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New Class of Distortion Risk Measures and Their Tail Asymptotics with Emphasis on VaR

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Abstract

Distortion risk measures are extensively used in finance and insurance applications because of their appealing properties. We present three methods to construct new class of distortion functions and measures. The approach involves the composting methods, the mixing methods and the approach that based on the theory of copula. We also investigate the tail subadditivity for VaR and other distortion risk measures. In particular, we demonstrate that VaR is tail subadditive for the case where the support of risk is bounded. Various examples are also presented to illustrate the results.

Keywords

Coherent Risk Measure, Copula, Distortion Risk Measure, Extreme Value Theory, GlueVaR, Tail Subadditivity, Tail Distortion Risk Measure

1. Introduction

A risk measure ρ is a mapping from the set of random variables $\mathcal{X}$, standing for risky portfolios of assets and/or liabilities, to the real line R. In the subsequent discussion, positive values of elements of $\mathcal{X}$ will be considered to represent losses, while negative values will represent gains. Distortion risk measures are a particular and most important family of risk measures that have been extensively used in finance and insurance as capital requirement and principles of premium calculation for the regulator and supervisor. Several popular risk measures belong to the family of distortion risk measures. For example, the value-at-risk (VaR), the tail value-at-risk (TVaR) and the Wang distortion measure. Distortion risk measures satisfy a set of properties including positive homogeneity, translation invariance and monotonicity. When the associated distortion function is concave, the distortion risk measure is also subadditive (Denneberg, 1994; Wang & Dhaene, 1998). VaR is one of the most

popular risk measures used in risk management and banking supervision due to its computational simplicity and for some regularity reasons, despite it has some shortcomings as a risk measure. For example, VaR is not a subadditive risk measure (see, for instance, Artzner et al., 1999; Denuit et al., 2005), it only concerns about the frequency of risk, but not the size of risk. TVaR, although being coherent, concerns only losses exceeding the VaR and ignores useful information of the loss distribution below VaR. Clearly, it is difficult to believe that a unique risk measure could capture all characteristics of risk, so that an ideal measure does not exist. Moreover, since risk measures associate a single number to a risk, as a matter of fact, they cannot exhaust all the information of a risk. However, it is reasonable to search for risk measures which are ideal for the particular problem under investigation. As all the proposed risk measures have drawbacks and limited applications, the selection of the appropriate risk measures continues to be a hot topic in risk management.

Zhu & Li (2012) introduced and studied the tail distortion risk measure which was reformulated by Yang (2015) as follows. For a distortion function g, the tail distortion risk measure at level p of a loss variable X is defined as the distortion risk measure with distortion function

$$g_p(x)=\begin{cases} g\left(\dfrac{x}{1-p}\right), & \text{if } 0\le x\le 1-p, \\ 1, & \text{if } 1-p<x\le 1. \end{cases}$$

Some properties and applications can be found in Mao, Lv, & Hu (2012), Mao & Hu (2013) and Lv, Pan, & Hu (2013).

As an extension of VaR and TVaR, Belles-Sampera et al. (2014a) proposed a new class of distortion risk measures called GlueVaR risk measures, which can be expressed as a combination of VaR and TVaR measures at different probability levels. They obtain the analytical closed-form expressions for the most frequently used distribution functions in financial and insurance applications, while a subfamily of these risk measures has been shown to satisfy the tail-subadditivity property which means that the benefits of diversification can be preserved, at least they hold in extreme cases. The applications of GlueVaR risk measures in capital allocation can be found in the recent paper Belles-Sampera et al. (2014b).

Cherubini & Mulinacci (2014) propose a class of distortion measures based on contagion from an external "scenario" variable. The dependence between the scenario and the variable whose risk is modeled with a copula function with horizontal concave sections, they give conditions to ensure that coherence requirements be met, and propose examples of measures in this class based on copula functions.

The first purpose of this paper is to construct new risk measures following Zhu & Li (2012), Belles-Sampera et al. (2014a) and Cherubini & Mulinacci (2014). The newly introduced risk measures are included the tail distortion risk measure and the GlueVaR as specials. The second goal of the paper is to

investigate the tail asymptotics of distortion risk measures for the sum of possibly dependent risks with emphasis on VaR. The rest of the paper is organized as follows. We review some basic definitions and notations such as distorted functions, distorted expectations and distortion risk measures in Section 2. In Section 3 several new distortion functions and risk measures are introduced. In Section 4 we investigate the tail asymptotics as well as subadditivity/superadditivity of VaR. In Section 5 we analyze the subadditivity properties of a class of distortion risk measures and Section 6 focuses on conclusion.

2. Distortion Risk Measures

2.1. Distorted Functions

A distortion function is a non-decreasing function $g:[0,1]\to[0,1]$ such that $g(0)=0, g(1)=1$. Since Yaari (1987) introduced distortion function in dual theory of choice under risk, many different distortions g have been proposed in the literature. Here we list some commonly used distortion functions. A summary of other proposed distortion functions can be found in Denuit et al. (2005).

- $g(x)=\mathbf{1}_{(x>1-p)}$, where the notation $\mathbf{1}_A$ to denote the indicator function, which equals 1 when A holds true and 0 otherwise.
- $g(x)=\min\left\{\frac{x}{1-p},1\right\}$.
- Incomplete beta function $g(x)=\frac{1}{\beta(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$, where $a>0$ and $b>0$ are parameters and $\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$. Setting $b=1$ gives the power distortion $g(x)=x^a$; setting $a=1$ gives the dual-power distortion $g(x)=1-(1-x)^b$.
- The Wang distortion $g(x)=\Phi\left(\Phi^{-1}(x)+\Phi^{-1}(p)\right), 0<p<1$, where Φ is the distribution function of the standard normal.
- The lookback distortion $g(x)=x^p(1-p\ln x), p\in(0,1]$.

Obviously, every concave distortion function is continuous on the interval $(0,1]$ and can have jumps in 0. In contrast, every convex distortion function is continuous on the interval $[0,1)$ and can have jumps in 1. The identity function is the smallest concave distortion function and also the largest convex distortion function; $g_0(x):=\mathbf{1}_{(x>0)}$ is concave on $[0,1]$ and is the largest distortion function. $g^0(x):=\mathbf{1}_{(x=1)}$ is convex on $[0,1]$ and is the smallest distortion function. For $0<p<1$, we remark that $g_1(x):=\min\left\{\frac{x}{1-p},1\right\}$ is the smallest concave distortion function such that $g_1(x)\geq\mathbf{1}_{(x>1-p)}$. In fact, we consider a concave distortion function g such that $g(x)\geq\mathbf{1}_{(x>1-p)}$, then $g\equiv 1$ on $(1-p,1]$. As g is concave, it follows that $g(x)\geq\frac{x}{1-p}$ for $x\leq 1-p$, and

thus $g(x) \geq \min\left\{\frac{x}{1-p}, 1\right\}$ for $0 < x < 1$. Any concave distortion function g gives more weight to the tail than the identity function $g(x) = x$, whereas any convex distortion function g gives less weight to the tail than the identity function $g(x) = x$.

