

Ground State Energy Density of the Quantum Harmonic Oscillator

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Abstract

The total energy of the ground state of the quantum harmonic oscillator is obtained with minimal assumptions. The vacuum energy density of the universe is derived and a cutoff frequency is obtained for the upper bound of the quantum harmonic oscillator.

Keywords

Planck Time, Quantum Harmonic Oscillator, Vacuum Energy Density, Cosmological Constant

1. Introduction

In this paper, we find an expression for the ground state energy of the quantum harmonic oscillator (QHO) to obtain the vacuum energy density of the universe. We use the addition of velocities of Einstein's Special Relativity theory to derive a contraction of the Planck time. Using Heisenberg's Uncertainty Principle, we obtain an expression for the total energy of the QHO ground state. We also obtain a cutoff frequency for the QHO. This paper is a parallel version of an earlier e-publication by the author [1] which was based on Carmeli's Cosmological Special Relativity theory [2].

2. Addition of Velocities and the Contracted Planck Time

Consider an inertial reference frame K_1 which moves at a velocity v_1 along the x axis of inertial frame K . If an object is measured to have a velocity v_2 along the x_1 axis relative to the origin of frame K_1 then the velocity v_{1+2} of the object relative to the origin of frame K is given by the addition of velocities of Einstein's Special Relativity theory [3],

$$v_{1+2} = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}, \quad (1)$$

where c is the speed of light in vacuum. Assume that $v_1 = v$, $v_2 = \delta v'$ and $v_{1+2} = v + \delta v$, where $\delta v' \ll c$. Substituting these variables into (1) gives

$$v + \delta v = \frac{v + \delta v'}{1 + v\delta v'/c^2} \approx (v + \delta v') \left(1 - \frac{v\delta v'}{c^2}\right) \approx v + \left(1 - \frac{v^2}{c^2}\right) \delta v', \tag{2}$$

where approximations are made due to $\delta v' \ll c$. Simplifying (2) yields

$$\delta v \approx \left(1 - \frac{v^2}{c^2}\right) \delta v'. \tag{3}$$

Assume velocity v is very close to the speed of light c such that it is given by

$$v = c - \delta v'. \tag{4}$$

For this value of v the second term in the first factor on the right hand side of (3) is given by

$$-\frac{v^2}{c^2} = -\frac{(c - \delta v')^2}{c^2} \approx -1 + \frac{2\delta v'}{c}, \tag{5}$$

where we make the approximation again since $\delta v' \ll c$. Substituting (5) into (3) and simplifying we obtain

$$\delta v \approx \frac{2(\delta v')^2}{c}. \tag{6}$$

Now, define the acceleration constant $a_0 = cH_0$, where H_0 is Hubble's constant, and use it to define the expansion velocity parameter

$$\mathcal{V} = a_0 \mathcal{T}_P, \tag{7}$$

where the Planck time [4] $\mathcal{T}_P = \sqrt{\hbar G/c^5}$, where \hbar is the reduced Planck constant and G is Newton's constant. Equation (7) is just Hubble's law [5] given by $\mathcal{V} = H_0 \mathcal{L}_P$ where $\mathcal{L}_P = c\mathcal{T}_P$ is the Planck length. It is a fundamental principle of Special Relativity that the laws of physics are invariant for all observers in inertial reference frames, from which it follows that the physical constants are the same for all inertial observers. Since \hbar , G , c and H_0 are physical constants, hence a_0 and therefore \mathcal{V} are both constants in any inertial reference frame. Then, substituting from (7) for the velocity in (6) such that $\delta v' = \mathcal{V} = \sqrt{\hbar G/c^3} H_0$, we obtain the expression for the contracted expansion velocity

$$\mathcal{V}_C = \delta v = \frac{2\mathcal{V}^2}{c} = \frac{2\hbar G H_0^2}{c^4}. \tag{8}$$

