cdot)=(v_0(t,\cdot),v_1(t,\cdot),\cdots,v_m(t,\cdot))^{\mathrm{T}}, \tag{3.1a}$$

we recover (2.6) succinctly as

$$v(t,\cdot,\boldsymbol{e}_i)=\boldsymbol{e}_i\cdot\boldsymbol{v}(t,\cdot), \tag{3.1b}$$

for $i\in\mathbb{I}_m$. By Itô's rule, the value function (2.6) is characterized via (3.1) as the solution to the ultraparabolic Hamilton-Jacobi system of equations

$$\frac{\partial v_0}{\partial t}=0$$

$$v_0(T,x)=\psi_0(T,x)$$

$$
\begin{aligned}
&\frac{\partial v_1}{\partial t} + rb\frac{\partial v_1}{\partial b} + \mathcal{A}_1 v_1 + p_{10}v_0 + \left(p_{11}-1\right)v_1 + \cdots + p_{1m}v_m = 0\\
&v_1\left(T,x\right) = \psi_1\left(T,x\right)\\
&\vdots\\
&\frac{\partial v_m}{\partial t} + rb\frac{\partial v_m}{\partial b} + \mathcal{A}_m v_m + p_{m0}v_0 + p_{m1}v_1 + \cdots + \left(p_{mm}-1\right)v_m = 0\\
&v_m\left(T,x\right) = \psi_m\left(T,x\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_i v = {} & \frac{1}{2}r\sigma_i^2\frac{\partial^2 v}{\partial r^2} + \alpha_i\left(\overline{r}_i - r\right)\frac{\partial v}{\partial r} + \sum_{1\le j<m}\frac{1}{2}\beta_j^2\left(\boldsymbol{P}_i\right)\sigma_{ij}^2\frac{\partial^2 v}{\partial p_{ij}^2}\\
& + \sum_{1\le j<m}\alpha_{ij}\left(\overline{p}_{ij} - p_{ij}\right)\frac{\partial v}{\partial p_{ij}} - \sum_{1\le j<m}\frac{1}{2}\beta_j^2\left(\boldsymbol{P}_i\right)\sigma_{ij}^2\frac{\partial^2 v}{\partial p_{im}^2}\\
& - \sum_{1\le j<m}\alpha_{ij}\left(\overline{p}_{ij} - p_{ij}\right)\frac{\partial v}{\partial p_{im}} - \rho\left(\boldsymbol{e}_\varepsilon\right)v.
\end{aligned}
$$

Let $\boldsymbol{t} = \left(t,b\right) \in \mathcal{Q} = \left(0,T\right)\times\left(0,\infty\right)$ denote the temporal variable and $\boldsymbol{x} = \left(r, p_{11}, p_{12}, \cdots, p_{mm}\right) \in \Omega = \left(0,\infty\right)\times\left\{\left(0,1\right)^m \middle| \sum_j p_{1j} \le 1\right\}\times\cdots\times\left\{\left(0,1\right)^m \middle| \sum_j p_{mj} \le 1\right\}$ the spatial, we define

$$
\mathcal{H} = \frac{\partial}{\partial t} + rb\frac{\partial}{\partial b}
$$

and $\boldsymbol{\mathcal{A}} = \left(A_1, A_2, \cdots, A_m\right)$, such that the above can be written

$$
\mathcal{H}\boldsymbol{v}\left(\boldsymbol{t},\boldsymbol{x}\right) + \boldsymbol{\mathcal{A}}\cdot\boldsymbol{v}\left(\boldsymbol{t},\boldsymbol{x}\right) + \left(\Pi - \boldsymbol{I}\right)\cdot\boldsymbol{v}\left(\boldsymbol{t},\boldsymbol{x}\right) = \boldsymbol{0}, \tag{3.2a}
$$

for all $\left(\boldsymbol{t},\boldsymbol{x}\right) \in \mathcal{Q}\times\Omega$, subject to the terminal constraint

$$
\boldsymbol{v}\left(T,\cdot\right) = \boldsymbol{\psi}\left(T,\cdot\right), \tag{3.2b}
$$

for $\left(b,\boldsymbol{x}\right) \in \left(0,\infty\right)\times\Omega$, where $\boldsymbol{\psi} = \left(\psi\left(\cdot,\boldsymbol{e}_0\right), \psi\left(\cdot,\boldsymbol{e}_1\right), \cdots, \psi\left(\cdot,\boldsymbol{e}_m\right)\right)^{\mathrm{T}}$.

## 4. Approximation Solvability

Towards obtaining a constructive approximation of (3.2), we consider an exhaustive sequence of bounded open domains $\left\{\Omega_k\right\}$ such that $\Omega_k \subset \Omega_{k+1}$ and $\bigcup\Omega_k = \Omega$ as well as a sequence of monotonically increasing real numbers $T_k \to \infty$, as $k \to \infty$. Let $\mathcal{Q}_k = \left(0,T\right)\times\left(0,T_k\right)$ and $\partial\mathcal{Q}_k = \left\{T\right\}\times\left(0,T_k\right)\bigcup\left(0,T\right)\times\left\{T_k\right\}$, we seek $\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right)$ satisfying

$$
\mathcal{H}\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right) + \boldsymbol{\mathcal{A}}\cdot\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right) + \left(\Pi - \boldsymbol{I}\right)\cdot\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right) = \boldsymbol{0}, \tag{4.1a}
$$

for all $\left(\boldsymbol{t},\boldsymbol{x}\right) \in \mathcal{Q}_k\times\Omega_k$, subject to the boundary condition

$$
\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right) = \boldsymbol{\psi}\left(\boldsymbol{t},\boldsymbol{x}\right), \tag{4.1b}
$$

for $\left(\boldsymbol{t},\boldsymbol{x}\right) \in \mathcal{Q}_k\times\partial\Omega_k$, and terminal constraint

$$
\boldsymbol{v}_k\left(\boldsymbol{T},\boldsymbol{x}\right) = \boldsymbol{\psi}\left(\boldsymbol{T},\boldsymbol{x}\right), \tag{4.1c}
$$

where $\left(\boldsymbol{T},\boldsymbol{x}\right) \in \partial\mathcal{Q}_k\times\Omega_k$. As (3.2) is an infinite horizon problem in $b$, we remark to the necessity of introducing the artificial terminal condition $\boldsymbol{v}_k = \boldsymbol{\psi}$ along the frontier $\left(T,b\right) \in \left\{T\right\}\times\left(0,T_k\right)$ (cf. [16]). In particular, $\boldsymbol{v}_k\left(\boldsymbol{t},\boldsymbol{x}\right) \to \boldsymbol{v}\left(\boldsymbol{t},\boldsymbol{x}\right)$ as $k \to \infty$, on any compact subset of $\Omega$, for any fixed $\boldsymbol{t} \in \overline{\mathcal{Q}}$.

We next place (4.1) into standard form by setting $\tau = T - t$, $\varsigma_k = T_k - b$, $\boldsymbol{\tau} = \left(\tau,\varsigma_k\right)$, in which case $\boldsymbol{u}_k\left(T-\tau, T_k - b, \cdot\right) = \boldsymbol{v}_k\left(t,b,\cdot\right)$. Letting

$$\mathcal{H}_k = \frac{\partial}{\partial \tau} + r\left(T_k - \varsigma_k\right)\frac{\partial}{\partial \varsigma_k},$$

Equation (4.1) becomes

$$-\mathcal{H}_k \boldsymbol{u}_k\left(\boldsymbol{\tau}, \boldsymbol{x}\right) + \boldsymbol{\mathcal{A}} \cdot \boldsymbol{u}_k\left(\boldsymbol{\tau}, \boldsymbol{x}\right) + \left(\Pi - \boldsymbol{I}\right)\boldsymbol{u}_k\left(\boldsymbol{\tau}, \boldsymbol{x}\right) = \boldsymbol{0}, \tag{4.2a}$$

for all $\left(\boldsymbol{\tau}, \boldsymbol{x}\right) \in \mathcal{Q}_k \times \Omega_k$, subject to the boundary condition

$$\boldsymbol{u}_k\left(\boldsymbol{\tau}, \boldsymbol{x}\right) = \boldsymbol{\psi}\left(\boldsymbol{\tau}, \boldsymbol{x}\right), \tag{4.2b}$$

for $\left(\boldsymbol{\tau}, \boldsymbol{x}\right) \in \mathcal{Q}_k \times \partial\Omega_k$, and initial condition

$$\boldsymbol{u}_k\left(\boldsymbol{\tau}_0, \boldsymbol{x}\right) = \boldsymbol{\psi}\left(\boldsymbol{\tau}_0, \boldsymbol{x}\right), \tag{4.2c}$$

where $\left(\boldsymbol{\tau}_0, \boldsymbol{x}\right) \in \partial\mathcal{Q}_{0,k} \times \Omega_k$, where $\partial\mathcal{Q}_{0,k} = \{0\} \times \left(0, T_k\right) \cup \left(0, T\right) \times \{0\}$.

We consider the discretization of (4.2) by the backward Euler method temporally and central differencing in space. To this end, we introduce the temporal step sizes $\left(\delta_\tau, \delta_\varsigma\right) \in \mathbb{R}_+^2$ and mesh sizes $\left(\mathcal{N}_\tau, \mathcal{N}_\varsigma\right) \in \mathbb{N}^2$, such that $T = \delta_\tau \cdot \mathcal{N}_\tau$ and $T_k = \delta_\varsigma \cdot \mathcal{N}_\varsigma$. Spatially, we utilize the step sizes $\left(\delta_r, \delta_p\right) \in \mathbb{R}_+^2$ and mesh sizes $\left(\mathcal{M}_r, \mathcal{M}_p\right) \in \mathbb{N}^2$; we denote the value of $\boldsymbol{u}_k$ on the grid by

$$\boldsymbol{u}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \boldsymbol{u}_k\left(\tau^{\nu_1}, \varsigma^{\nu_2}, r^{\mu_0}, p_{11}^{\mu_1}, \cdots, p_{mm}^{\mu_{m^2}}\right),$$

where $\tau^{\nu_1} = \nu_1 \cdot \delta_\tau$, $\varsigma^{\nu_2} = \nu_2 \cdot \delta_\varsigma$, $r^{\mu_0} = \mu_0 \cdot \delta_r$, $p_{11}^{\mu_1} = \mu_1 \cdot \delta_p$, and so forth. Notationally, we let $\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) \in \mathcal{Q}_\delta \times \Omega_{k,\delta}$, where $\boldsymbol{\nu} = \left(\nu_1, \nu_2\right)$, $\boldsymbol{\mu} = \left(\mu_0, \mu_1, \cdots, \mu_{m^2}\right)$, $\mathcal{Q}_\delta = \left[0, 1, \cdots, \mathcal{N}_\tau\right] \times \left[0, 1, \cdots, \mathcal{N}_\varsigma\right]$, and $\Omega_{k,\delta} = \left[0, 1, \cdots, \mathcal{M}_r\right] \times \left\{\left[0, 1, \cdots, \mathcal{M}_p\right]^m \middle| \sum_j p_{1j}^{\mu_1} \le 1\right\}^m \times \cdots \times \left\{\left[0, 1, \cdots, \mathcal{M}_p\right]^m \middle| \sum_j p_{mj}^{\mu_m} \le 1\right\}^m$. For

$$\boldsymbol{u}_k\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) = \boldsymbol{u}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}},$$

the difference quotients are then backward first order in time:

$$\nabla_\tau \boldsymbol{u}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{\delta_\tau}\left[\boldsymbol{u}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} - \boldsymbol{u}_k^{\nu_1-1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}}\right]$$

$$\nabla_\varsigma \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{\delta_\tau}\left[\boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} - \boldsymbol{\mu}_k^{\nu_1,\nu_2-1,\mu_0,\mu_1,\cdots,\mu_{m^2}}\right]$$

and central second-order in space:

$$\delta_0^2 \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{\delta_r^2}\left[\boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0+1,\mu_1,\cdots,\mu_{m^2}} - \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} + \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0-1,\mu_1,\cdots,\mu_{m^2}}\right]$$

$$\delta_1^2 \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{\delta_r^2}\left[\boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1+1,\cdots,\mu_{m^2}} - \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} + \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1-1,\cdots,\mu_{m^2}}\right]$$

and so forth, and

$$\delta_0 \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{2\delta_r}\left[\boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0+1,\mu_1,\cdots,\mu_{m^2}} - \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0-1,\mu_1,\cdots,\mu_{m^2}}\right]$$

$$\delta_1 \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1,\cdots,\mu_{m^2}} = \frac{1}{2\delta_r}\left[\boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1+1,\cdots,\mu_{m^2}} - \boldsymbol{\mu}_k^{\nu_1,\nu_2,\mu_0,\mu_1-1,\cdots,\mu_{m^2}}\right]$$

and so forth.