2.2. Distorted Risk Measures

Let (Ω, F, P) be a probability space on which all random variables involved are defined. Let F_X be the cumulative distribution function of random variable X and the decumulative distribution function is denoted by $\bar{F}_X$, i.e. $\bar{F}_X(x) = 1 - F_X(x) = P(X > x)$. Let g be a distortion function. The distorted expectation of the random variable X, notation $\rho_g[X]$, is defined as

$$\rho_g[X] = \int_0^{+\infty} g\left(\bar{F}_X(x)\right)\mathrm{d}x + \int_{-\infty}^{0}\left[g\left(\bar{F}_X(x)\right) - 1\right]\mathrm{d}x,$$

provided at least one of the two integrals above is finite. If X a non-negative random variable, then ρ_g reduces to

$$\rho_g[X] = \int_0^{+\infty} g\left(\bar{F}_X(x)\right)\mathrm{d}x.$$

From a mathematical point of view, a distortion expectation is the Choquet integral (see Denneberg (1994)) with respect to the nonadditive measure $\mu = g \circ P$. That is $\rho_g[X] = \int X \mathrm{d}\mu$. In view of Dhaene et al. (2012: Theorems 4 and 6) we know that, when the distortion function g is right continuous on $[0,1)$, then $\rho_g[X]$ may be rewritten as

$$\rho_g[X] = \int_{[0,1]} VaR_{1-q}^{+}[X]\mathrm{d}g(q),$$

where $VaR^{+}p[X] = \sup\{x \mid F_X(x) \leq p\}$, and when the distortion function g is left continuous on $(0,1]$, then $\rho_g[X]$ may be rewritten as

$$\rho_g[X] = \int_{[0,1]} VaR_{1-q}[X]\mathrm{d}g(q) = \int_{[0,1]} VaR_q[X]\mathrm{d}\bar{g}(q),$$

where $VaR_p[X] = \inf\{x \mid F_X(x) \geq p\}$ and $\bar{g}(q) := 1 - g(1-q)$ is the dual distortion of g. Obviously, $\bar{\bar{g}} = g$, g is left continuous if and only if $\bar{g}$ is right continuous; g is concave if and only if $\bar{g}$ is convex. The distorted expectation $\rho_g[X]$ is called a distortion risk measure with distortion function g. Distortion risk measures are a particular class of risk measures which as premium principles were introduced by Denneberg (1994) and further developed by Wang (1996, 2000) among others.

Distortion risk measures satisfy a set of properties including positive homogeneity, translation invariance and monotonicity. A risk measure is said to be coherent if it satisfies the following set of four properties (see, e.g., Artzner et al., 1997, 1999):

(M) Monotonicity: $\rho(X) \leq \rho(Y)$ provided that $P(X \leq Y) = 1$.

(P) Positive homogeneity: For any positive constant $c > 0$ and loss X, $\rho(cX) = c\rho(X)$.

(S) Subadditivity: For any losses X, Y, then $\rho(X+Y) \le \rho(X)+\rho(Y)$.

(T) Translation invariance: If c is a constant, then $\rho(X+c)=\rho(X)+c$.

It is furthermore shown by Artzner et al. (1999) that all mappings satisfying the above properties allow a representation:

$$\rho(X)=\sup_{p\in\mathcal{P}} E_p[X],$$

where $\mathcal{P}$ is a collection of "generalised scenarios". A risk measure ρ is called a convex risk measure if it satisfies monotonicity, translation invariance and the following convexity (C):

$$\rho(\lambda X+(1-\lambda)Y) \le \lambda\rho(X)+(1-\lambda)\rho(Y), 0 \le \lambda \le 1.$$

Clearly, under the assumption of positive homogeneity, monotonicity and translation invariance, the convexity of a risk measure is equivalent subadditivity.

The most well-known examples of distortion risk measures are the above-mentioned VaR and TVaR, corresponding to the distortion functions, respectively, are $g(x)=\mathbf{1}_{(x>1-p)}$ and $g(x)=\min\left\{\frac{x}{1-p},1\right\}$. Notice that $TVaR_p[X]$ can be alternatively expressed as the weighted average of VaR and losses exceeding VaR:

$$TVaR_p[X]=VaR_p[X]+\frac{1-F_X(VaR_p[X])}{1-p}E[X-VaR_p[X]\,|\,X>VaR_p[X]]. \quad (2.1)$$

For continuous distributions, $TVaR$ coincide with the expected loss exceeding p-Value-at Risk, i.e., the mean of the worst $(1-p)100\%$ losses in a specified time period which defined by

$$CTE_p[X]=E[X\,|\,X>VaR_p[X]].$$

Detailed studies of distortion risk measures and their relation with orderings of risk and the concept of comonotonicity can be found in, for example, Wang (1996), Wang & Young (1998), Hua & Joe (2012) and the references therein. The following lemma will be used in proofs of later results, which characterizes an ordering of distortion risk measures in terms of their distortion functions.

Lemma 2.1 (Belles-Sampera et al., 2014b). *If* $g(x) \le g^*(x)$ *for* $x\in[0,1]$, *then* $\rho_g[X] \le \rho_{g^*}[X]$ *for any random variable X.*

3. Generating New Distortion Functions and Measures

Distortion functions can be considered as a starting point for constructing families of distortion risk measures. Thus, constructions of distortion functions play an important role in producing various families of risk measures. Using the technique of mixing, composition and copula allow the construction of new class of distortion functions and measures.

3.1. Composting Methods

The first approach to construct distortion functions is the composition of

distortion functions.

Let $h_1, h_2, \cdots$ be distortion functions, define $f_1(x) = h_1(x)$ and composite functions $f_n(x) = f_{n-1}(h_n(x)), n = 1, 2, \cdots$. It is easy to check that $f_n(x), n = 1, 2, \cdots$ are all distortion functions. If $h_1, h_2, \cdots$ are concave distortion functions, then each $f_n(x)$ is concave and satisfies that

$$f_1 \le f_2 \le f_3 \le \cdots$$

and

$$\lim_{n\to\infty} f_n(x) = \mathbf{1}_{(x>0)}, \ x \in [0,1].$$

The associated risk measures satisfy (by Lemma 2.1)

$$\rho_{f_1}[X] \le \rho_{f_2}[X] \le \rho_{f_3}[X] \le \cdots$$

and

$$\lim_{n\to\infty} \rho_{f_n}[X] = VaR_1[X] = \operatorname{esssup}(X).$$

If $h_1, h_2, \cdots$ are convex distortion functions, then each $f_n(x)$ is convex and satisfies that

$$f_1 \ge f_2 \ge f_3 \ge \cdots$$

and

$$\lim_{n\to\infty} f_n(x) = \mathbf{1}_{(x=1)}, \ x \in [0,1].$$

The associated risk measures satisfy (by Lemma 2.1)

$$\rho_{f_1}[X] \ge \rho_{f_2}[X] \ge \rho_{f_3}[X] \ge \cdots$$

and

$$\lim_{n\to\infty} \rho_{f_n}[X] = VaR_0[X] = \operatorname{essinf}(X).$$

Consider two distortion functions g_1 and g_2. If

$$g_2(x) = \begin{cases} \dfrac{x}{1-p}, & \text{if } 0 \le x \le 1-p, \\ 1, & \text{if } 1-p < x \le 1, \end{cases}$$

then we get

$$g_p(x) := g_1(g_2(x)) = \begin{cases} g_1\left(\dfrac{x}{1-p}\right), & \text{if } 0 \le x \le 1-p, \\ 1, & \text{if } 1-p < x \le 1. \end{cases}$$