What this means is that while the observer in K_1 detects that the object has an expansion velocity \mathcal{V} , the observer in K will say that the object actually has an expansion velocity of \mathcal{V}_C relative to the origin of K_1 so that the expansion velocity of the object in K is $v_{2K} = c - \mathcal{V} + \mathcal{V}_C$. In order to obtain a time value from the contracted velocity, divide a_0 into (8) to get the time interval

$$\mathcal{T}_{PC} = \frac{\mathcal{V}_C}{a_0} = \frac{2\hbar G H_0}{c^5}. \tag{9}$$

We call \mathcal{T}_{PC} the contracted Planck time since it has the form

$$\mathcal{T}_{PC} = 2(\mathcal{T}_P)^2 H_0. \tag{10}$$

With $\mathcal{T}_P \approx 5.39 \times 10^{-44} \text{s}$ and $H_0 \approx 71.9 \text{ km} \cdot \text{s}^{-1} \text{ Mpc}^{-1} \approx 2.33 \times 10^{-18} \text{s}^{-1}$ [6], we get $\mathcal{T}_{PC} \approx 1.35 \times 10^{-104} \text{s}$. The expansion velocity parameter has a value $\mathcal{V} \approx 3.77 \times 10^{-51} \text{ cm} \cdot \text{s}^{-1}$ while the contracted expansion velocity has a value $\mathcal{V}_C \approx 9.46 \times 10^{-112} \text{ cm} \cdot \text{s}^{-1}$. The value of the acceleration constant is $a_0 \approx 6.99 \times 10^{-8} \text{ cm} \cdot \text{s}^{-2}$.

3. Quantum Harmonic Oscillator Ground State Energy

For a simple linear quantum harmonic oscillator (QHO) in one dimension the energy levels for the plane wave modes \mathbf{k} and frequencies $\omega_{\mathbf{k}}$ are given by [7] [8]

$$E(\mathbf{k})_n = \left(n + \frac{1}{2}\right) \hbar \omega_{\mathbf{k}}, \tag{11}$$

where the quantum number $n = 0, 1, 2, \dots$ is the state number. The ground state energy for the frequency $\omega_{\mathbf{k}}$ is given for $n = 0$,

$$E(\mathbf{k})_0 = \frac{1}{2} \hbar \omega_{\mathbf{k}}. \tag{12}$$

For three linear QHO's oriented along orthogonal axes, and for two polarizations, the ground state energy of the electromagnetic field is given by

$$E(\mathbf{k})_0 = 2 \left(\frac{3}{2}\right) \hbar \omega_{\mathbf{k}} = 3 \hbar \omega_{\mathbf{k}}. \tag{13}$$

To get the total energy density ρ_{vac} for all modes \mathbf{k} in the ground state 0 we must sum over all oscillator mode frequencies between zero and a finite cut-off frequency ω_{max} to obtain the expected value

$$\rho_{vac} = \langle 0 | \hat{\rho} | 0 \rangle = \frac{E}{V} = \frac{1}{V} \sum_{\mathbf{k}} (3 \hbar \omega_{\mathbf{k}}) \approx \frac{1}{8\pi^3 c^3} \int_0^{\omega_{max}} 3 \hbar \omega (4\pi \omega^2) d\omega, \tag{14}$$

where E is the total energy of the ground state and V is the volume of the universe. Performing the integration in (14), the relationship between ρ_{vac} and ω_{max} is given by

$$\rho_{vac} = \left(\frac{3 \hbar \omega_{max}^4}{8\pi^2 c^3}\right). \tag{15}$$