Given the above, we define the method-of-lines finite difference discretization of (4.2) such that

$$-\mathcal{H}_\delta \boldsymbol{u}_k\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) + \boldsymbol{\mathcal{A}}_\delta \cdot \boldsymbol{u}_k\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) + \left(\Pi - \boldsymbol{I}\right)\boldsymbol{u}_k\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) = \boldsymbol{0}, \tag{4.3a}$$

for all $\left(\boldsymbol{\nu}, \boldsymbol{\mu}\right) \in \mathcal{Q}_\delta \times \Omega_{k,\delta}$, subject to the boundary condition

$$\boldsymbol{u}_k(\boldsymbol{\nu},\boldsymbol{\mu}) = \boldsymbol{\psi}(\boldsymbol{\tau},\boldsymbol{x}), \tag{4.3b}$$

for $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{Q}_\delta \times \partial\Omega_{k,\delta}$, and initial condition

$$\boldsymbol{u}_k(\boldsymbol{\nu}_0,\boldsymbol{\mu}) = \boldsymbol{\psi}(\boldsymbol{\tau}_0,\boldsymbol{x}), \tag{4.3c}$$

where $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \partial\mathcal{Q}_\delta \times \Omega_{k,\delta}$, $\partial\mathcal{Q}_\delta = \{0\} \times [0,1,\cdots,\mathcal{N}_\varsigma] \cup [0,1,\cdots,\mathcal{N}_\tau] \times \{0\}$,

$$\mathcal{H}_\delta u = \mathcal{H}_\delta(\boldsymbol{\nu},\boldsymbol{\mu})u = \nabla_\tau u + r^{\mu_0}\left(T_k - \varsigma^{\nu_2}\right)\nabla_\varsigma u,$$

$$\begin{aligned}\mathcal{A}_\varepsilon u = \mathcal{A}_\varepsilon(\boldsymbol{\nu},\boldsymbol{\mu})u &= \frac{1}{2} r^{\mu_0}\sigma_\varepsilon^2\delta_0^2 u + \alpha_\varepsilon\left(\overline{r}_\varepsilon - r^{\mu_0}\right)\delta_0 u + \frac{1}{2}\beta_1^2(\boldsymbol{P}_1)\sigma_{11}^2\delta_1^2 u \\ &+ \cdots + \frac{1}{2}\beta_{m-1}^2(\boldsymbol{P}_{m-1})\sigma_{m-1,m-1}^2\delta_{(m-1)^2}^2 u - \sum_{1\le j<m}\frac{1}{2}\beta_j^2(\boldsymbol{P}_j)\sigma_{\varepsilon j}^2\delta_m^2 u \\ &+ \alpha_{11}\left(\overline{p}_{11} - p_{11}\right)\delta_1 u + \cdots + \alpha_{m-1,m-1}\left(\overline{p}_{m-1,m-1} - p_{m-1,m-1}\right)\delta_{(m-1)^2} u \\ &- \sum_{1\le j<m}\alpha_{\varepsilon j}\left(\overline{p}_{\varepsilon j} - p_{\varepsilon j}\right)\delta_m u - \rho(\boldsymbol{e}_\varepsilon)u,\end{aligned}$$

and $\boldsymbol{\mathcal{A}}_\delta = (A_1, A_2, \cdots, A_m)$. We solve (4.3) utilizing the pseudo-code (cf. [16], [17]):

*do* $\nu_1 = 1,\cdots,\mathcal{N}_\tau$

*do* $\eta_2 = 0,\cdots,\mathcal{N}_\varsigma$

*solve for* $\boldsymbol{u}_k(\boldsymbol{\nu},\boldsymbol{\mu})$ *via* (4.3).

## 5. Numerical Experiment

In this section, we present a representative computation for the valuation of a callable bond relative to three credit ratings:

$$\boldsymbol{\mathcal{E}}(t) = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \text{Default} \\ A \text{ rating} \\ B \text{ rating} \end{pmatrix}$$

and rating's dependent pay-off contract

$$\psi(T,b,\boldsymbol{\mathcal{E}}) = \begin{cases} 0 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_0 \\ \max\{b-0.70, 0\} & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_1 \\ \max\{b-0.68, 0\} & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_2 \end{cases}$$

with expiry $T = 0.5$. We suppose a solvent risk-free rate of return of $\rho = 0.02$. For simplicity, we will consider the following transition matrix

$$\boldsymbol{\Pi} = \begin{pmatrix} 1.00 & 0.00 & 0.00 \\ 0.0 & 0.95 & 0.05 \\ P_{def} & 0.10 - P_{def} & 0.9 \end{pmatrix}$$

in which only the default probability $P_{20} = P_{def}$ is stochastic.

For $0 < t < s < T < \tilde{T}$, we have the economy;

$$\mathrm{d}P_{def}(s) = 0.01\left(0.05 - P_{def}\right)\mathrm{d}s + 0.05\sqrt{P_{def}}\sqrt{0.10 - P_{def}}\,\mathrm{d}W_{def}(s);\ P_{def}(t) = p_{def} \tag{5.1a}$$

$$\mathrm{d}R(s;\boldsymbol{\mathcal{E}}) = \alpha(\boldsymbol{\mathcal{E}})\cdot\left(\overline{r}(\boldsymbol{\mathcal{E}}) - R\right)\mathrm{d}s + \sigma(\boldsymbol{\mathcal{E}})\sqrt{R}\cdot\mathrm{d}W(s);\ R(t;\boldsymbol{e}) = r(\boldsymbol{e}), \tag{5.1b}$$

$$\mathrm{d}B(s)\mathrm{d}s = R(s)B(s)\mathrm{d}s;\ B(t) = b, \tag{5.1c}$$

where

$$\alpha(\boldsymbol{\mathcal{E}}) = \begin{cases} 0 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_0 \\ 0.010 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_1 \\ 0.005 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_2 \end{cases};\quad \sigma(\boldsymbol{\mathcal{E}}) = \begin{cases} 0 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_0 \\ 0.20 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_1 \\ 0.25 & \text{if } \boldsymbol{\mathcal{E}} = \boldsymbol{e}_2 \end{cases}.$$

and

$$\bar{r}(\mathcal{E}) = \begin{cases} 0 & \text{if } \mathcal{E} = \boldsymbol{e}_0 \\ 0.03 & \text{if } \mathcal{E} = \boldsymbol{e}_1 \\ 0.06 & \text{if } \mathcal{E} = \boldsymbol{e}_2 \end{cases}$$

Letting $\Gamma_k = 1.5$ and $\Omega_k = [0.0, 0.5] \times [0.0, 0.10]$, the ultraparabolic Hamilton-Jacobi system of Equations (4.1) for the value function $\boldsymbol{v}(t, b, r, p_{def}) = (v_0, v_1, v_2)$ associated with the ultradiffusion (5.1) is then

$$v_0(t, b, r, p_{def}) = 0 \tag{5.2}$$

for all $(t, b, r, p_{def}) \in [0, T] \times [0, \Gamma_k] \times \bar{\Omega}_k$,

$$\begin{aligned} &\frac{\partial v_1}{\partial t} + rb\frac{\partial v_1}{\partial b} + \frac{1}{2} r\sigma_\varepsilon^2 \frac{\partial^2 v_1}{\partial r^2} + \alpha_\varepsilon (\bar{r}_\varepsilon - r)\frac{\partial v_1}{\partial r} + \frac{1}{2} p_{def}(0.1 - p_{def})(0.05)^2 \frac{\partial^2 v_1}{\partial p_{def}^2} \\ &+ 0.01(0.05 - p_{def})\frac{\partial v_1}{\partial p_{def}} - 0.02 v_1 + (0.95 - 1) v_1 + 0.05 v_2 = 0 \end{aligned} \tag{5.3a}$$

for all $(t, b, r, p_{def}) \in (0, T) \times (0, \Gamma_k) \times \Omega_k$, such that

$$v_1(t, b, r, p_{def}) = \max\{b - 0.70, 0\}, \tag{5.3b}$$

for $(t, b, r, p_{def}) \in (0, T) \times (0, \Gamma_k) \times \partial\Omega_k$ and

$$v_1(T, b, r, p_{def}) = \max\{b - 0.70, 0\}, \ (b, r, p_{def}) \in (0, \Gamma_k) \times \Omega_k \tag{5.3c}$$

$$v_1(t, \Gamma_k, r, p_{def}) = \max\{b - 0.70, 0\}, \ (t, r, p_{def}) \in (0, T) \times \Omega_k \tag{5.3d}$$

and

$$\begin{aligned} &\frac{\partial v_2}{\partial t} + rb\frac{\partial v_2}{\partial b} + \frac{1}{2} r\sigma_\varepsilon^2 \frac{\partial^2 v_2}{\partial r^2} + \alpha_\varepsilon (\bar{r}_\varepsilon - r)\frac{\partial v_2}{\partial r} + \frac{1}{2} p_{def}(0.1 - p_{def})(0.05)^2 \frac{\partial^2 v_2}{\partial p_{def}^2} \\ &+ 0.01(0.05 - p_{def})\frac{\partial v_2}{\partial p_{def}} - 0.02 v_1 + (0.10 - p_{def}) v_1 + (0.9 - 1) v_2 = 0 \end{aligned} \tag{5.4a}$$

for all $(t, b, r, p_{def}) \in (0, T) \times (0, \Gamma_k) \times \Omega_k$, such that

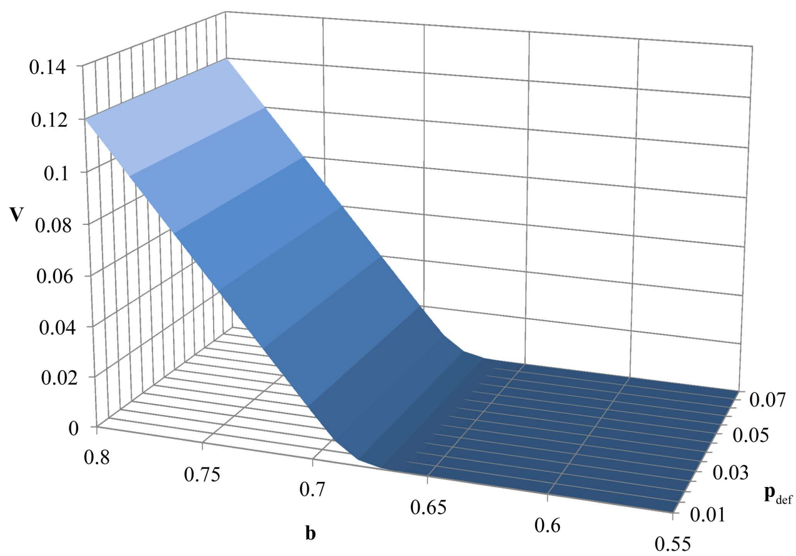


**Figure 1.** $v_1(0, b, 0.05, p_{def})$.

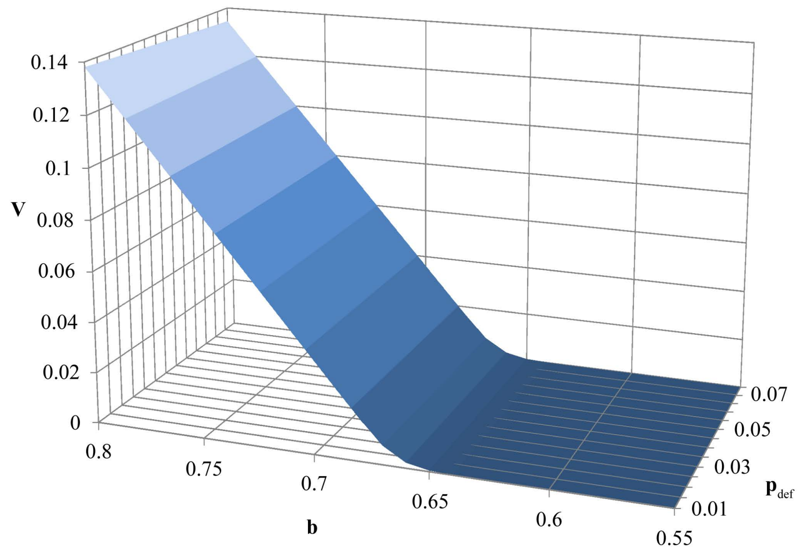


**Figure 2.** $v_2$ (0, $b$, 0.05, $p_{def}$).

$$v_2\left(t,b,r,p_{def}\right)=\max\left\{b-0.68,0\right\}, \tag{5.4b}$$

for $\left(t,b,r,p_{def}\right)\in\left(0,T\right)\times\left(0,\Gamma_k\right)\times\partial\Omega_k$ and

$$v_2\left(T,b,r,p_{def}\right)=\max\left\{b-0.68,0\right\},\ \left(b,r,p_{def}\right)\in\left(0,\Gamma_k\right)\times\Omega_k \tag{5.4c}$$

$$v_2\left(t,\Gamma_k,r,p_{def}\right)=\max\left\{b-0.68,0\right\},\ \left(t,r,p_{def}\right)\in\left(0,T\right)\times\Omega_k. \tag{5.4d}$$

**Figure 1** and **Figure 2** show the value function components $v_1\left(t,b,r,p_{def}\right)$ and $v_2\left(t,b,r,p_{def}\right)$, respectively, for $r=0.05$. Relative to the discretization of (5.2)-(5.4), we utilized $\delta_\tau=0.001$, $\delta_\varsigma=0.001$, $\delta_r=0.005$, $\delta_p=0.005$. In particular, we note the effect of the rating based exercise prices on $v_1$ and $v_2$ and the decreasing value of $v_2$ with increasing $p_{def}$, as expected.