The corresponding risk measure $\rho_{g_p}[X]$ is the tail distortion risk measure which was first introduced by Zhu & Li (2012), and was reformulated by Yang (2015). In particular, on the space of continuous loss random variables X,

$$\rho_{g_p}[X] = \int_0^\infty g_p\left(1 - P\left(X \le x \mid X > VaR_p[X]\right)\right)\mathrm{d}x.$$

If $g_1(x) = x^r, 0 < r < 1$ and

$$g_2(x) = \begin{cases} \dfrac{x}{1-p}, & \text{if } 0 \le x \le 1-p, \\ 1, & \text{if } 1-p < x \le 1, \end{cases}$$

then

$$g_{12}(x) := g_1(g_2(x)) = \begin{cases} \left(\dfrac{x}{1-p}\right)^r, & \text{if } 0 \le x \le 1-p, \\ 1, & \text{if } 1-p < x \le 1, \end{cases}$$

and

$$g_{21}(x) := g_2(g_1(x)) = \begin{cases} \dfrac{x^r}{1-p}, & \text{if } 0 \le x \le (1-p)^{\frac{1}{r}}, \\ 1, & \text{if } (1-p)^{\frac{1}{r}} < x \le 1. \end{cases}$$

Clearly, $g_1 < g_{21}$ and $g_2 < g_{12}$, so that, by Lemma 2.1, $\rho_{g_1}[X] < \rho_{g_{21}}[X]$ and $\rho_{g_2}[X] < \rho_{g_{12}}[X]$.

In practice, sometimes one needs distort the initial distribution more than one times.

Example 3.1 Consider two risks X and Y with distributions, respectively, are:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.6, & \text{if } 0 \le x < 100, \\ 0.975, & \text{if } 100 \le x < 500, \\ 1, & \text{if } x \ge 500, \end{cases}$$

and

$$F_Y(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.6, & \text{if } 0 \le x < 100, \\ 0.99, & \text{if } 100 \le x < 1100, \\ 1, & \text{if } x \ge 1100. \end{cases}$$

Then $EX = EY = 50$, $VaR_{0.95}[X] = VaR_{0.96}[X] = 100$, $VaR_{0.95}[Y] = VaR_{0.96}[Y] = 100$.

$TVaR$ can be calculated by formula (2.1):

$TVaR_{0.95}[X] = TVaR_{0.95}[Y] = 300$, $TVaR_{0.96}[X] = TVaR_{0.96}[Y] = 350$. So that when $\alpha = 0.95$ and $\beta = 0.96$, according to the measures of VaR and TVaR, both X and Y bear the same risk! However, the maximal loss for Y (1100) is more than double than for loss X (500), clearly, risk Y is more risky than risk X. Now we consider distortion expectation ρ_{g_p} with

$$g_1(x) = g_2(x) = \begin{cases} \dfrac{x}{1-p}, & \text{if } 0 \le x \le 1-p, \\ 1, & \text{if } 1-p < x \le 1. \end{cases}$$

One can easily find that, with $p = 0.95$, $\rho_{g_p}[X] = 500$ and $\rho_{g_p}[Y] = 1100$.

3.2. Mixing Methods

One of the easiest ways to generate distortion functions is to use the method of mixing along with finitely distortion functions or infinitely many distortion functions. Specifically, if g_w ($w \in \langle a,b \rangle$) is a one-parameter family of distortion functions, ψ is an increasing function on $\langle a,b \rangle$ such that

$\int_{\langle a,b\rangle} \mathrm{d}\psi(w)=1$, then the function $g=\int_{\langle a,b\rangle} g_w \mathrm{d}\psi(w)$ is a distortion function, the associated risk measure is given by

$$\rho_g[X]=\int_{\langle a,b\rangle}\rho_{g_w}[X]\mathrm{d}\psi(w). \tag{3.1}$$

In particular, if ψ is discrete distribution, then (3.1) can be written as the form of convex linear combination $g=\sum_i w_i g_i$ $\left(w_i\geq 0, \sum_i w_i=1\right)$, the associated risk measure is given by

$$\rho_g[X]=\sum_i w_i \rho_{g_i}[X]. \tag{3.2}$$

Further studies on this line can be found recent papers He et al. (2015) and Wei (2017).

The following lemma is well known (cf. Kriele & Wolf (2014: Theorem 2.1, p. 33)).

Lemma 3.1 *If all* ρ_{g_w} $(w\in\langle a,b\rangle)$ *are monotone, positively homogeneous, subadditive and translation invariant, then* $\rho_g[X]$ *also has the corresponding properties. That is, if all* g_w $(w\in\langle a,b\rangle)$ *are coherent, then* $\rho_g[X]$ *is also coherent.*

Now we list three interesting special cases:

- If $[a,b)=[0,\infty)$, $g_i(x)=1-(1-x)^i, i\geq 1$ and $w_i\geq 0, \sum_i w_i=1$, then v in (3.2) is coherent since $g_i(x)$ is concave. As in Tsukahara (2009), if we take w_i from Bin v $(0<\theta<1)$, then $g_\theta(u)=u+u\theta-u^2\theta$. If we take

$$w_i=\frac{\theta^i}{\left(\mathrm{e}^\theta-1\right)i!}, \theta>0,$$

then

$$g_\theta(u)=\frac{\mathrm{e}^\theta\left(1-\mathrm{e}^{-\theta u}\right)}{\mathrm{e}^\theta-1}.$$

Also, if we take $w_i=(1-\theta)^{i-1}\theta$ $(0<\theta<1)$, the geometric distribution, then

$$g_\theta(u)=\frac{u}{u+\theta(1-u)},$$

which is the proportional odds distortion; see Example 2.1 in Cherubini & Mulinacci (2014).

- If $[a,b]=[0,1]$, $\rho_{g_w}=VaR_w[X]$ and $\mathrm{d}\psi(w)=\phi(w)\mathrm{d}w$, then $\rho_g[X]$ in (3.1) reduces to

$$\rho_\phi[X]=\int_0^1 VaR_w[X]\phi(w)\mathrm{d}w, \tag{3.3}$$

which is spectral risk measure (see Acerbi, 2002). Here ϕ is called a weighting function satisfies the following properties: $\phi\geq 0$, $\int_0^1\phi(w)\mathrm{d}w=1$. The following lemma gives a sufficient condition for $\rho_\phi[X]$ to be a coherent risk measure (cf. Kriele & Wolf (2014)).

Lemma 3.2 *Spectral risk measure* $\rho_\phi[X]$ *is coherent if* ϕ *is* (*almost everywhere*) *monotone increasing.*

Clearly, there exists a one-to-one correspondence between distortion function g and weighting function ϕ, namely, $g(1-t)=1-\int_0^t \phi(s)\mathrm{d}s$.