We can obtain an expression for the density by defining the energy E in terms of the contracted Planck time \mathcal{T}_{PC} according to Heisenberg's Uncertainty Principle [9] of the form

$$\Delta E \Delta t \geq \hbar. \tag{16}$$

Taking $\Delta t = \mathcal{T}_{PC}$ from (9) and using (16) with $E = \Delta E$, we assume for the ground state the total energy

$$E = \frac{\hbar}{\mathcal{T}_{PC}}. \tag{17}$$

For a universe with Hubble radius $R_H = c/H_0$ the volume is given by

$$V = \frac{4\pi}{3} \left(\frac{c}{H_0}\right)^3. \tag{18}$$

Substituting for E and V from (17) and (18) into (14) we get the vacuum energy density [14]

$$\rho_{vac} = \frac{E}{V} = \left(\frac{\hbar}{\mathcal{T}_{PC}} \right) \left(\frac{3H_0^3}{4\pi c^3} \right) = \frac{3c^2 H_0^2}{8\pi G}, \quad (19)$$

where the last right hand side expression is identical in form to the critical energy density ρ_c of standard cosmology. We note that this expression for ρ_{vac} was obtained through a general application of the Hubble law, Special Relativity theory and the Heisenberg Uncertainty Principle. Solving (15) for ω_{max} using (19) we obtain

$$\omega_{max} = \left(\frac{\pi H_0^2}{\mathcal{T}_P^2} \right)^{1/4}. \quad (20)$$

Substituting values for the parameters, the value of the cut-off frequency (20) is given by

$$\omega_{max} = \left(\frac{\pi c^5 H_0^2}{\hbar G} \right)^{1/4} \approx 8.75 \times 10^{12} \text{ Hz}. \quad (21)$$

The oscillator ground state energy at the cutoff frequency is

$$\epsilon_{max} = 3\hbar\omega_{max} \approx 0.0173 \text{ eV}, \quad (22)$$

which is 0.067% of the upper limit for the electron neutrino rest mass energy of 26 eV of the *Kamiokande II* experiment [10], also just 0.69% of the upper limit for the electron antineutrino rest mass energy of 2.5 eV of the *Troitsk neutrino mass* experiment [11] and between 4.3% to 8.6% of the upper limit of the summed neutrino rest mass energies $\sum m_\nu c^2 < (0.2 - 0.4) \text{ eV}$ from [12] and the *Planck* experiment [13]. The value of the vacuum energy density $\rho_{vac} \approx 8.73 \times 10^{-9} \text{ erg cm}^{-3} \approx 9.71 \times 10^{-30} \text{ gm cm}^{-3}$ or equivalent to about 5.8 Hydrogen (H) atoms per cubic meter. The cosmological constant [14] [15] [16] is expressed by

$$\Lambda = \kappa\rho_{vac}, \quad (23)$$

where $\kappa = 8\pi G/c^4$ is Einstein's constant. Substituting for the parameters we get a value $\Lambda \approx 1.81 \times 10^{-56} \text{ cm}^{-2}$, for a flat universe with no matter. This compares with the *High-Z Supernova Search Team* experiment [17] where $\Lambda \approx 1.07 \times 10^{-56} \text{ cm}^{-2}$ for a flat universe with $H_0 \approx 65.2 \text{ km} \cdot \text{s}^{-1} \text{ Mpc}^{-1}$, with vacuum density parameter $\Omega_\Lambda \approx 0.72$ and mass density parameter $\Omega_M \approx 0.28$.

4. Conclusions

From a general application of the Hubble law and Special Relativity, we obtained a time interval \mathcal{T}_{PC} which is 60 orders of magnitude smaller than the Planck time \mathcal{T}_P . By the Heisenberg Uncertainty Principle, we associated this time with energy and derived the vacuum energy density which, from (15) and (20), can be put in the form

$$\rho_{vac} = \frac{3\hbar}{8\pi c^3} \frac{H_0^2}{\mathcal{T}_P^2}. \quad (24)$$

Since the ground state energy cutoff (22) is approximately 4% to 8% of the present day upper limits of the sum of neutrino rest mass energies, and as the neutrino mass upper limits appear to be diminishing by an order of magnitude with more sensitive experiments, it is probably not too far-fetched to speculate [18] [19] that $\epsilon_{max} \approx \sum m_\nu c^2$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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