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**Journal of Mathematical Finance, 2015, 5, 58-72**
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51006

# Interest Rate Volatility: A Consol Rate Approach

**Vincent Brousseau[1], Alain Durré[2*]**

[1]Visiting Scholar at Lille 2-Skema Management Research Center-Centre National de la Recherche Scientifique (E.A. 4112), Lille, France
[2]IéSEG-School of Management (Lille Catholic University) and LEM-Centre National de la Recherche Scientifique (U.M.R. 8179), Paris, France
Email: vincent.brousseau@univ-lille2.fr, *a.durre@ieseg.fr

Received 22 January 2015; accepted 10 February 2015; published 13 February 2015

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## Abstract

**In this paper, we propose a new methodology to estimate the volatility of interest rates in the euro area money market. In particular, our approach aims at avoiding the limitations of market implied volatilities, *i.e.* the dependency on arbitrary choices in terms of maturity and frequencies and/or of other factors like credit and liquidity risks. The measure is constructed as the implied instantaneous volatility of a consol bond that would be priced on the EONIA swap curve over the sample period from 4 January 1999 to 21 November 2013. Our findings show that this measure tracks well the historical volatility since, by dividing the consol excess returns by our volatility measure. This removes nearly entirely excess of kurtosis and volatility clustering, bringing the excess returns close to an ordinary Gaussian white noise.**

## Keywords

**Consol Rate, Historical Volatility, Overnight Money Market, Interbank Offered Interest Rates**

## 1. Introduction

It is common that the logarithm of asset prices is not well depicted by a simple Brownian motion. The first feature of a Brownian motion, which is a Gaussian white noise, is the absence of excess of kurtosis, and the square or the absolute value of this first difference exhibits no autocorrelation. By contrast, the returns of an asset exhibit noticeable excess of kurtosis, and their square or their absolute returns exhibit positive autocorrelations. Attempts to find a better modelling for the log-prices may include the adjunction of jumps or the specification of a

*Corresponding author.

**How to cite this paper:** Brousseau, V. and Durré, A. (2015) Interest Rate Volatility: A Consol Rate Approach. *Journal of Mathematical Finance*, **5**, 58-72. http://dx.doi.org/10.4236/jmf.2015.51006

variable, generally random, volatility. A variable volatility is sufficient to produce the excess of kurtosis and the positive autocorrelation of squared and absolute values—known as "volatility clustering". Therefore, it is legitimate to wonder whether the recourse to variable volatilities is sufficient to capture the observed features of kurtosis excess and volatility clustering.

To answer this question, one may start with the following thought experiment. Consider the observed time series of a liquidly traded asset price, for instance the euro-dollar exchange rate. Ignoring the short-term interest rates that prevail on those two currencies, we assume for simplicity that two subsequent observations are never identical. The corresponding return is thus the difference of the log-price at time t + 1 and at time t. Then, we define the proxy of its instantaneous volatility, at time t, by the absolute value of that return. The normalised return is therefore defined as the ratio of the return and of that proxy. By construction, this normalized return is a random sequence of +1 and −1. It will exhibit no autocorrelation, and its distribution will be even less leptokurtic than a Gaussian one, which means that it has a negative excess of kurtosis. This can be the basis of a procedure for constructing the instantaneous volatility of the exchange rate, if and only if the value of that rate at time t + 1 is known at time t, which is obviously not the case. So one is led to reformulate the initial question as follows: is it possible to construct a proxy of the instantaneous volatility at time t with data available at time t, in such a way that the return normalized by this proxy exhibits neither excess of kurtosis nor autocorrelation of its square or of its absolute value?

Here, there are two possible ways forward. One might construct that proxy on the basis of the history of the series available at time t, or, on the other hand, one might take recourse to exogenous data available at time t, namely the corresponding market-implied volatilities. The second way should be favoured because it does not assume anything about the statistical properties of the price series. In the example of the euro-dollar exchange rate, the exercise could be described by the following steps: a) to reconstruct an instantaneous volatility from the available market implied volatilities; b) to calculate the excess return of the exchange rate taking into account short-term interest rates of the two currencies; c) to calculate the normalised excess return by dividing the excess return by the instantaneous volatility; and d) to compare the statistical properties of the raw excess returns and of the normalised one. If the raw excess returns exhibit excess of kurtosis and volatility clustering whereas the normalised retruns do not, then it means that it is possible to model the exchange rate as an Ito process with variable volatility.

This paper follows the same approach with an application to the money market yield curve. The case of the yield curve is substantially more complex than the case of the exchange rate since it has a structure in contrast with a single interest rate which is reduced to a number. Furthermore, it is impossible to ignore the short-term interest rate in the case of the yield curve as it is a constitutive element of the curve under study. Finally, there is not one traded asset that could be representative of the yield curve.

The contribution of our analysis to the literature is to suggest a simple way to circumvent the above-mentioned technical difficulties. In particular, it uses as the representative asset a perpetuity paying a constant rate of dividend, called the consol rent, priced from the curve. It defines its excess return, constructs an instantaneous volatility from market-implied ones, constructs the corresponding normalised excess return and compares the statistical features of raw and normalized excess returns. Our results show that the consol rate can be described as an Ito process with variable volatility, hence avoiding the inclusion of jumps.

The paper is organised as follows. After this introduction, Section 2 summarizes the purpose and the main steps of our approach before Section 3 presents the useful definitions of key parameters, namely the consol excess return, the consol volatility and the consol normalized excess returns. Section 4 then compares the statistical features of consol excess return and consol normalized excess return when the process of the curve is simulated according to two arbitrage-free theoretical models. Section 5 compares the statistical features of consol excess return and consol normalized excess return for the EONIA swap curve. Finally, Section 6 concludes while the reconstruction of the consol normalized excess returns and volatility for that empirical EONIA swap curve is described in the annex.

## 2. Presentation of the Exercise

The proposed exercise requires two preliminary steps: first, an accurate definition of the excess return relevant for the consol rate; second, a construction of the consol rate, excess return and volatility. Then, a comparison between the excess return and the normalized excess return of excess of kurtosis and of ACF of squared and absolute values for an empirical yield curve will allow to evaluate the quality of the measure. The empirical yield curve chosen in the paper is the one implied by the EONIA swaps, which is considered as the riskless curve for the euro currency.

The central idea is to claim that if some computed volatility does lead to normalized excess returns free of excess of kurtosis and of autocorrelation in squared and absolute values, then it can be regarded as the volatility of the consol rent[1]. In order to support the previous conclusion, it is therefore useful to see what happens in cases where the true volatility is effectively known. It is effectively known in the case of some arbitrage-free yield curve models. So a way to judge the validity of the proposed criterion is to apply it in the case of simulated trajectories of the yield curve, when the simulation follows an arbitrage-free model for which the volatility can be effectively computed, and for which the length of the sample is the same as the length of the available sample of empirical data.

## 3. Consol Rate, Consol Excess Return, Consol Volatility

Let us first recall some key mathematical notions related to the concepts of yield curve and of consol rate, before discussing the basic properties of a consol volatility. From those basic properties, it follows that the consol volatility reduces the consol excess returns to a Gaussian white noise, within the mathematical frameworkof a continuous-time arbitrage-free model. This hints that, in the real world, a correct measure of volatility should also be able to reduce the excess returns to a Gaussian white noise. This can be tested by examining whether dividing the excess returns by the volatility actually reduces the leptokurticity and the volatility clustering. This test will be illustrated with simulated data, for which the true volatility can be known ex ante, before being presented for the actual euro data. In this case, a success of the test indictes that the true volatility can be, and has effectively been, recovered.

### 3.1. Notational Convention

The purpose of this paragraph is to recall and define the key notions of this paper within the continuous-time framework, as well as their basic mathematical properties. For convenience, the following convention is adopted: Latin letters denote dates while Greek letters refer to delay/duration between two dates[2].

#### 3.1.1. Zero-Coupon and Forward Rates

Yield curves are formally defined as functions of a continuous time parameter, which associate an interest rate to a maturity within a (theoretically unbounded) maturity set. Yield curves are also usually expressed in mainly two ways: a) as *zero-coupon* interest rate curves; or b) as *instantaneous forward* interest rate curves. These two ways are equivalent and convey the same quantity of information. With $z(\tau)$ the zero-coupon interest rate at maturity $\tau$, $f(\tau)$ the forward interest rate at maturity $\tau$ (whereby both interest rates are continuously compounded) and $P(\tau)$ the spot price of the zero coupon of maturity $\tau$, *i.e.* the present value of one currency unit to be paid over $\tau$, one can write that:

$$P(\tau) = \mathrm{e}^{-\tau z(\tau)} \tag{1}$$

and

$$\frac{P'(\tau)}{P(\tau)} = -f(\tau), \tag{2}$$

which implies the following mathematical relationship between zero-coupon interest rate and the forward interest rate:

$$f(\tau) = z(\tau) + \tau \frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} \tag{3}$$

or, conversely, as the change of variables is duly invertible:

$$z(\tau) = \frac{1}{\tau}\int_0^\tau f(\theta)\,\mathrm{d}\theta \tag{4}$$

From Equations (3) and (4), it follows two basic but important properties follow:

[1]See also the discussion on the relevance and the scope of the various volatility measures in [1].

[2]For example, a bond observed at time t and having maturity date T shall have a time to maturity $\tau$ satisfying to the relation: $\tau = T - t$ .

a) $z(.)$ is constant if and only if $f(.)$ is constant. In this particular case, it means that the constant value taken by $f(.)$ is the same as the constant value taken by $z(.)$, which would imply a *flat* yield curve while the constant value taken by both $z(.)$ and $f(.)$ is called the *level* of the flat yield curve;

b) irrespective of the shape of the yield curve, $z(0)$ and $f(0)$ are always equal and the corresponding unique value defines the short-term interest rate $r$, *i.e.* $r := z(0) = f(0)$.

Note also that one defines as parallel shift in the remainder of the analysis a transformation of the yield curve such as a constant is added to the zero-coupon interest rates, or, equivalently, of the forward rates following from Equation (3).

### 3.1.2. Consol Bond and Consol Rate

A consol bond is defined as a perpetual (infinite horizon) bond paying continuously a constant rate of money, which is called the coupon flow. By definition, the consol price $C$ is defined as the price of the consol bond divided by the coupon flow[3], which can be expressed, in terms of the yield curve, as follows:

$$C = \int_0^{\infty} P(\theta)\mathrm{d}\theta \tag{5}$$

By substituting $P(\theta)$ by its value using Equation (1), the consol price becomes:

$$C = \int_0^{\infty} \mathrm{e}^{-\theta z(\theta)}\mathrm{d}\theta \tag{6}$$

The consol rate, $y$, defined as the yield of the consol bond, is the inverse value of the consol price, *i.e.*[4]:

$$y = \frac{1}{\int_0^{\infty} \mathrm{e}^{-\theta z(\theta)}\mathrm{d}\theta} \tag{7}$$

The consol duration, $D$, that is the duration (in the sense of [2]) of the consol bond reflecting the sensitivity of the logarithm of the consol price to a parallel shift of the consol yield curve, can be expressed as:

$$D = \frac{\int_0^{\infty} \theta \mathrm{e}^{-\theta z(\theta)}\mathrm{d}\theta}{\int_0^{\infty} \mathrm{e}^{-\theta z(\theta)}\mathrm{d}\theta} \tag{8}$$

where the sensitivity of the consol rate to a parallel shift of the yield curve, $\xi$, is given by the following product:

$$\xi = yD \tag{9}$$

This dimensionless number $\xi$ is equal to 1 in case of a flat curve. Empirical evidence shows that yield curves have usually $\xi$ smaller than, but close to, 1. We say that that a financial price is in constant terms, by opposition to current terms, when it is expressed in currency values of a fixed reference date in the past rather than in currency values of the current date.

Finally, the consol wealth process, denoted $A$ hereafter, is defined as the wealth, expressed in constant terms, of an ideal investor facing no transaction costs or short-selling restrictions who holds a portfolio of consol bonds in which he reinvests automatically the whole coupon flow at the prevailing market price. Therefore, the corresponding (infinitesimal) consol excess return, denoted $(\mathrm{d}A_t/A_t)$, is written:

$$\frac{\mathrm{d}A_t}{A_t} = \frac{\mathrm{d}C_t}{C_t} + \left(\frac{1}{C_t} - r_t\right)\mathrm{d}t = y_t \mathrm{d}\frac{1}{y_t} + (y_t - r_t)\mathrm{d}t \tag{10}$$

whereby the first two terms refer to the nominal gain (or loss) due respectively to the change in the market price of the consol bond $(\mathrm{d}C_t/C_t)$ and to the coupon flow $(\mathrm{d}t/C_t)$ while the third term $(r\mathrm{d}t)$ refers to the carry cost of the position, *i.e.* holding the portfolio of consol bonds.