- If $[a,b]=[0,1]$, $\rho_{g_w}=TVaR_w[X]$ and $\psi=\mu$ is a probability measure on $[0,1]$, then $\rho_g[X]$ in (3.1) reduces to

$$\rho_\mu[X]=\int_0^1 TVaR_w[X]\mathrm{d}\mu(w), \tag{3.4}$$

which is the weighted $TVaR$ (see Cherny (2006)). $TVaR_p$ is a special weighted TVaR with $\mu(w)=1(w\geq p)$. According to Lemma 3.1, since each $TVaR_w[X]$ is coherent risk measure, the weighted TVaR is coherent risk measure. The weighted $TVaR$ can be rewritten as the form of spectral risk measure as following:

$$\begin{aligned}\rho_\mu[X]&=\int_0^1 TVaR_w[X]\mathrm{d}\mu(w)=\int_0^1\left(\frac{1}{1-w}\int_w^1 VaR_q[X]\mathrm{d}q\right)\mathrm{d}\mu(w)\\&=\int_0^1\left(VaR_q[X]\int_0^q\frac{1}{1-w}\mathrm{d}\mu(w)\right)\mathrm{d}q\ \text{(by the Fubini theorem)}\\&=\int_0^1 VaR_q[X]\phi(q)\mathrm{d}q=\int_0^1 VaR_{1-q}[X]\mathrm{d}g(q),\end{aligned}$$

where g is a function with $g(0)=0$ and satisfies

$$g'(1-q)=\phi(q)=\int_0^q\frac{1}{1-w}\mathrm{d}\mu(w).$$

Because $\phi(q)$ is increasing function of q, it follows from Lemma 3.2 that the weighted $TVaR$ $\rho_\mu[X]$ is coherent. Or, equivalently, $g'(q)$ is decreasing function of q, i.e. g is a concave function, moreover, g is increasing and

$$\begin{aligned}g(1)&=\int_0^1 g'(1-w)\mathrm{d}w=\int_0^1\mathrm{d}q\int_0^q\frac{1}{1-w}\mathrm{d}\mu(w)\\&=\int_0^1\frac{1}{1-w}\mathrm{d}\mu(w)\int_w^1\mathrm{d}q=\int_0^1\mathrm{d}\mu(w)=1.\end{aligned}$$

so that g is a concave distortion function, and hence the weighted TVaR is coherent.

Conversely, the distortion measure with concave distortion function g can be expressed by the weighted TVaR. In fact, note that $\phi(q)=g'(1-q)$ is monotone increasing, we define a measure $\nu([0,q])=\phi(q)$. As in the proof of Theorem 2.4 in Kriele & Wolf (2014) we have

$$\rho_g[X]=\int_0^1 TVaR_w[X]\mathrm{d}\mu(w),$$

where

$$\mathrm{d}\mu(w)=(1-w)\mathrm{d}\nu(w).$$

It can be shown that μ is a probability measure. In fact,

$$\int_0^1\mathrm{d}\mu(w)=\int_0^1\nu([0,w])\mathrm{d}w=\int_0^1\phi(w)\mathrm{d}w=\int_0^1 g'(w)\mathrm{d}w=1.$$

We now give some examples of interesting distortion functions and risk measures.

Example 3.2 If $w_1, w_2, w_3, w_4 \geq 0, \sum_{i=1}^{4} w_i = 1$, then

$$g_{\alpha\beta}(x) = w_1 \nu_\beta(x) + w_2 \nu_\alpha(x) + w_3 \psi_\beta(x) + w_4 \psi_\alpha(x),$$

is a distortion function, where $\nu_\beta, \nu_\alpha, \psi_\beta, \psi_\alpha$ are the distortion functions of TVaR and VaR at confidence levels β and α, respectively. Then the corresponding risk measure

$$\rho_{g_{\alpha\beta}}[X] = w_1 TVaR_\beta[X] + w_2 TVaR_\alpha[X] + w_3 VaR_\beta[X] + w_4 VaR_\alpha[X],$$

is called the GlueVaR risk measure, which were initially defined by Belles-Sampera et al. (2014a) (in the case $w_4 = 0$) and the closed-form expressions of GlueVaR for Normal, Log-normal, Student's t and Generalized Pareto distributions are provided. Two new proportional capital allocation principles based on GlueVaR risk measures are studied in Belles-Sampera et al. (2014b).

Although GlueVaR has superior mathematical properties than VaR and TVaR, however, the GlueVaR risk measure may also fails to recognize the differences between two risks. For example, consider two risks X and Y in Example 3.1, we have computed that $VaR_{0.95}[X] = VaR_{0.96}[X] = 100$, $VaR_{0.95}[Y] = VaR_{0.96}[Y] = 100$. $TVaR_{0.95}[X] = TVaR_{0.95}[Y] = 300$, $TVaR_{0.96}[X] = TVaR_{0.96}[Y] = 350$. So that when $\alpha = 0.95$ and $\beta = 0.96$, we have $\rho_{g_{\alpha\beta}}[X] = \rho_{g_{\alpha\beta}}[Y]$. Thus according to $\rho_{g_{\alpha\beta}}$, both X and Y bear the same risk! However, the maximal loss for Y (1100) is more than double than for loss X (500), clearly, risk Y is more risky than risk X.

Example 3.3 Let $\lambda \in [0,1]$, define a distortion function

$$g_\lambda(x) = \lambda g_0(x) + (1-\lambda) g(x),$$

where $g_0(x) := \mathbf{1}_{(x>0)}$ and g is an arbitrary distortion function. Note that g_λ can be rewritten as

$$g_\lambda(x) = \begin{cases} 0, & \text{if } x = 0, \\ \lambda + (1-\lambda) g(x), & \text{if } 0 < x \leq 1. \end{cases}$$

In particular, if $g(x) = x$, then we get the esssup-expectation convex combination distortion function with weight λ on the essential supremum, which was introduced in Bannör & Scherer (2014). The corresponding risk measure

$$\rho_{g_\lambda}[X] = \lambda \operatorname{esssup}(X) + (1-\lambda) E(X),$$

which is a convex combination of the essential supremum of X and the ordinary expectation of X w.r.t. P.

If

$$g(x) = \begin{cases} \left[\alpha(1-\beta) + \dfrac{1-\alpha}{1-p}\right] x, & \text{if } 0 \leq x \leq 1-p, \\ \alpha\beta + 1 - \alpha + \alpha(1-\beta) x, & \text{if } 1-p < x \leq 1, \end{cases}$$

where $0 \leq \alpha, \beta \leq 1$, $0 < p < 1$ are constants, then we get

$$\rho_{g_\lambda}[X]=\lambda\,\text{esssup}(X)+(1-\lambda)\rho_g[X],$$

where

$$\rho_g[X]=\alpha(1-\beta)E(X)+\alpha\beta VaR_p[X]+(1-\alpha)TVaR_p[X].$$

As illustration, we consider the risks X and Y in Example 3.1, if $p=0.95$, then $\rho_g[X]=\rho_g[Y]=50\alpha\beta-250\alpha+300$. It follows that

$$\rho_{g_\lambda}[X]=500\lambda+(1-\lambda)(50\alpha\beta-250\alpha+300)$$

and

$$\rho_{g_\lambda}[Y]=1100\lambda+(1-\lambda)(50\alpha\beta-250\alpha+300).$$

Taking $\lambda=\frac{1}{2},\alpha=1,\beta=0$, then $\rho_{g_\lambda}[X]=275$ and $\rho_{g_\lambda}[Y]=575$. Taking $\lambda=\alpha=\beta=\frac{1}{2}$, then $\rho_{g_\lambda}[X]=437.5$ and $\rho_{g_\lambda}[Y]=737.5$. Thus ρ_{g_λ} can measure the differences between two risks X and Y.