### 3.1.3. Volatility of a Consol Bond

The quadratic variation of a process $X_t$ is denoted $[X_t]$, so that Ito's formula is:

[3]Such a normalisation is needed given the perpetual nature of this bond.
[4]By construction, the consol rate corresponding to a flat consol yield curve is equal to the level of that yield curve.

$$\mathrm{d}f_t\left(X_t\right)=f'\left(X_t\right)\mathrm{d}X_t+\frac{1}{2}f''\left(X_t\right)\mathrm{d}\left[X_t\right] \tag{11}$$

The volatility of the consol bond, $\sigma_t$, is defined as:

$$\sigma_t^2\mathrm{d}t=\frac{\mathrm{d}\left[C\right]_t}{C_t^2}=\frac{\mathrm{d}\left[y\right]_t}{y_t^2} \tag{12}$$

The differential $\mathrm{d}L_t$ (with $L_t$ denoting the logarithm of the consol wealth process, hereafter referenced as to consol performance) can be obtained by applying the Ito's iteration to Equation (10), *i.e.*:

$$\mathrm{d}L_t=y_t\mathrm{d}\frac{1}{y_t}+\left(y_t-r_t-\frac{\sigma_t^2}{2}\right)\mathrm{d}t \tag{13}$$

which leads to the following identity using the Ito's formula:

$$y_t\mathrm{d}\frac{1}{y_t}=-\frac{\mathrm{d}y_t}{y_t}+\sigma_t^2\mathrm{d}t \tag{14}$$

Substituting Equation (14) into Equation (13) leads to express the consol excess return as:

$$\frac{\mathrm{d}A_t}{A_t}=-\frac{\mathrm{d}y_t}{y_t}+\left(y_t-r_t+\sigma_t^2\right)\mathrm{d}t \tag{15}$$

which, by applying Ito's iteration to Equation (15), yields:

$$\mathrm{d}L_t=-\frac{\mathrm{d}y_t}{y_t}+\left(y_t-r_t+\frac{\sigma_t^2}{2}\right)\mathrm{d}t \tag{16}$$

Finally, the normalized excess return is defined as:

$$\mathrm{d}N_t=\frac{\mathrm{d}A_t}{\sqrt{\frac{\mathrm{d}\left[A_t\right]}{\mathrm{d}t}}} \tag{17}$$

which, by combining Equations (15), (16) and (17), yields:

$$\mathrm{d}N_t=\frac{\mathrm{d}L_t}{\sigma_t}+\frac{\sigma_t}{2}\mathrm{d}t \tag{18}$$

We will then apply the mathematical specification of the consol rate, and the calculation of the corresponding volatility as discussed from Equations (11) to (18), to standardise the volatility measure for interest rates in the money market.

#### 3.1.4. Risk-Neutral Probability

Within the framework of Heath-Jarrow-Morton [3], the dynamics of $y_t$, $L_t$ and $A_t$ are fully specified by the stochastic process of $\sigma_t$ under risk-neutral probability. $A_t$ must be a martingale, so Equations (10) and (12) imply:

$$\frac{\mathrm{d}A_t}{A_t}=\sigma_t\mathrm{d}W_t \tag{19}$$

with $W_t$ denoting a Wiener process, from which it immediately follows that:

$$\mathrm{d}L_t=\sigma_t\mathrm{d}W_t-\frac{\sigma_t^2}{2}\mathrm{d}t \tag{20}$$

By combining Equations (15) and (19) under the risk-neutral probability, the change of the yield of the consol, bond, $\mathrm{d}y_t$, becomes:

$$\mathrm{d}y_t=-\sigma_t y_t\mathrm{d}W_t+y_t\left(y_t-r_t+\sigma_t^2\right)\mathrm{d}t \tag{21}$$

By combining Equations (12), (15), (17) and (21), it follows:

$$\mathrm{d}N_t = \mathrm{d}W_t \tag{22}$$

where $N_t$ itself is also a Wiener process under risk-neutral probability. This implies that:

$$\mathrm{d}\left[N\right]_t = \mathrm{d}t \tag{23}$$

Equation (23) is valid not only under risk-neutral probability but also under any equivalent probability to the risk neutral probability, as it pertains to the quadratic variation only. Finally, Equations (18) and (22) yield that:

$$\frac{\mathrm{d}L_t}{\sigma_t} + \frac{\sigma_t}{2}\mathrm{d}t = \mathrm{d}W_t \tag{24}$$

Note the left-hand side of Equation (24) is defined as the normalized excess return.

It follows from Equation (24) that the the normalized excess returns—*i.e*. the excess returns divided by the true volatility—should be akin a Gaussian white noise under risk-neutral probability. As we know, empirical excess returns of any financial asset usually differ from a Gaussian white noise by two properties: a) their empirical distribution more kurtosis than the Gaussian distribution (the property of *leptokurticity*); and b) their absolute values (or their square values) have a positive serial correlation (the property of *volatility clustering*). To the extent that the Ito-process modelisation is a realistic representation of the consol rate dynamics, one should be able to remove those two properties, by dividing the excess returns by a correct measure of the underlying volatility, and that operation of normalization would recover the underlying Gaussian white noise process.

## 4. Data

The dataset contains only TARGET working days; it contains all the TARGET working days from 4 January 1999 to 21 November 2013, which represents 3816 TARGET working days. The financial instruments taken into account are handled in the OTC market. They consist into: short-term unsecured deposit of maturity 1-day (overnight, tom-next and spot-next), EONIA swaps from 1-week to 30-year, 6-month EURIBOR swaps for the corresponding maturities, at-the-money implied volatilities of options on the EURIBOR swaps.

We used the options on 6-month EURIBOR swaps with option maturity 1-month, 3-month, 6-month, 1-year, 2-year, 3-year, 4-year and 5-year, and with underlying EURIBOR swap maturity 1-year, 2-year, 3-year, 4-year, 5-year, 7-year, 10-year, 15-year, 20-year, 25-year and 30-year. Other maturities of options or of underlying swaps are represented in the quotes contributed by brokers, but their history may start at relatively recent dates, which makes preferable not to use them. Besides, the EONIA fixing is included in the dataset. Each instrument or fixing is identified in the Reuters database by a unique RIC (Reuters Instrument Code).

As the financial instruments taken into account are handled on the OTC market, we made use of quoted data, generally given as a bid-ask spread from which we retained only the mid. We gave a preference to quotes issued by the broker ICAP, and when not available, our primary fallback was the generic quote of Reuters, which contains the latest quote issued by a bank or broker at the time of its snapshot or of its contribution. In case of missing data, the data set is completed by a reconstruction of data as described in detail in Annex..

## 5. Testing the Robustness of the Benchmark Rule

To assess the correctness of a measure of consol volatility ("benchmark rule" hereafter), we will test whether the excess returns of the consol bond, when normalized by our volatility measure, resemble a Gaussian white noise process, *i.e*. that both leptokurticity and volatility clustering are essentially reduced. Removing the volatility clustering is not sufficient to assess the correctness of the measure. It is easy to see that for an indicator $X$ that oscillates rapidly enough, the absolute value, and the square, of the ratio $\mathrm{d}L/X$ have small autocorrelations: the rapid oscillations may remove the volatility clustering but only at the expense of an increase of the leptokurticity. This makes necessary to require the reduction of *both* elements (volatility clustering and leptokurticity) in order to assess the quality of the volatility indicator.

Two formal tests are presented in this section. First, we conduct simulation of affine factor models (which allow the knowledge of the true volatility ex ante) and apply the test on the result of those simulations: this aims at checking that our implementation of the test actually behaves as it is supposed to. Second, we apply the test to the actual history of the EONIA curve, and to our reconstruction of the consol volatility, and assess by this way the quality of this reconstruction of the consol volatility.

## 5.1. Test Based on Simulation

The consol performance $L$ is also well defined in the case of simulated data. In simulating the evolution of the yield curve under an arbitrage-free affine factor model, one can check whether a) $\mathrm{d}L_t$ exhibits (or not) leptokurticity and volatility clustering; and b) $\frac{\mathrm{d}L_t}{\sigma_t}+\frac{\sigma_t}{2}\mathrm{d}t$ does exhibit less (or no) leptokurticity and less less (or no) volatility clustering. This type of exercise is of particular interest as it would provide a benchmark for the results of our test and hence a natural comparison point for the results based on the empirical data set.

### 5.1.1. General Setting of the Simulation Exercise

Denote with $t_1$ and $t_2$ two consecutive TARGET working days. Excess returns are constructed by setting the cost of carry equal to the EONIA observed at the close of business of $t_1$. Normalized excess returns are constructed using the volatility computed at the close of business of $t_1$, (and not $t_2$).

To assess the existence of leptokurticity, we examine whether the excess of kurtosis differs from zero (contrary to a normal distribution where its value is zero). To test the existence of volatility clustering, we use the correlation of the absolute values of two consecutive returns, and the correlation of the square of two consecutive returns.

Both simulations are run on 3816 TARGET working days.

### 5.1.2. Simulation Models

For the sake of simplicity, we will use the special case of arbitrage-free models known as affine models with constant parameters, with continuous time setting, and continuous trajectories. We will perform our simulations in the case of the generic 1-factor[5] affine model and in the case of the generic 2-factor affine model. The 1-factor model is in effect the simplest choice, and the 2-factor model is its more natural generalisation.

In an arbitrage-free model, the zero-coupon bond price takes necessarily the form of the expectation:

$$P(t)=E_u\left[\mathrm{e}^{-\int_u^t r_s \mathrm{d}s}\right] \tag{25}$$

whereby the expectation refers to the so-called *risk-neutral probability*. Furthermore, the probability under which the short-term rate $r_s$ actually diffuses should be equivalent, in the probabilistic sense[6], to the risk-neutral one. We will refer to that second probability as to the *data-generating probability*.

We have performed several attempts with different choices of parameters. The results are always that the normalised excess returns behave close to a gaussian white noise, and that the raw excess returns behave less close to a gaussian white noise. Yet the contrast between the normalized excess return's and the raw excess return's behaviours may be more or less pronounced. Typically, we find a excess of kurtosis ranging between zero and six for the raw case, and close to zero for the normalized case. The simulations that we present here will roughly correspond to a median case.

**Simulation with 1-factor model**—The generic 1-factor affine model is coincident with the Duffie and Kan [4] 1-factor model (hereafter DK1). In the DK1 model, the risk-neutral probability is the solution[7] of the stochastic differential equation (SDE):

$$\mathrm{d}r_s=(a-br_s)\mathrm{d}s+\sqrt{c+r_s\nu^2}\,\mathrm{d}W_s \tag{26}$$

with $a$ and $b$ positive constants, $c$ a real constant, and $\nu$ a positive or zero constant, $c$ and $\nu$ not being both equal to zero at the same time. By construction, this model has thus four parameters ($a$, $b$, $c$ and $\nu$)

[5]Affine models represent the yield curve as determined by a finite number of state variables, called factors, in such a way that the curve is an affine function of those factors.

[6]By definition, two probabilities are said to be equivalent if and only if they are defined on the same $\sigma$-algebra and, furthermore, they attribute the weight zero or the weight one to the same measurable sets.

[7]The solution of a SDE consists of a measure on the set of future trajectories—endowed with some suitable $\sigma$-algebra—and is therefore the relevant concept when defining expectations, which are essentially integrations w.r.t. that measure.

and one unique factor which can be identified to the short-term interest rate, $r_s$. It evolves between $-c/v^2$ and $+\infty$.

The data-generating probability should be equivalent (in probabilistic terms) to the risk-neutral probability. Since we are here in a continuous-time setting, the equivalence condition implies that the Brownian part of the data-generating probability is also provided by the expression $\sqrt{c+r_s v^2}\,\mathrm{d}W_s$ with the same $c$ and $v$ whereas it has no implication whatsoever as regards the form of the drift terms of the data-generating probability. Nevertheless, again for the sake of simplicity, one focuses on the specific case for which the stochastic differential equation defining the data-generating probability takes a form similar to Equation (26), only allowing for different values for the drift parameters $a^*$ and $b^*$. The historical, denoted with stars, determine the computation of the diffusion of factors. The risk neutral, denoted without stars, determine the computation of the curve at each step.

Note that the DK1 model is the generic case of the 1-factor affine model with constant coefficients. It is also the simplest possible arbitrage-free model of the yield curve, which includes two specific cases: a) the original Vasicek model [5], obtained when $v$ is set to zero; b) the original Cox-Ingersoll-Ross model [6] (the CIR model hereafter), obtained when $c$ is set to zero[8].

The functional form resulting from the DK1 model happens to be realistic (see [7]). Yet, the DK1 model appears in practice relatively far from the real motion of actual yield curves.

To perform the simulation, we need to choose values for the parameters and for the initial value of the factor. There is no compelling reason to choose one set of parameters rather than another. We have adopted values producing yields which are realistic for the euro, but this, strictly speaking, is not a constraint for the current purpose. We perform the simulation on the basis of the values for the parameters and initial values of the factors listed in **Table 1**.

Based on this simulation model, the following **Table 2** reports the results for the (raw) consol excess returns and the normalized consol excess returns:

**Simulation with 2-factor model**—The generic 2-factor affine model is coincident with the model presented in [8] (hereafter GS2), but we will rephrase it with another choice of parameters and factors, in order to ensure formal consistency with the previous discussion.

In the GS2 model, excluding again the case where the short-term rate is bounded away from zero, the risk-neutral probability can be seen as the solution of SDE described in Equation (26):

**Table 1.** Parameters and initial values of the 1-factor model simulation.

| Parameters | Annualized values |
|---|---|
| $a^*$ | 0.028 |
| $b^*$ | 0.5 |
| $a$ | 0.022 |
| $b$ | 0.35 |
| $c$ | 0.0002 |
| $v$ | 0.25 |
| $r_s$ initial | 0:03 |

**Table 2.** Results of the 1-factor model simulation.

| | Raw | Normalized |
|---|---|---|
| Std. deviation | 0.009 | 0.982 |
| Excess of kurtosis | 3.28 | 0.04 |
| ACF lag 1 of abs. values | 35% | 1% |
| ACF lag 1 of squares | 23% | 2% |

[8]When $v$ is not equal to zero. The DK1 model can be rewritten as a parallel shift of the CIR model.

$$\begin{pmatrix} \mathrm{d}r_s \\ \mathrm{d}p_s \end{pmatrix} = \left( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \begin{pmatrix} r_s \\ p_s \end{pmatrix} \right) \mathrm{d}s + \sqrt{\begin{pmatrix} c + r_s \nu^2 & \frac{p_s \nu^2}{2} \\ \frac{p_s \nu^2}{2} & \frac{\nu^2}{4} \end{pmatrix}} \mathrm{d}\mathbf{W}_s \tag{27}$$

with $a_1$, $a_2$, $b_{11}$, $b_{12}$, $b_{22}$, $c$, $\nu$, $p_s$ and $r_s$ satisfying certain constraints, namely:

$$\begin{aligned} & \nu > 0 \\ & b_{11} > 0 \\ & c + \left(r - p^2\right)\nu^2 > 0 \end{aligned} \tag{28}$$

and either:

$$\begin{aligned} & 2b_{22} - b_{11} > 0 \\ & a_1 \ge a_{1\min}^{(0)} \end{aligned} \tag{29}$$

where:

$$a_{1\min}^{(0)} = b_{11} r_{\min} + \frac{\nu^2}{4} + \frac{\left(2a_2 + b_{12}\right)^2}{4\left(2b_{22} - b_{11}\right)} \tag{30}$$

or:

$$\begin{aligned} & 2b_{22} - b_{11} = 0 \\ & b_{12} = -a_2 \frac{b_{11}}{b_{22}} \\ & a \ge a_{\min}^{(1)} \end{aligned} \tag{31}$$

where:

$$a_{\min}^{(1)} = b_{11} r_{\min} + \frac{\nu^2}{4} \tag{32}$$

The square root sign over the matrix has to be interpreted as the matrix square root operator, as opposed to an operator acting component by component. The model has seven parameters, $a_1$, $a_2$, $b_{11}$, $b_{12}$, $b_{22}$, $c$ and $\nu$, and two factors of which the first one is identified to the short-term rate $r_s$. As was the case for the 1-factor affine model, $r_s$ evolves between $-c/\nu^2$ and $+\infty$. The second factor, denoted with $p$, has no particular economic interpretation. It evolves between $-\infty$ and $+\infty$, and its physical dimension is the same one as for a volatility, or equivalently, as the square root of a rate.

From the Equation (27), it appears that if $b_{12}$ is set to zero, the GS2 model is reduced to the DK1 model. The factor $r_s$ in this case is not influenced by the dynamics of the other factor $p$ and follows simply the solution of Equation (26).

Again, while the mathematic structure of the model only obliges us to have the same Brownian part for the risk-neutral and data-generating probabilities, we focus nevertheless on data-generating probabilities sharing the same algebraic form with the risk-neutral one. The data-generating probability is then given by an equation similar to (27), in which parameters $a_1$, $a_2$, $b_{11}$, $b_{12}$, $b_{22}$ are replaced by counterparts denoted with an asterisk, following similar constraints than the parameters without asterisk.

We perform the simulation on the basis of the following values for the parameters and initial values listed in **Table 3**.

The results for the consol excess returns on the left side and for the normalized excess return on the right side of Equation (24) become those listed in **Table 4**.

We obtain again the expected results, regarding the fact that leptokurticity and volatility clustering are present in the excess returns and removed from the normalized excess returns. Leptokurticity and volatility clustering reach values comparable to those of the 1-factor model. Yet, as we will see, they still cannot be compared with what is observed on empirical excess returns.

### 5.2. Test Based on Empirical Data

Similarly to the general specifications recalled in the previous sections, the consol rate and the corresponding volatility for the EONIA is calculated over the period from 4 January 1999 to 21 November 2013, *i.e.* 3816 TARGET working days. **Table 5** reports the results based on actual consol volatility calculatd from the empirical data. A graphical representation of the volatility measure based on the consol rate specification together with the level of the consol interest rate is reported in **Figure 1**.

As shown by **Table 5**, the excess of kurtosis and the volatility clustering exhibited by the normalised consol excess returns is substantially lower than those exhibited by (raw) excess returns. It is also interesting to underline that the excess of kurtosis and the volatility of the normalized consol excess returns based on empirical data appears even lower that the value they take for the raw excess returns (*i.e.* before nomalisation) in the case of the simulations.

## 6. Concluding Remarks

This paper proposes a new measure of volatility derived from the specification of the consol rate for the EONIA swap curve in order to have an accurate estimation of volatility free from any model-based specifications and relaxed from maturity and frequency constraints. We demonstrate that this volatility measure is very close to the true (unobserved implicit) instantaneous volatility as it allows the excess returns of the consol rate to display a Gaussian white noise process (under risk-neutral probability or any similar probability) once normalised by this measure. This finding is quite powerful for several reasons.

First, it legitimates the use of yield curve dynamics being free of jumps. It is thus more parsimonious. The discrepancy between the statistical features of the excess returns and those of a Gaussian white noise can be brought back to the mere variability of the volatility and do not require the intervention of jumps, at least for what regards the particular EONIA swap curve.

**Table 3.** Parameters and initial values of the 2-factor model simulation.

| | | | |
|---|---|---|---|
| $a_1^*$ | 0.028 | $b_{11}$ | 0.35 |
| $a_2^*$ | 0.03 | $b_{12}$ | 0.045 |
| $b_{11}^*$ | 0.5 | $b_{22}$ | 0.5 |
| $b_{12}^*$ | −0.06 | $c$ | 0.0002 |
| $b_{22}^*$ | 0.37 | $v$ | 0.25 |
| $a_1$ | 0.022 | *rinit.* | 0.03 |
| $a_2$ | 0.027 | *pinit.* | −0.15 |

**Table 4.** Results of the 2-factor model simulation.

| Variables | Raw data | Normalized data |
|---|---|---|
| Standard deviation | 0.009 | 0.995 |
| Excess of kurtosis | 3.59 | 0.12 |
| ACF lag 1 of abs. v. | 34% | 0% |
| ACF lag 1 of squares | 20% | 1% |

**Table 5.** Results from the empirical data.

| | Raw data | Normalized data |
|---|---|---|
| Standard deviation | 0.013 | 0.964 |
| Excess of kurtosis | 14.95 | 1.15 |
| ACF lag 1 of abs. variation | 34% | 6% |
| ACF lag 1 of squares | 41% | 6% |

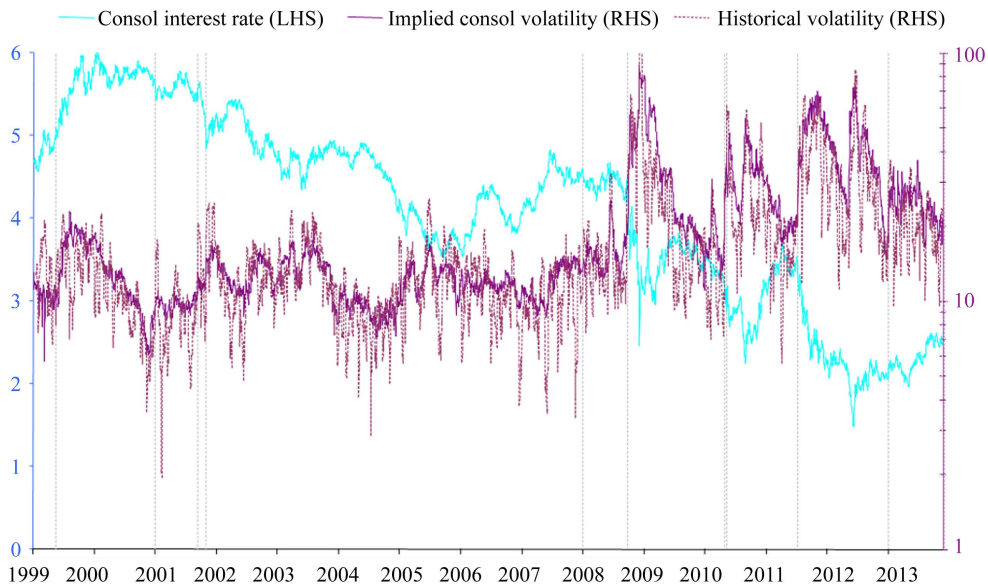


**Figure 1.** Consol rate and consol volatilities.

Second, our findings allow an homogeneisation of volatility measure (with a forward-looking feature), providing information for the entire market without being restricted to one particular maturity or to the frequency of coupons/cash flows idiosyncratic to a particular benchmark instrument.

The restrictive nature of standard volatility measures (due to the strong link to a certain maturity and/or frequency) usually limit the use of volatility measure in times series regressions. The consol volatility provides new research avenues as regards volatility transmission and/or assessment of market stress.

## Acknowledgements

The authors would like to thank colleagues of the ECB and the participants at the 7th Methods in International Finance Network (MIFN) Conference on 23-24 September 2013 for fruitful discussion and suggestions. A special thank to the anonymous referees of the ECB Working Paper Series and of the journal which allow to improve the paper with their helpful remarks.

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# Annex

# Determination of the Consol Volatility

## 1. Data

### 1.1. Working Days and Instruments

The dataset contains all the TARGET working days from 4 January 1999 to 21 November 2013, so 3816 TARGET working days. The financial instruments belong to the OTC marke: short-term unsecured deposit of maturity 1-day, EONIA swaps from 1-week to 30-year, 6-month EURIBOR swaps for the corresponding maturities, at-the-money implied volatilities of EURIBOR swaptions with maturity 1-month to 5-year, and with underlying EURIBOR swap maturity 1-year to 30-year. Besides, the EONIA fixing is included in the dataset.

### 1.2. Raw Data

As the financial instruments taken into account are handled on the OTC market, we made use of quoted data, from which we retained only the mid. We gave a preference to quotes issued by the broker ICAP, and when not available, our primary fall-back was the generic quote of Reuters.

### 1.3. Completion

The data have been completed by reconstructed numbers in three cases.
- When the history of long-term EONIA swaps was missing.
- When the history of the options was missing.
- When the history of an instrument was missing due to a London closing day, a case which occurred only in the recent years.

### 1.4. Descriptive Statistics of the Sample

The resulting data sample is described by the following descriptive statistics:

| | |
|---|---|
| Number of TARGET working days | 3816 |
| Start | 4 January 1999 |
| End | 21 November 2013 |
| Number of RICs | 187 |
| Number of recomputed rates | 72 |
| Size | 570,542 |

## 2. Algorithms

### 2.1. Yield Curves

#### 2.1.1. Composition

The EONIA curve is made of short-term unsecured deposits of maturity 1-day (overnight, tom-next and spot-next) and of EONIA swaps from 1-week to 30-year.

The EURIBOR swap curve is made of short-term unsecured deposits of maturity 1-day (overnight, tom-next and spot-next) and longer (between 1-week and 6-month) and of swaps versus 6-month EURIBOR from 1-year to 30-year.

#### 2.1.2. Bootstrapping

We turn now to the construction of the yield curve from a collection of interest rate instruments. The construction of the yield curve consists into the successive calculation of zero-coupon prices or "discount factors", is termed "bootstrapping". For a detailed description of the bootstrapping, we refer to [7][9], [9]. The algorithm

[9]See paragraph 3.2.2., pp. 19-20.

should be such that:

- it re-prices all the instruments contained in the above described yield curves to their exact original observed price;
- it can be entirely described through a finite (albeit not a priori specified) number of rates and dates.

The instruments are sorted by ascending maturity. One constructs the curve by recurrence up to each maturity. Each of those maturities is termed a “knot point”. Rates at intermediate points on the curve can be estimated by assuming a shape for the curve either in zero-coupon price or rate space. The choice of that interpolation rule constitutes the signature of the bootstrapping method. This choice is not conditioned by any theoretical reason, but by several practical reasons. The chosen rule should be such that:

1) The curve is smooth.

2)The curve does not have strong oscillations.

To those requirements, we add the supplementary one that:

3) The integral of the zero-coupon price between two knots can be computed in closed analytical form. The same holds for the integral of the zero-coupon price multiplied by the time to maturity.

The two first conditions are antagonist: an interpolation rule that favours one requirement will generally disfavour the other one.

The third requirement reflects the necessity of performing consol-related calculations, to be described in paragraph 2.3.2..

We have tested four different rules, satisfying to those three criteria, among which the “unsmoothed Fama-Bliss” bootstrapping, whose interpolation rule results into stepwise constant forward rates. All of the methods performed close in term of the realistic aspect of the constructed curve. On the basis of some minute differences, we have chosen as default bootstrapping method one of the three other methods, namely, the one used in [9].

#### 2.1.3. Extrapolation

It order to handle swaptions of maturity 5-year on 30-year swaps, one needs a yield curve covering a range of 35 years. Yet, quotes for the EONIA swaps stop at the 30-year tenor, at least for the price source that we have chosen to privilege. We have therefore extrapolated the EONIA curve.We have chosen to also extrapolate the EURIBOR swap curve to 35-year, to ensure the similarity of their treatment. The extrapolation of a curve has been achieved by adding the zero-coupon rate $z(35)$ defined as $2 \times z(30) - z(25)$.

### 2.2. Implied Volatilities

#### 2.2.1. Converted Volatilities

The primary input for the calculation of the consol volatility is a set of implied volatilities quoted in the market. The implied volatilities that we have used as raw data are those of EURIBOR swaptions, which are standardized options on 6 month EURIBOR swaps. They cannot be directly used, and this, for two reasons.