3.3. A Copula-Based Approach

If F is a distribution function on $[0,1]$, then F can be used as a distortion function. The well-known examples are the PH transform and the dual power transform and, more generally, the beta transform; see Wirch & Hardy (1999) for details. Similarly, we use this technique to a distribution function on $[0,1]^2$. We first introduce the notion of copula in the two-dimensional case.

Definition 3.1. A two-dimensional copula $C(u,v)$ is a bivariate distribution on the square $[0,1]^2$ having uniform margins. That is a function $C(u,v):[0,1]^2\to[0,1]$ is right-continuous in each variable such that $C(0,v)=C(u,0)=0$, $C(u,1)=u$, $C(1,v)=v$ and for $u_2>u_1,v_2>v_1$,

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\geq 0.$$

For an introduction to copula theory and some of its applications, we refer to Joe (1997), Denuit et al. (2005) and Nelsen (1999).

The well-known examples of copulas are $C^+(u,v)=\min(u,v)$, $C^\perp(u,v)=uv$ and $C^-(u,v)=\max(u+v-1,0)$ describing, respectively, comonotone dependence, independence and countermonotone dependence between two random variables X and Y. The copula version of the Fréchet-Hoeffding bounds inequality tells us

$$C^-(u,v)\leq C(u,v)\leq C^+(u,v).$$

Any copula has the following decomposition (cf. Yang et al. (2006))

$$C(u,v)=\alpha C^+(u,v)+\beta C^\perp(u,v)+\gamma C^-(u,v)+lG(u,v),$$

where $\alpha,\beta,\gamma,l\geq 0$, $\alpha+\beta+\gamma+l=1$. Here G is a copula which called the indecomposable part.

For a given two-dimensional copula $C(\cdot,\cdot)$, define one-parameter family $\{g_p\}_{p\in(0,1]}$ by $g_p(u)=\frac{C(u,p)}{p}$ or $\frac{C(p,u)}{p}$. Clearly, for each p, g_p is a right

continuous distortion function. For example,

- $g_p(u)=\dfrac{C^{\perp}(u,p)}{p}=u$ is continuous and both convex and concave, the associated risk measure is EX;
- $g_{1-p}(u)=\dfrac{C^{+}(u,1-p)}{1-p}=\min\left\{\dfrac{x}{1-p},1\right\}$ is continuous and concave, the corresponding risk measure is $TVaR_p$;
- $g_p(u)=\dfrac{C^{-}(u,p)}{p}=\max\left\{\dfrac{u+p-1}{p},0\right\}$ is continuous and convex, the corresponding risk measure is $\rho_{g_p}[X]=\dfrac{1}{p}\int_0^p VaR_q[X]\mathrm{d}q$.

Conversely, if $\{g_p\}_{p\in(0,1]}$ is a family of distortion functions, then, however, $C(u,p)=pg_P(u)$ is not a copula in general; A sufficient condition can be found in Cherubini & Mulinacci (2014).

We give below the most common bivariate copulas and the corresponding distortion functions.

- The Archimedean copulas:

$$C_{\Psi}(u,v)=\Psi^{[-1]}\left(\Psi(u)+\Psi(v)\right)$$

for some generator $\Psi:(0,1]\to R^{+}$ with $\Psi(1)=0$ such that Ψ is convex. The pseudo-inverse of Ψ is the function $\Psi^{[-1]}$ with $\mathrm{Dom}\Psi^{[-1]}=[0,\infty]$ and $\mathrm{Ran}\Psi^{[-1]}=[0,1]$ given by

$$\Psi^{[-1]}(t)=\begin{cases}\Psi^{-1}(t), & \text{if } 0\le t\le\Psi(0),\\ 0, & \text{if } \Psi(0)\le t\le\infty.\end{cases}$$

If Ψ is twice differentiable and $\Psi(0)=\infty$, then C_{Ψ} is componentwise concave if, and only if $\dfrac{1}{\Psi'}$ is concave, where Ψ' is the derivative of Ψ (see Dolati & Nezhad (2014)). Aa a consequence, we have

Theorem 3.1 *For each* $v>0$, *the distortion function*

$$g_v(u)=\frac{1}{v}\Psi^{-1}\left(\Psi(u)+\Psi(v)\right)$$

is concave if, and only if $\dfrac{1}{\Psi'}$ is concave.

Some examples of the Archimedean copulas and the corresponding distortion functions:

a) The Clayton copula with parameter $\alpha>0$ is generated by $\Psi(t)=\dfrac{1}{\alpha}\left(t^{-\alpha}-1\right)$ and takes the form

$$C_{\alpha}(u,v)=\left(u^{-\alpha}+v^{-\alpha}-1\right)^{-1/\alpha},\ \alpha>0.$$

The limit of $C_{\alpha}(u,v)$ for $\alpha\downarrow 0$ and $\alpha\uparrow\infty$ leads to independence and comonotonicity respectively (Nelsen, 1999). The corresponding distortion functions are

$$g_{\alpha,v}(u)=\frac{1}{v}\left(u^{-\alpha}+v^{-\alpha}-1\right)^{-1/\alpha},\ \alpha>0, v\in(0,1].$$

In particular, if $\alpha=1$, we get the proportional odds distortion which is found by Cherubini & Mulinacci (2014):

$$g_{1,v}(u)=\frac{u}{u+v(1-u)}.$$

Since $\left(\frac{1}{\Psi'(t)}\right)''=-\alpha(1+\alpha)t^{\alpha-1}<0$, $g_{\alpha,v}(u)$ is concave.

b) In case $\Psi(t)=-\ln\left(\frac{\mathrm{e}^{-\alpha t}-1}{\mathrm{e}^{-\alpha}-1}\right)$, we get the Frank copulas:

$$C_\alpha(u,v)=-\frac{1}{\alpha}\ln\left(1+\frac{\left(\mathrm{e}^{-\alpha u}-1\right)\left(\mathrm{e}^{-\alpha v}-1\right)}{\mathrm{e}^{-\alpha}-1}\right),\ \alpha\neq 0.$$

The corresponding distortion functions are

$$g_{\alpha,v}(u)=-\frac{1}{\alpha v}\ln\left(1+\frac{\left(\mathrm{e}^{-\alpha u}-1\right)\left(\mathrm{e}^{-\alpha v}-1\right)}{\mathrm{e}^{-\alpha}-1}\right).$$

Since $\left(\frac{1}{\Psi'(t)}\right)''=-\alpha\mathrm{e}^{\alpha t}$, $g_{\alpha,v}(u)$ is convex if $\alpha<0$ and concave if $\alpha>0$.

c) In case $\Psi(t)=t^{-1/\alpha}-1$, we get the Pareto survival copulas:

$$\hat{C}_\alpha(u,v)=\left(u^{-1/\alpha}+v^{-1/\alpha}-1\right)^{-\alpha},\ \alpha>0.$$