Firstly, the swaptions volatilities are implied by a Black and Scholes model in which the logarithm of the swap rate is assumed to be a Brownian motion. By contrast, while the consol volatility, resulting from the equations presented in the text, has to be implied by a Black and Scholes model. A conversion will thus be necessary, changing the raw swaptions volatilities into other ones implied by the second model.

Secondly, the swaptions volatilities pertain to EURIBOR-linked instruments, whereby the consol volatility pertains to the EONIA curve.

The change of model cannot be done by an exact calculation (or that exact calculation would be too complicated). However, as we already mentioned, we know that the motion of the empirical yield curves are primarily composed of parallel shifts. We then make an approximation and assume that those movements consist purely of parallel shifts. Thus, we only need to do the calculation at the first order, *i.e.* to multiply the raw swaptions volatility by the sensitivity of the log zero coupon price w.r.t. the log swap rate.

The two conversions are simultaneously achieved as follows:

The underlying of the option—which is a forward EURIBOR swap—is priced from the EURIBOR curve. Then, one computes the quantity of parallel shift to apply to the EONIA curve to let it price the forward swap at its present market price. That quantity is called the “z-spread”, we denote it with h.

The sensitivity of the log zero coupon price (of the EONIA curve) w.r.t. the log swap rate (non compounded, with day-count actual/360) is then given by:

$$\frac{(\tau_2-\tau_1)S(\tau_1,\tau_2,x)}{\left.\frac{\partial S(\tau_1,\tau_2,x)}{\partial x}\right|_{x=h}} \tag{1}$$

where:

- $\tau_1$ refers to the time-to-maturity of the option.
- $\tau_2$ is the time-to-maturity of the swap (for instance 35-year for a 30-year swap forward in 5-year).
- $S(\tau_1,\tau_2)$ is the swap rate (non compounded, with day-count actual/360), $h$ is the swap's z-spread to EONIA curve (continuously compounded, with day-count actual/365), and
- $S(\tau_1,\tau_2,x)$ is the swap rate (non compounded, with day-count actual/360) of a forward swap having z-spread $x$ w.r.t. the EONIA curve (so that $S(\tau_1,\tau_2,0)=S(\tau_1,\tau_2)$).

The product of the quoted volatility by this sensitivity yields a first order approximation of the "converted volatility". To be on the safe side, we actually used a higher order approximation (up to the 5-th power of the quoted volatility). However, that higher precision does not bring any visible difference.

We denote the "converted volatility" with $\sigma(\tau_1,\tau_2)$. For an instantaneous volatility $\sigma(0,\tau)$, we will use the short-hand $\sigma(\tau)$.

### 2.2.2. Instantaneous Volatilities

We are then left with a collection of implied volatilities for options tenors ranging between 1-month and 5-year.

We need to obtain instantaneous volatilities, *i.e*. the limit of the at the money volatility tends to zero. Instantaneity as meaning, in concrete terms, the interval between two subsequent TARGET working days. This implies that we reconstruct at the money volatilities of tenor one TARGET working day.

This is achieved by creating the cubic spline of the converted volatilities of tenors ranging between 1-month and 5-year and by extrapolating the resulting splined function to the tenor 1-day. The quantity to be splined *is not directly the converted volatility, but the squared converted volatility multiplied by the time to maturity*.

## 2.3. Consol-Related Calculations

### 2.3.1. Formulas

The consol rate, consol duration, consol ksi and consol volatility rely on the computation of three integrals:

$$I_1=\int_0^\infty P(\tau)\,\mathrm{d}\tau \tag{2}$$

$$I_2=\int_0^\infty \tau P(\tau)\,\mathrm{d}\tau \tag{3}$$

$$I_3=\int_0^\infty \sigma(\tau)P(\tau)\,\mathrm{d}\tau \tag{4}$$

where $P(\tau)$ is the zero coupon price for time-to-maturity $\tau$, and $\sigma(\tau)$ is the zero coupon price instantaneous volatility for time-to-maturity $\tau$. The notation $\sigma$, without argument, designates the consol volatility.

The computation of the consol rate, denoted with $y$, the consol duration, denoted with $D$, and the consol ksi, denoted with $\xi$, boils down to the formulas:

$$y=\frac{1}{I_1} \tag{5}$$

$$D=\frac{I_2}{I_1} \tag{6}$$

$$\xi=\frac{I_2}{I_1^2} \tag{7}$$

Consistently with what we did for the conversion of volatilities, we will assume that the yield curves movements consist purely of parallel shifts. With the help of that approximation, that we use for the second time, the $\sigma(\tau)$ under the integral sign in (4) can be interpreted as scalars instead of vectors. The consol volatility is given by the formula:

$$\sigma = \frac{I_3}{I_1} \tag{8}$$

The $\sigma(\tau)$, even when we made them scalar numbers, have to be determined on the basis of the instantaneous volatilities resulting from the procedure described in 6. Yet those ones exist for eleven tenors of the underlying swaps, ranging from 1-year to 30-year. We take recourse again to a spline procedure, whereby we add to the list of available tenors the artificial tenor zero-day. For now, the quantity to be splined is *directly the instantaneous volatility*, and not the squared one multiplied by the time to maturity.

To compute the consol volatility of the theoretical yield curves of the affine models, used in the simulations, we start from the identity:

$$\sigma(\tau) = V \cdot \nabla \log\left(P(\tau)\right) \tag{9}$$

where $V$ designates the volatility of the vector of factors: $V$ is either $\sqrt{c + rv^2}$ for the 1-factor case or $\sqrt{\begin{pmatrix} c + rv^2 & \frac{pv^2}{2} \\ \frac{pv^2}{2} & \frac{v^2}{4} \end{pmatrix}}$ for the 2-factor case. The operator $\nabla$ represents the gradient w.r.t. the vector of factors.

For the 1-factor model, the $\sigma(\tau)$ is anyway a scalar, and for the 2-factor model, the $\boldsymbol{\sigma}(\tau)$ is a vector of dimension 2, but both components can be explicitly computed.

#### 2.3.2. Numerical Implementation

For what regards the consol rate, duration and ksi, in the case of the curves bootstrapped from empirical data, we have analytically computed the integrals between subsequent knot points of the bootstrapping. To complement the integral between 35-year and infinity, we have assumed that the zero-coupon rate remained constant.

For what regards the consol rate, duration and ksi, in the case of the affine model curves, and for what regards the consol volatility, in both cases of empirical curve and model curve, we have discretized the integrals with a step of 30 calendar days and we have computed 1000 steps, reaching a maturity of circa 80-year. To complement the integral between that latest maturity and infinity, we have assumed that the zero-coupon rate remained constant, and that the instantaneous volatility remained strictly proportional to the time-to-maturity.

#### 2.3.3. Consol Performance and Numerical Performance

Let $t_1$ and $t_2$ to consecutive TARGET working days. The excess return between $t_1$ and $t_2$ then writes:

$$\log\left(\frac{\frac{100}{CONSOL(t_2)} + \frac{t_2 - t_1}{365}}{1 + \frac{(t_2 - t_1)EONIA(t_1)}{36000}} \frac{CONSOL(t_1)}{100}\right) \tag{10}$$

where $CONSOL(t)$, is the consol rate observed at the close of business of $t$, continuously compounded, with a day count actual/365, and expressed in percentage points. The normalized excess return is the quotient of the excess return between $t_1$ and $t_2$ and of the consol volatility observed at the close of business of $t_1$.

Journal of Mathematical Finance, 2015, 5, 73-81
Published Online February 2015 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2015.51007

# A Real Options Approach to Distressed Property Borrower-Lender Reconciliation

**David J. Moore***, **Nuriddin Ikromov**

College of Business Administration, California State University, Sacramento, CA, USA
Email: *djmphd@csus.edu

Received 5 February 2015; accepted 23 February 2015; published 25 February 2015

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## Abstract

**We propose a real option framework to value distressed properties and restructure their loans. Our approach reconciles the interests of borrowers and lenders through a constrained optimization model yielding mutually beneficial restructure terms. Borrowers receive lower loan balances and payments, while lenders replace non-performing loans with performing loans that have higher market values. A numerical illustration shows that the market value of a restructured loan can exceed that of the original non-performing loan and the post-foreclosure cash flows when the lender repossesses the property.**

## Keywords

**Real Option, Real Estate, Restructure, Distressed Property, Market Value**

## 1. Introduction

Many studies have applied option theory to real estate investment, abandonment, and timing decisions over the past three decades. [1] is one of the first studies to applying real options theory to real estate development and to offer a theoretical model of valuation of vacant lots. [2] offers an analytical and numerical solution for the option to develop or abandon real estate. [3]-[5] derive models of pricing lease contracts in the option pricing context. [6] applies real option theory to value adjustable-rate mortgages in the presence of default, and [7] surveys the theoretical studies on the option pricing methodology of mortgage valuation. [8] applies option theory to explain the cyclical nature of real estate markets. [9] finds that individual investors are unable to apply real options theory to investment decisions in a laboratory experiment. However, prior real estate applications of option theory are from the perspective of developers or financial institutions. To our knowledge, this study is the first to apply option theory to real estate from a household perspective. This study extracts home values from the option

*Corresponding author.

**How to cite this paper:** Moore, D.J. and Ikromov, N. (2015) A Real Options Approach to Distressed Property Borrower-Lender Reconciliation. *Journal of Mathematical Finance*, **5**, 73-81. http://dx.doi.org/10.4236/jmf.2015.51007

pricing framework. We then use the estimated market value in a constrained maximization problem to produce mutually beneficial loan restructure terms.

**Figure 1** illustrates that the market-to-book ratio for non-performing loans is consistently lower than that of performing loans from 1990 to 2013. Overall, the average market-to-book value of non-performing loans (35%) is less than half of performing loans (83%). We investigate whether or not a non-performing loan can be restructured into a performing loan, if the restructured loan (likely with a lower balance) has a higher market value than the original non-performing loan, and if the new loan terms are in the best interest of the borrower and lender.

The precipitous drop in home loan market values coupled with increased reserve requirements has necessitated loan restructuring. However lenders are often unwilling to restructure loans. We suggest one reason lenders are apprehensive is that they are unaware of the related issues of current home valuation and optimum restructuring terms. [10] refers to this scenario as "a lender's dilemma." Our study proposes a constrained optimization model that provides optimal loan restructuring terms and distressed property valuation. We obtain these results by observing that borrowers have a real option and incorporating the value of that option in the lender's maximization problem.

To illustrate the potential benefits to borrowers and lenders, consider the average market-to-book values for performing and non-performing loans, 83% and 35%, respectively. The restructured loan market value will exceed the non-performing loan market value as long as the restructured principal balance is greater than $0.35/0.83 \approx 42\%$ of the original principal balance. For example, assume anon-performing loan balance of \$100,000. Further assume that a reduction of principal balance to \$70,000 produces a performing loan. The market values before and after restructuring are:

$$\mathrm{MV}_{\text{before}} = \mathrm{MV}_{\text{nonperf}} = 0.35(100,000) = 35,000;$$
$$\mathrm{MV}_{\text{after}} = \mathrm{MV}_{\text{restructured}} = 0.83(70,000) = 58,100.$$

The lender can increase the market value of the non-performing loans 66% by restructuring into performing loans. Of course there are many other choice variables for the lender such as the loan rate and term. The trio of principal reduction percentage, loan rate, and loan term are used as the lender's choice variables in our model that follows.

## 2. Model

For the purposes of this study, a "distressed" property is a property where a homeowner has ceased repayment due an "underwater" condition: the current loan balance exceeds the current market value of the home. Our

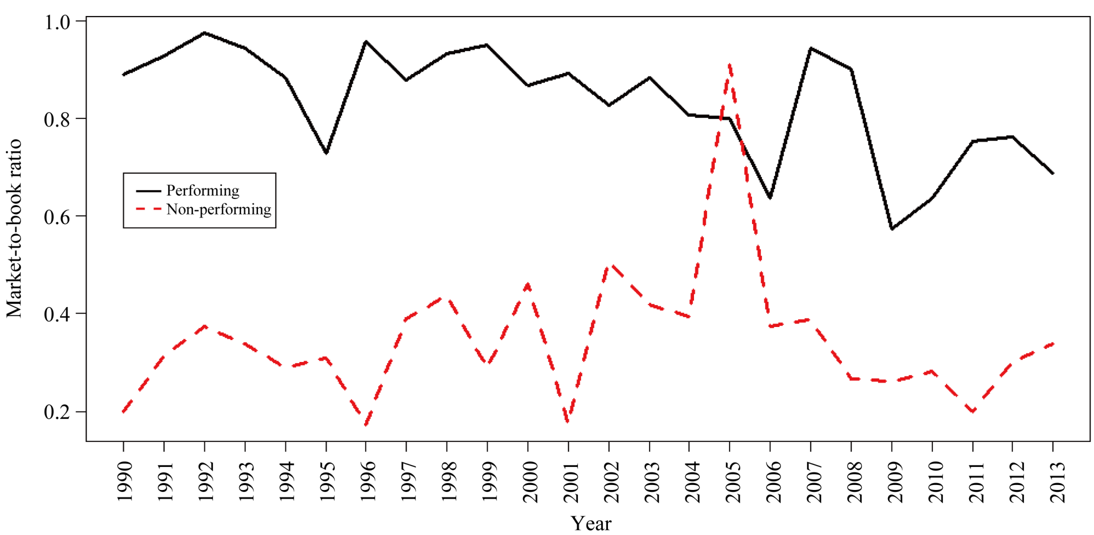


**Figure 1.** Average market-to-book ratio for performing and nonperforming loans from 1990 to 2013. Source: FDIC historical loan sales data (http://www.fdic.gov/buying/historical/index.html).

model first implementsthe underwater condition by setting the real option value to the borrower to zero, and then solving for the current market value of the home $S_0$ ("stock price" in standard options nomenclature). Given the current distressed property market value $S_0$ and existing loan amount $K_0$, the lender chooses the principal reduction amount *a*, new interest rate *r*, and new loan term *T* that maximizes the value of the now-performing loan $b_p a K_0$ subject to several constraints. The constraints are defined to ensure the real option to the home-owner is positive (the performing condition), the market value of the restructured performing loan exceeds that of the existing non-performing loan, and the loan terms are within reasonable market conditions. We begin by describing the assumptions and constraints in detail then follow with the model specification.