The corresponding distortion functions are

$$g_{\alpha,v}(u)=\frac{1}{v}\left(u^{-1/\alpha}+v^{-1/\alpha}-1\right)^{-\alpha}.$$

Since $\left(\frac{1}{\Psi'(t)}\right)''=-\left(1+\frac{1}{\alpha}\right)t^{\frac{1}{\alpha}-1}<0$, $g_{\alpha,v}(u)$ is concave.

d) In case $\Psi(t)=\ln\frac{1+\alpha(t-1)}{t}$, $\alpha\in[-1,1)$, we get the Ali-Mikhail-Haq copulas:

$$C_\theta(u,v)=\frac{uv}{1-\alpha(1-u)(1-v)}.$$

The corresponding distortion functions:

$$g_{\theta,v}(u)=\frac{u}{1-\alpha(1-u)(1-v)}.$$

Since $\left(\frac{1}{\Psi'(t)}\right)''=\frac{2\alpha}{\alpha-1}$, $g_{\theta,v}(u)$ is convex if $\alpha\in[-1,0]$ and concave if $\alpha\in[0,1)$.

e) In case $\Psi(t)=(-\ln t)^{\alpha}$, we get the Gumbel-Hougaard copulas:

$$C_{\alpha}(u,v)=\exp\left\{-\left((-\ln u)^{\alpha}+(-\ln v)^{\alpha}\right)^{1/\alpha}\right\},\ \alpha\geq 1.$$

The corresponding distortion functions:

$$g_{\alpha,v}(u)=\frac{1}{v}\exp\left\{-\left((-\ln u)^{\alpha}+(-\ln v)^{\alpha}\right)^{1/\alpha}\right\}.$$

The value $\alpha=1$ gives independence and the limit for $\alpha\uparrow\infty$ leads to comonotonicity. Since

$$\left(\frac{1}{\Psi'(t)}\right)''=\frac{\alpha-1}{\alpha t}(-\ln t)^{\alpha-2}\left(-1+\frac{2-\alpha}{\ln t}\right)\begin{cases}\leq 0, & \text{if } 0<t\leq 1,\ 1\leq\alpha\leq 2,\\ >0, & \text{if } \mathrm{e}^{2-\alpha}<t\leq 1,\ \alpha>2,\\ <0, & \text{if } 0<t<\mathrm{e}^{2-\alpha},\ \alpha>2,\end{cases}$$

$g_{\alpha,v}(u)$ is concave if $1\leq\alpha\leq 2$ and, if $\alpha>2$, $g_{\alpha,v}(u)$ is convex on $\left(\mathrm{e}^{2-\alpha},1\right]$ and concave on $\left(0,\mathrm{e}^{2-\alpha}\right)$.

Among other copulas, which do not belong to Archimedean family, it is worth to mention the following three copulas, given in the bivariate case as:

- The Farlie-Gumbel-Morgenstern copulas:

$$C_{\alpha}(u,v)=uv+uv\alpha(1-u)(1-v),\ \alpha\in[-1,1],$$

The corresponding distortion functions are

$$g_{\alpha,v}(u)=u+u\alpha(1-u)(1-v),\ \alpha\in[-1,1],v\in[0,1],$$

which is convex if $\alpha\in[-1,0]$ and concave if $\alpha\in[0,1]$.

- The Marshall-Olkin copulas:

$$C_{\alpha,\beta}(u,v)=\min\left\{u^{1-\alpha}v,uv^{1-\beta}\right\},\ \alpha,\beta\in[0,1].$$

Note that this copula is not symmetric for $\alpha\neq\beta$. The corresponding distortion functions are

$$g_{\alpha,\beta,v}(u)=\min\left\{u^{1-\alpha},uv^{-\beta}\right\},\ \alpha,\beta\in[0,1],v\in(0,1],$$

which is concave. In particular, $g_{\alpha,0,v}(u)=g_{0,\beta,v}(u)=u$,

$$g_{1,1,v}(u)=\min\left\{1,\frac{u}{v}\right\}.$$

- The normal copulas:

$$C_{\rho}(u,v)=\Phi_{\rho}\left(\Phi^{-1}(u),\Phi^{-1}(v)\right),$$

where Φ_{ρ} is a bivariate normal distribution with standard normal marginal distributions and the correlation coefficient $-1<\rho<1$, Φ^{-1} is the inverse function of the standard normal distribution. The corresponding distortion functions:

$$g_{\rho,v}(u)=\frac{1}{v}\Phi_{\rho}\left(\Phi^{-1}(u),\Phi^{-1}(v)\right).$$

4. Tail-Asymptotics for VaR

Subadditivity is an appealing property when aggregating risks in order to

preserve the benefits of diversification. Subadditivity of two risks is not only dependent on their dependence structure but also on the marginal distributions. Value at risk is one of the most popular risk measures, but this risk measure is not always subadditive, nor convex, exception of elliptically distributed risks. This family consists of many symmetric distributions such as the multivariate normal family, the multivariate Student-t family, the multivariate logistic family and the multivariate exponential power family, and so on. A recent development in the VaR literature concerns the subadditivity in the tails (see Daníelsson et al. (2013)) which demonstrate that VaR is subadditive in the tails of all fat tailed distributions, provided the tails are not super fat. However, in most practical models of interest the support of loss is bounded so that the maximum loss is simply finite. We will also show that for this class losses VaR is subadditive in the tail. We can illustrate the ideas here with three simple examples. In Examples 4.1 and 4.3, X and Y are independent, while in Example 4.2, X and Y are dependent.

Example 4.1 Let X and Y be i.i.d. random variables which are Bernoulli (0.02) distributed, i.e. $P(X=1)=1-P(X=0)=0.02$. Then

$$P(X+Y=0)=P(X=0)P(Y=0)=0.98^2=0.9604,$$

$$P(X+Y=1)=P(X=1)P(Y=0)+P(X=0)P(Y=1)=0.0392,$$

$$P(X+Y=2)=P(X=1)P(Y=1)=0.0004.$$

Dhaene et al. (2006) verified that VaR is not subadditive since $VaR_{0.975}[X]=VaR_{0.975}[Y]=0$ and $VaR_{0.975}[X+Y]=1$. However, for $p\geq 0.98$, $VaR_p[X]=VaR_p[Y]=1$ and

$$VaR_p[X+Y]=\begin{cases}1, & \text{if } 0.98\leq p<0.9996,\\ 2, & \text{if } p\geq 0.9996.\end{cases}$$

Thus for $p\geq 0.98$,

$$VaR_p[X+Y]\leq VaR_p[X]+VaR_p[Y].$$

Example 4.2 Suppose we have losses X and Y, both dependent on the same underlying Uniform (0,1) random variable U as follows.

$$X=\begin{cases}1000, & \text{if } U\leq 0.04\\ 0, & \text{if } U>0.04\end{cases}$$

$$Y=\begin{cases}0, & \text{if } U\leq 0.96\\ 1000, & \text{if } U>0.96\end{cases}$$

Note that

$$P(X+Y=0)=P(X=0,Y=0)=P(U>0.04,U\leq 0.96)=0.92,$$

$$P(X+Y=1000)=P(X=0,Y=1000)+P(X=1000,Y=0)=0.08.$$

Hardy (2006) found that $VaR_{0.95}[X]=VaR_{0.95}[Y]=0$, $VaR_{0.95}[X+Y]=1000$. Thus

$$VaR_{0.95}[X+Y]\geq VaR_{0.95}[X]+VaR_{0.95}[Y].$$

However, for any $\alpha > 0.96$, $VaR_\alpha[X] = VaR_\alpha[Y] = 1000$, $VaR_\alpha[X+Y] = 1000$. Thus,

$$VaR_\alpha[X+Y] \le VaR_\alpha[X] + VaR_\alpha[Y].$$

Example 4.3 Let X and Y be i.i.d. random variables which are Uniform (0,1) distributed. Then

$$F_{X+Y}(z) = \begin{cases} 0, & \text{if } z < 0, \\ \frac{1}{2}z^2, & \text{if } 0 \le z < 1, \\ 1 - \frac{1}{2}(2-z)^2, & \text{if } 1 \le z < 2, \\ 1, & \text{if } z \ge 2, \end{cases}$$

and for $p \in (0,1]$, $VaR_p[X] = VaR_p[Y] = p$,

$$VaR_p[X+Y] = \begin{cases} \sqrt{2p}, & \text{if } p \in \left(0, \frac{1}{2}\right], \\ 2 - \sqrt{2(1-p)}, & \text{if } p \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Thus for $p \in \left[\frac{1}{2}, 1\right]$,

$$VaR_\alpha[X+Y] \le VaR_\alpha[X] + VaR_\alpha[Y].$$

Generally, we have the following result.

Theorem 4.1 *If the risks* $X_1, X_2, \cdots, X_k$ *have finite upper endpoints, then*

$$\limsup_{p \to 1} \frac{VaR_p\left[\sum_{i=1}^k X_i\right]}{\sum_{i=1}^k VaR_p[X_i]} \le 1.$$

Proof The proof is very simple. Denote by $\operatorname{esssup}(X_i) = \sup\{x : P(X_i \le x) < 1\}$. Then $\operatorname{esssup}(X_i) < \infty$ and $P(X_i \le \operatorname{esssup}(X_i)) = 1, i = 1, 2, \cdots, k$, which lead to

$$P\left(\sum_{i=1}^k X_i \le \sum_{i=1}^k \operatorname{esssup}(X_i)\right) = 1.$$

Hence

$$\operatorname{esssup}\left(\sum_{i=1}^k X_i\right) \le \sum_{i=1}^k \operatorname{esssup}(X_i),$$

and the result follows.

Next theorem consider the random variables $X_1, X_2, \cdots, X_k$ that are not necessarily has finite upper endpoint, we first recall the notion of (extended) regularly varying function:

Definition 4.1 *A function f is called regularly varying at some point* x^- *(or* x^+*, respectively) with index* $\alpha \in R$ *if for all* $t > 0$,

$$\lim_{s \uparrow x} \frac{f(st)}{f(s)} = t^\alpha$$

(or $\lim_{s\downarrow x}\frac{f(st)}{f(s)}=t^{\alpha}$, respectively). We write $f\in R_{\alpha}^{x^-}$ ($f\in R_{\alpha}^{x^+}$, respectively). For $\alpha=0$ we say f is slowly varying; for $\alpha=-\infty$ rapidly varying.

Definition 4.2 *Assume that F is the distribution function of a nonnegative random. We say F belongs to the extended regular variation class, if there are some* $0<\alpha\le\beta<\infty$ *such that*

$$s^{-\beta}\le\liminf_{x\to\infty}\frac{\bar{F}(sx)}{\bar{F}(x)}\le\limsup_{x\to\infty}\frac{\bar{F}(sx)}{\bar{F}(x)}\le s^{-\alpha},\ \text{for all } s\ge 1,$$

or equivalently

$$s^{-\alpha}\le\liminf_{x\to\infty}\frac{\bar{F}(sx)}{\bar{F}(x)}\le\limsup_{x\to\infty}\frac{\bar{F}(sx)}{\bar{F}(x)}\le s^{-\beta},\ \text{for all } 0<s\le 1.$$

We write $F\in ERV(-\alpha,-\beta)$.

A standard reference to the topic of (extended) regular variation is Bingham et al. (1987) while main results are summarized by Embrechts et al. (1997).

Theorem 4.2 *We assume that* $X_1,X_2,\cdots,X_k$ *have the same absolutely continuous marginal distributions F with infinite upper endpoint.*

1) If

$$\lim_{z\to\infty}\frac{P\left(\sum_{i=1}^{k}X_i>z\right)}{P\left(X_1>\frac{z}{k}\right)}<1, \tag{4.1}$$

then

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^{k}X_i\right]}{\sum_{i=1}^{k}VaR_p\left[X_i\right]}<1; \tag{4.2}$$

(2) If

$$\lim_{z\to\infty}\frac{P\left(\sum_{i=1}^{k}X_i>z\right)}{P\left(X_1>\frac{z}{k}\right)}=1,$$

then

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^{k}X_i\right]}{\sum_{i=1}^{k}VaR_p\left[X_i\right]}=1;$$

3) If

$$\lim_{z\to\infty}\frac{P\left(\sum_{i=1}^{k}X_i>z\right)}{P\left(X_1>\frac{z}{k}\right)}>1,$$

then

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^{k}X_i\right]}{\sum_{i=1}^{k}VaR_p\left[X_i\right]}>1.$$

Proof We prove (1) only since the other cases follow immediately in the same way. Because all the marginal distributions are absolutely continuous, so we have for any $p \in (0,1)$,

$$P\left(X_1 > VaR_p\left[X_1\right]\right) = P\left(\sum_{i=1}^{k} X_i > VaR_p\left[\sum_{i=1}^{k} X_i\right]\right) = 1-p.$$

This, together with (4.1), implies that

$$\lim_{p\to 1} \frac{P\left(X_1 > VaR_p\left[X_1\right]\right)}{P\left(X_1 > \frac{1}{k} VaR_p\left[\sum_{i=1}^{k} X_i\right]\right)} < 1. \tag{4.3}$$

The absolute continuity of F implies that $\bar{F}$ is continuous and strictly monotone decreasing. Then from (4.3) we have

$$\lim_{p\to 1} \frac{VaR_p\left[X_1\right]}{\frac{1}{k} VaR_p\left[\sum_{i=1}^{k} X_i\right]} > 1,$$

which is (4.2). This completes the proof.

Example 4.4 Suppose that each X_i is regularly varying with index $-\alpha < 0$. When the X_i are mutually independent, it follows from (Feller, 1971: p. 279) that

$$\lim_{s\to\infty} \frac{P\left(\sum_{i=1}^{k} X_i > s\right)}{P\left(X_1 > \frac{s}{k}\right)} = \frac{k}{k^{\alpha}}.$$

Thus we get

$$\lim_{p\to 1} \frac{VaR_p\left[\sum_{i=1}^{k} X_i\right]}{\sum_{i=1}^{k} VaR_p\left[X_i\right]} \begin{cases} <1, & \text{if } \alpha > 1, \\ =1, & \text{if } \alpha = 1, \\ >1, & \text{if } \alpha < 1. \end{cases}$$

Suppose that the X_i are comonotonic, i.e. $P\left(X_1 = \cdots = X_k\right) = 1$, then

$$\lim_{s\to\infty} \frac{P\left(\sum_{i=1}^{k} X_i > s\right)}{P\left(X_1 > \frac{s}{k}\right)} = 1.$$

So that in the case $\alpha = 1$ the result for the independent and the comonotonic case are the same.