## 2.1. Assumptions

We make several simplifying assumptions for model tractability. These assumptions are intentionally restrictive in that we do not intend for this model to apply to all non-performing loans. However, loans that meet the criteria identified in the following assumptions do represent a subset of all non-performing loans.

Assumption 1: The non-performing loan represents a negative equity scenario where the real option value on the home is equal to zero. Admittedly, it is possible that a loan is non-performing for reasons other than negative equity. The borrower may have experienced unemployment or under-employment recently or an investor may walk away from a property in which mortgage payments exceed rental income. In either case, borrowers and investors can avoid default by selling unless the home has negative equity. Hence, this study focuses on negative equity distressed properties.

Assumption 2: The loan will become performing after restructuring. The restructured loan, by construction of our model, has a lower balance, lower monthly payments, and a positive real option value. Assuredly, not all home loans can be restructured. However, we consider successful convergence to a solution, subject to several constraints, an indication of potential applicability. We deem the loan a restructure candidate when the model converges to a solution. We do not consider the loan a restructure candidate when the model does not converge.

Assumption 3: The home price process follows a lognormal distribution. We make this assumption to apply the Black-Scholes pricing valuation to our model. This is a standard assumption with option pricing made in [11], and implemented in [12] and [13].

Assumption 4: The The borrower's time horizon is *t* years. We acknowledge that the same agreement was made in the non-performing original loan. However, since the output of our model is a restructured loan beneficial to both the borrower and the lender, we consider *t*, which is distinct from the loan term *T*, a required commitment from the borrower for restructuring to occur.

Additional simplifying assumptions.We further assume the market-to-book-value of non-performing loans and performing loans remains constant. In line with option pricing practice, we also assume that price volatility and the risk free rate are stationary, there are no transaction costs or taxes, and the stock (home) does not pay a dividend. It is true that these assumptions do not always hold in the real world. However, the transaction costs and taxes will be minimal for a distressed property due to a major decline in value. In addition, although rental revenue may be positive, it is likely offset by costs of maintaining the property. Finally, [1] describes, if real estate investments of publicly-traded firms "are chosen in a manner consistent with value maximization, then real estate prices will be determined in equilibrium as if markets were really frictionless."

## 2.2. Current Home Value Based on Real Option Value under Distress

The current valueof the distressed property $S_0$ is obtained by setting value of the distressed home real option $c_{\text{dis}}$ equal to zero:

$$c_{\text{dis}} = S_0 N\left[d_{\text{dis},1}\right] - K_{\text{dis},t} \mathrm{e}^{-r_f t} N\left[d_{\text{dis},2}\right] = 0 \tag{1}$$

$$d_{\text{dis},1} = \frac{\ln\left[S_0 / K_{\text{dis},t}\right] + \left(r_f + \dfrac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{2}$$

$$d_{\text{dis},2} = d_{\text{dis},1} - \sigma\sqrt{t} \tag{3}$$

where $K_{\text{dis},t}$ is future value of the distressed loan if the borrower continued making payments. $K_{\text{dis},t}$ is computed as

$$K_{\text{dis},t} = K_0 \frac{1-\mathrm{e}^{-r_{\text{dis}}(T-t)}}{1-\mathrm{e}^{-r_{\text{dis}}T}} \tag{4}$$

where $r_{\text{dis}}$ is the current interest rate on the distressed loan. We use numerical methods to solve Equation (1) for $S_0$.

## 2.3. Real Option Value under Restructuring

The real option value of the restructured loan to the borrower is computed in the same fashion as under distress with changes to the interest rate $r_{\text{dis}}$ and strike price $K_{\text{res}}$ using the Black-Scholes option pricing model:

$$c_{\text{res}} = S_0 N\left[d_{\text{res},1}\right] - K_{\text{res},t}\mathrm{e}^{-r_f t} N\left[d_{\text{res},2}\right] \tag{5}$$

$$d_{\text{res},1} = \frac{\ln\left[S_0 / K_{\text{res},t}\right] + \left(r_f + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{6}$$

$$d_{\text{res},2} = d_{\text{res},1} - \sigma\sqrt{t} \tag{7}$$

where $S_0$ is the current market value of the distressed property, $t$ represents the time commitment of the borrower (see Assumption 4), and $K_{\text{res},t}$ is the principal balance at time $t$ represented by:

$$K_{\text{res},t} = \left((1-a)K_0\right)\frac{1-\mathrm{e}^{-r(T-t)}}{1-\mathrm{e}^{-rT}} \tag{8}$$

## 2.4. Real Option Value under Foreclosure

Should the lender reposses the home and sell on the open market (referred to Real Estate Owned or REO sale), the foreclosed home will sell at a discount to its current market value. This discount can be attributed to expected repairs associated with a property that has been vacant for six months or more (RealtyTrac). In a REO sale both the lender and third party buyer incur repair expenses [14]. Let $z$ be the prevailing market discount of REO sales such that the amount paid for the foreclosed home by a third party is $S_{\text{for},0} = (1-z)S_0$, $E_b$ be the repair expense for the buyer, $E_l$ be the repair expense for the lender, $\gamma$ be the down payment requirement dictated by current market conditions, and $r_{\text{mkt}}$ be the current market rate for 30 year home loans. The real option value to the third party purchaser of this REO home is computed as:

$$c_{\text{for}} = S_{\text{for},0} N\left[d_{\text{for},1}\right] - K_{\text{for},t}\mathrm{e}^{-r_f t} N\left[d_{\text{for},2}\right] \tag{9}$$

$$d_{\text{for},1} = \frac{\ln\left[S_0 / K_{\text{for},t}\right] + \left(r_f + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{10}$$

$$d_{\text{for},2} = d_{\text{for},1} - \sigma\sqrt{t} \tag{11}$$

where

$$K_{\text{for},t} = (1-\gamma)\left(S_{\text{for},0} + E_b\right)\frac{1-\mathrm{e}^{-r_{\text{mkt}}(T-t)}}{1-\mathrm{e}^{-r_{\text{mkt}}T}} \tag{12}$$

Note that the cash flow to the lender in the REO sale is not the sale price $S_{\text{for},0}$, rather, the sale price less any repair expenses incurred to make the home available for sale $S_{\text{for},0} - E_l$.

## 2.5. Constrained Optimization Model

The lender chooses the loan balance discount *a*, the new loan interest rate $r$, and the new loan term $T$ to maximize the current value of the newly-formed performing loan subject to several constraints.

$$\max_{a,r,T} b_p\left((1-a)K_0\right) \text{ subject to } \begin{cases} 0<a<1 & (C1) \\ c_{\text{res}}=0.5c_{\text{for}} & (C2) \\ b_p\left((1-a)K_0\right)>b_{np}K_0 & (C3) \\ r\geq r_{\text{mkt}} & (C4) \\ T_{\text{orig}}<T\leq 30 & (C5) \end{cases} \tag{13}$$

where $b_p$ is the proportion of performing loan balance that translates into market value, $b_{np}$ is the proportion of the non-performing loan balance that translates into market value, $K_0$ is the original non-performing loan balance, $(1-a)K_0$ represents the restructured loan balance, $c_{\text{res}}$ is the value of the real option on the home if under restructuring, $c_{\text{for}}$ is the value of the real option on the home if it were purchased from the lender after foreclosure, and $r_{\text{mkt}}$ is the average market rate for 30 year conforming loans. Note that the monthly mortgage payment of the new performing loan $\text{PMT}_p$ is less than the monthly payment of the old non-performing loan $\text{PMT}_{np}$ by construction. The endogenous variables are the lender's choice variables $(a,r,T)$, *i.e.*, the new loan terms. The exogenous variables are the current home price $S_0$, the market rate $r_{\text{mkt}}$, volatility $\sigma$, time commitment to remain in the home $t$, and loan market-to book value ratios $b_p$ and $b_{np}$. We now briefly discuss each constraint.

Constraint C1: The restructured loan must be smaller than the original loan. This constraint follows Assumption 1 and the suggestion by [15] to reduce loan balances and split potential gains (Constraint C2) to fix the US mortgage crisis. In addition, this constraint focuses our model on negative equity non-performing loans and ensures a lower monthly payment. The combination of lower loan balance and lower monthly payments follows Assumption 2: the loan will become performing after restructuring.

Constraint C2: The borrower and lender evenly split potential gains. This constraint guides our model to a solution that is mutually beneficial to borrowers and lenders. Simply reducing a principal balance just to bring a borrower current may be insufficient incentive for a lender. We incorporate the suggestion of shared gains [15] via this constraint.

Constraint C3: The market value of the restructured performing loan must exceed the market value of the current non-performing loan. This constraint is included to ensure that restructuring is beneficial to the lender. Without this constraint it would be possible to restructure into a performing loan whose market value is lower than the current non-performing loan.

Constraint C4: The new loan rate must be greater than or equal to the market rate for performing loans.A lender may be unable to fund loans at rates below the current market rate. Therefore, we require the restructured rate tobe at least the average market rate for performing 30 year loans.

Constraint C5: The new loan term is between the current loan term and 30 years. Given the finite lives of borrowers and customary industry practice we restrict the maximum new loan term to 30 years. Also, setting the minimum term to the remaining term on the existing loan follows Assumption 2 (the loan will become performing due to lower balance and payment).

Again, we utilize numerical optimization methods to solve the constrained optimization problem in Equation (13) to produce optimum values of the lender choice variables $(a^*,r^*,T^*)$.

## 3. Numerical Illustration

### 3.1. Data and Parameterization

We obtain the average market rate for 30 year conforming loans $r_{\text{mkt}}$ from the 2014 Freddie Mac Primary Mortgage Market Survey® (PMMS) and the 30 year risk free rate $r_f$ from the United States Treasury. We use the Federal Housing Finance Agency (FHFA) US Housing Price Index (HPI) to obtain the annual US home price volatility $\sigma$. The Federal Deposit Insurance Corporation (FDIC) home loan sales data is our source for the market-to-book ratios of non-performing $b_{np}$ and performing $b_p$ loans. Our average length of home ownership $t$, average down payment requirement $\gamma$, and average Real Estate Owned (REO) discount $z$ values are from the National Association of Realtors, mortgageqna.com, and RealtyTrac.com, respectively. Finally, average REO repair expense for the buyer $E_b$ and the lender $E_l$ are from [14].

**Table 1** summarizes the exogenous variables, values, and sources used in this study. **Figure 2** depicts the FHFA US home price index level data. The rapid growth in housing prices from 1997 to the 2006 peak, the

**Table 1.** Parameters, values, and sources.

| Symbol | Description | Value | Source |
|---|---|---|---|
| $r_{mkt}$ | US average market rate of 30 year conforming loan | 4.20% | Freddie Mac PMMS, September 2014 |
| $r_f$ | Risk free rate. | 3.21% | US Treasury, September 2014 |
| $\sigma$ | Annual US home price volatility | 4.77% | FHFA HPI 1991Q1 to 2014Q3 |
| $b_{np}$ | Non-performing loan market to book ratio | 0.3379 | FDIC 2014Q3 home loan sales data |
| $b_p$ | Performing loan market to book ratio | 0.6877 | FDIC 2014Q3 home loan sales data |
| $t$ | Average length of home ownership | 6 years | National Association of Realtors |
| $\gamma$ | Average down payment requirement | 10% | mortgageqna.com |
| $z$ | Average REO discount from market value | 35.90% | RealtyTrac |
| $E_b$ | Average buyer REO repair expense | 7663 | La Jeunesse [14] |
| $E_l$ | Average lender REO repair expense | 2252 | La Jeunesse [14] |

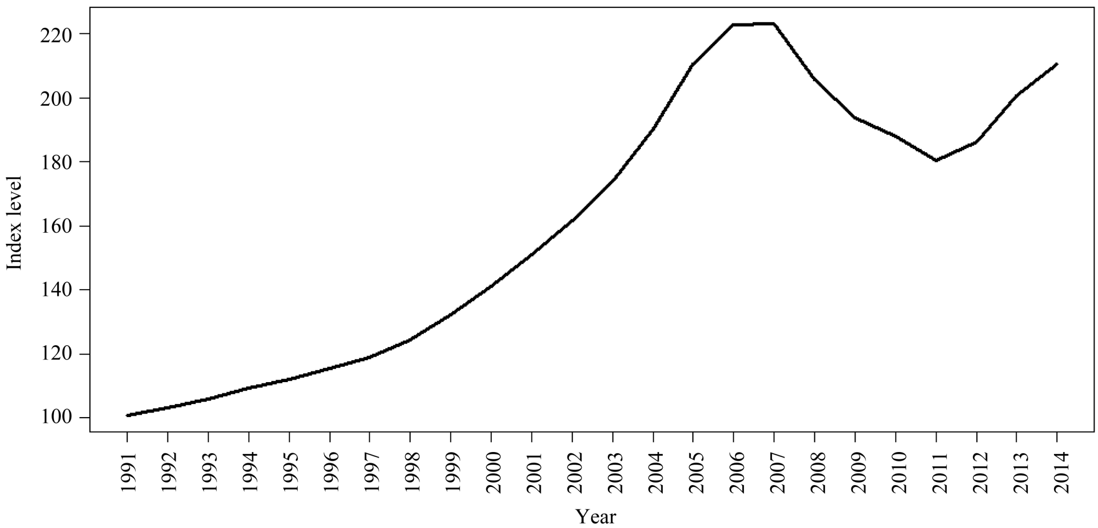


**Figure 2.** US housing price index level from 1991Q1 to 2014Q3. Source: Federal Housing Finance Agency.

correction from 2006 to mid-2011, and the post-2011 rebound are evident. **Figure 3** depicts the returns calculated from the FHFA US home price index level data. Returns averaged 3.36% with a standard deviation of $\sigma = 4.77\%$ during 1991Q1 to 2014Q3 time period.

## 3.2. Distressed Property Valuation

We begin our illustration with the median values of negative equity loans from [16]. Specifically, we establish our base case by setting the currently non-performing loan balance to the median mortgage balance at termination (default) $K_{dis,0} = \$359,000$, the original interest rate to the median rate at origination $r_{dis} = 7.5\%$, and the original loan term to the median time remaining at termination $T_{dis} = 28.5$. Given the parameterization of **Table 1** and the median characteristics of non-performing loans from [16], our model in Equation (12) produces an estimate of the current value of the distressed property $S_0 = \$189,427$.

Next, we perform a sensitivity analysis to investigate factors that impact distressed property values. The results of the sensitivity analysis are presented in **Figure 4**. Several observations are of note. First, home valuesare most sensitive to the current non-performing loan balance $K_{dis}$. This result reveals that even though the intial home purchase may have been at an inflated price, the associated loan balance still reflects information about the fundamental home value. Second, homes with more mature negative equity loans (lower $T_{dis}$) are of

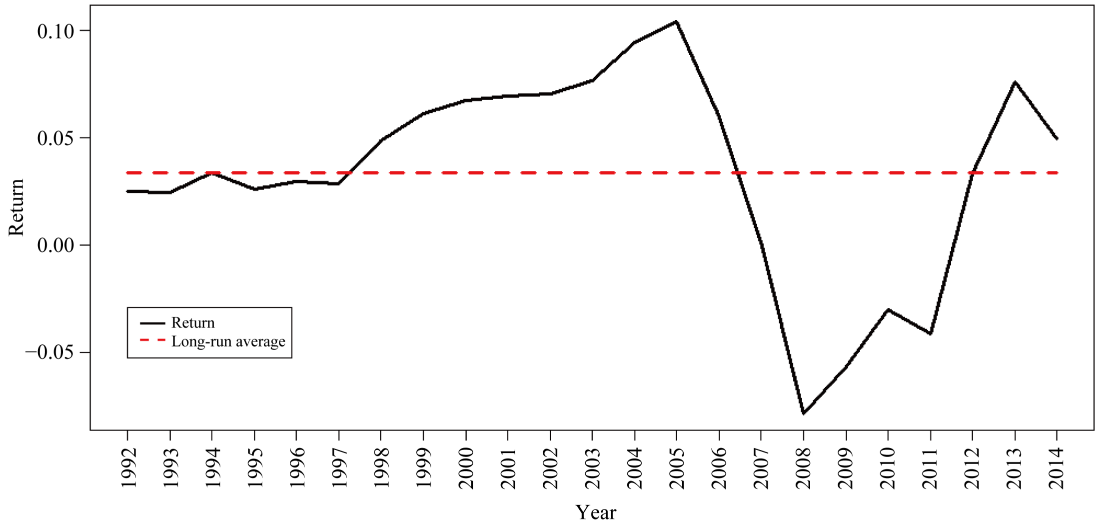


**Figure 3.** US housing price index returns from 1991Q1 to 2014Q3. Source: Federal Housing Finance Agency.

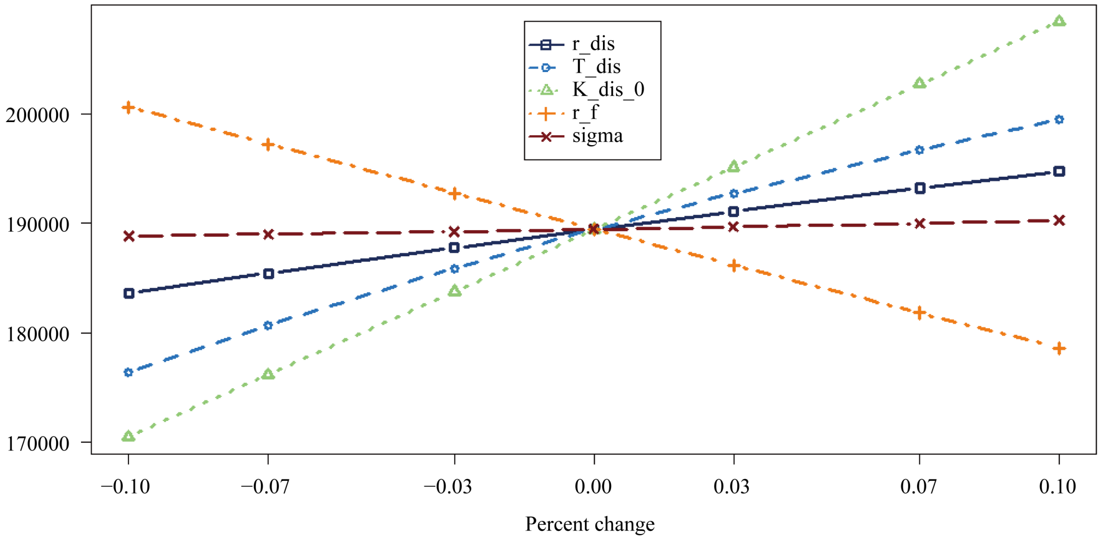


**Figure 4.** Home value sensitivity to changes in model inputs.

lower value. This could reveal that additional years of payments were unable to bring the loan balance of an extremely inflatedpurchase price in line with the fundamental value. Third, home values are relatively insensitive to the original interest rate $r_{\text{dis}}$. This result is not surprising given the median time from purchase to default is 18 months [16]. Fourth, home valuesare relatively insensitive to volatility. This result suggests inflated purchase prices drive the negative equity position rather than volatility. Finally, home value declines as the risk free rate increases. This result is consistent with the known inverse relationship between interest rates and housing prices.

### 3.3. Borrower-Lender Reconciliation

We determine optimum loan restructuring terms for the base case using our real options-based distressed property value estimate. We employ numerical methods [17] to solve the constrained nonlinear optimization model of Equation (13). The optimized loan reduction amount, loan rate, and loan term are $a^* = 47.10\%$, $r^* = 4.20\%$ and $T^* = 28.5$, respectively. The resultant performing loan market value of $130,599 represents a 9.59% improvement over the status quo non-performing loan market value of $119,171. Figure 5 shows the value of a restructured loan $V_{\text{res}} = b_p\left(\left(1-a\right)K_0\right)$ is larger than both the non-performing loan $V_{\text{dis}} = b_{np}K_0$ and the value to the lender post foreclosure sale $V_{\text{for}} = (1-z)S_0 - E_l$. Interestingly, the lender is better off selling the non-performing loan for $121,306 than foreclosing on the home to receive $119,171.

We now examine optimality conditions for different non-performing loan balances. **Table 2** details the impact of initial loan balance $K_0$ on optimality conditions $(a^*, r^*, T^*)$, option value to borrower $c_{\text{res}}$, option value to third-party purchaser at foreclosure $c_{\text{for}}$, value to the lender if non-performing loan is sold $V_{\text{dis}}$, value to the lender if they chose to foreclose $V_{\text{for}}$, and finally the value to the lender if they chose to restructure $V_{\text{res}}$. The lender realizes more value via restructuring for all non-performing loan balances shown. However, the benefit to the lender decreases as the non-performing loan balance increases. Restructuring a relatively small non-performing loan balance of $251,300 results in a 25.44% improvement over foreclosure. In contrast, the improvement over foreclosure is just 1.29% when restructuring the relatively larger non-performing loan balance of $466,700.

### 3.4. Analysis of Results

The restructured value $V_{\text{res}}$ exceeds the foreclosure value $V_{\text{for}}$ for all initial non-performing loan balances examined. Restructuring benefits borrowers since they retain their residence, become current on their mortgage, mitigate negative credit impacts, and minimize legal fees. We do not include transaction costs to the lender. However, inclusion of the lender costs associated with REO sales (legal fees, real estate commissions, etc.) would only increase the attractiveness of restructuring over foreclosure. In addition, our results show that lenders are better off selling non-performing loans than proceeding through foreclosure $(V_{\text{dis}} > V_{\text{for}})$. Finally, we find it particularly interesting that the benefit to lenders diminishes as the home value increases. These findings can serve as guidance to lenders as they wrestle with non-performing loans now and into the future.

## 4. Conclusion

We present a method to establish the market value of distressed properties and a model to reconcile the interests

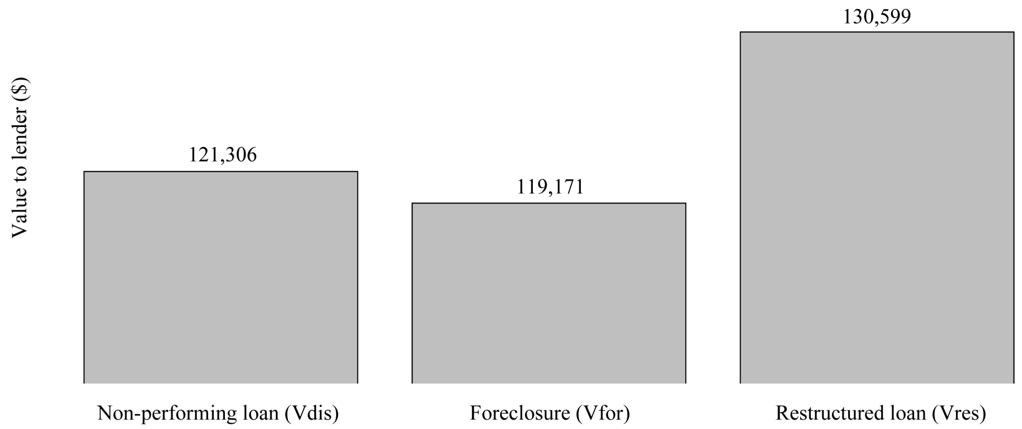


**Figure 5.** Value to lender under base case.

**Table 2.** Model outputs.

| $K_0$ | $S_0$ | $a^*$ | $r^*$ | $T^*$ | $c_{\text{res}}$ | $c_{\text{for}}$ | $V_{\text{dis}}$ | $V_{\text{for}}$ | $V_{\text{res}}$ |
|---|---|---|---|---|---|---|---|---|---|
| 251,300 | 132,599 | 0.3994 | 0.0420 | 28.5 | 23,825 | 47,649 | 84,914 | 82,744 | 103,797 |
| 287,200 | 151,542 | 0.4297 | 0.0420 | 28.5 | 33,296 | 66,592 | 97,045 | 94,886 | 112,644 |
| 323,100 | 170,485 | 0.4527 | 0.0420 | 28.5 | 42,767 | 85,535 | 109,175 | 107,029 | 121,604 |
| 359,000 | 189,427 | 0.4710 | 0.0420 | 28.5 | 52,239 | 104,477 | 121,306 | 119,171 | 130,599 |
| 394,900 | 208,370 | 0.4859 | 0.0420 | 28.5 | 61,710 | 123,420 | 133,437 | 131,313 | 139,606 |
| 430,800 | 227,313 | 0.4984 | 0.0420 | 28.5 | 71,181 | 142,363 | 145,567 | 143,455 | 148,617 |
| 466,700 | 246,255 | 0.5089 | 0.0420 | 28.6 | 80,563 | 161,306 | 157,698 | 155,598 | 157,608 |

of borrowers and lenders. Our approach applies real option theory using the Black-Scholes option pricing formula, a constrained maximization framework, and numerical methods. Sensitivity analysis reveals distressed property values are most sensitive to the non-performing loan balance and least sensitive to market volatility. The non-performing loan balance sensitivity indicates information remains in the non-performing balance. The volatility insensitivity suggests the resulting negative equity position of distressed homes is driven more by inflated purchase prices than overall market volatility. We present evidence that restructuring is the highest value alternative among the lender's choice to sell the loan, foreclose, or restructure. Borrowers also benefit from principal and interest rate reductions. We show that the real option under restructuring is a significant improvement over the zero-value option during non-performance. Overall, we believe our approach can be used to arrive at mutually beneficial loan terms thereby relieving current and future negative equity positions.

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- Solutions to PDEs
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