The following result generalizes the result in Jang & Jho (2011) in which all Y_i's are assumed identically distributed.

Theorem 4.3 *Suppose* $Y_1, \cdots, Y_k$ *are nonnegative random variables* (*but not necessarily independent or identically distributed.*) *If* Y_1 *has distribution* F *satisfying* $1 - F(x) = x^{-\alpha} L(x), \alpha > 0, x > 0$, *where* $L \in R_0^{\infty}$ *is slowly varying at infinity. If* $\frac{P\left(Y_i > x\right)}{\bar{F}(x)} \to c_i$ *and* $\frac{P\left(Y_i > x, Y_j > x\right)}{\bar{F}(x)} \to 0, i \neq j$, *as* $x \to \infty$, $i, j = 1, 2, \cdots, k$, *then*

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^k Y_i\right]}{\sum_{i=1}^k VaR_p\left[Y_i\right]}\begin{cases}<1, & \text{if } \alpha>1,\\ =1, & \text{if } \alpha=1,\\ >1, & \text{if } \alpha<1.\end{cases}$$

Proof It follows from Lemma 2.1 in Davis & Resnick (1996) that

$$\frac{P\left(\sum_{i=1}^k Y_i>x\right)}{1-F(x)}\to\sum_{i=1}^k c_i,\ \text{as } x\to\infty.$$

This leads to

$$\lim_{p\to 1}\frac{P\left(\sum_{i=1}^k Y_i>VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}{P\left(Y_1>VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}=\sum_{i=1}^k c_i. \tag{4.4}$$

Because

$$P\left(Y_1>VaR_p\left[Y_1\right]\right)=P\left(\sum_{i=1}^k Y_i>VaR_p\left[\sum_{i=1}^k Y_i\right]\right)=1-p.$$

Thus from (4.4) that

$$\lim_{p\to 1}\frac{P\left(Y_1>VaR_p\left[Y_1\right]\right)}{P\left(Y_1>VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}=\sum_{i=1}^k c_i,$$

which is equivalent to

$$\lim_{p\to 1}\frac{P\left(Y_1>VaR_p\left[Y_1\right]\right)}{P\left(Y_1>\left(\sum_{i=1}^k c_i\right)^{-\frac{1}{\alpha}}VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}=1.$$

This implies that

$$\lim_{p\to 1}\frac{VaR_p\left[Y_1\right]}{\left(\sum_{i=1}^k c_i\right)^{\frac{1}{\alpha}}VaR_p\left[\sum_{i=1}^k Y_i\right]}=1,$$

since $\overline{F}$ is continuous and strictly monotone decreasing. Note that $c_1=1$, $c_i^{1/\alpha}VaR_p\left[Y_1\right]\sim VaR_p\left[Y_i\right]$ (as $p\to 1$) and

$$\left(\sum_{i=1}^k c_i\right)^{1/\alpha}\begin{cases}<\sum_{i=1}^k c_i^{1/\alpha}, & \text{if } \alpha>1,\\ =\sum_{i=1}^k c_i, & \text{if } \alpha=1,\\ >\sum_{i=1}^k c_i^{1/\alpha}, & \text{if } \alpha<1,\end{cases}$$

completing the proof.

Remark 4.1 *The above result is obtained by* Embrechts et al. (2009) *for identically distributed and Archimedean copula dependent* Y_i*'s. However, our result can not obtained from their's due to the following fact: The famous Farlie-Gumbel-Morgenstern family, does not belong to Archimedean family, which has the form*

$$F(x,y)=F_1(x)F_2(y)\left(1+\alpha\overline{F_1}(x)\overline{F_2}(y)\right)$$

where F_1,F_2 are two distributions and $\alpha\in[-1,1]$ is a constant. When

$F_1 = F_2$, it satisfying $\dfrac{\bar{F}(x,x)}{\bar{F}_1(x)} \to 0$ as $x \to \infty$.

In the next theorem we consider the extended regularly varying instead of regularly varying.

Theorem 4.4 *Suppose* $Y_1, \cdots, Y_k$ *are nonnegative random variables with the common identical distribution function F. If* $F \in ERV(-\alpha, -\beta)$ *and* $\dfrac{P(Y_i > x, Y_j > x)}{\bar{F}(x)} \to 0, i \neq j$, *as* $x \to \infty$, $i, j = 1, 2, \cdots, k$, *then*

1) If $\beta < 1$,

$$\limsup_{p \to 1} \frac{VaR_p\left[\sum_{i=1}^k Y_i\right]}{\sum_{i=1}^k VaR_p[Y_i]} > 1;$$

2) If $\alpha > 1$,

$$\liminf_{p \to 1} \frac{VaR_p\left[\sum_{i=1}^k Y_i\right]}{\sum_{i=1}^k VaR_p[Y_i]} < 1;$$

3) If $\alpha = \beta = 1$,

$$\lim_{p \to 1} \frac{VaR_p\left[\sum_{i=1}^k Y_i\right]}{\sum_{i=1}^k VaR_p[Y_i]} = 1.$$

Proof It follows from Lemma 2.2 in Zhang et al. (2009) that

$$\frac{P\left(\sum_{i=1}^k Y_i > x\right)}{1 - F(x)} \to k, \text{ as } x \to \infty.$$

This leads to

$$\lim_{p \to 1} \frac{P\left(\sum_{i=1}^k Y_i > VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}{P\left(Y_1 > VaR_p\left[\sum_{i=1}^k Y_i\right]\right)} = k,$$

from which and using the same argument as that in the proof of Theorem 4.3 leads to

$$\lim_{p \to 1} \frac{P\left(Y_1 > VaR_p[Y_1]\right)}{P\left(Y_1 > VaR_p\left[\sum_{i=1}^k Y_i\right]\right)} = k. \tag{4.5}$$

If $\beta < 1$, then

$$\limsup_{p \to 1} \frac{\bar{F}\left(k^{-\frac{1}{\beta}} VaR_p\left[\sum_{i=1}^k Y_i\right]\right)}{\left(k^{-\frac{1}{\beta}}\right)^{-\beta} \bar{F}\left(VaR_p\left[\sum_{i=1}^k Y_i\right]\right)} \leq 1.$$

This and (4.5) imply that

$$\limsup_{p\to 1}\frac{\overline{F}\left(VaR_p\left[Y_1\right]\right)}{\overline{F}\left(k^{-\frac{1}{\beta}}VaR_p\left[\sum_{i=1}^{k}Y_i\right]\right)}\geq 1.$$

It follows that

$$\limsup_{p\to 1}\frac{k^{\frac{1}{\beta}}VaR_p\left[Y_1\right]}{VaR_p\left[\sum_{i=1}^{k}Y_i\right]}\leq 1. \tag{4.6}$$

Thus

$$\limsup_{p\to 1}\frac{kVaR_p\left[Y_1\right]}{VaR_p\left[\sum_{i=1}^{k}Y_i\right]}<1.$$

Similarly, if $\alpha>1$,

$$\liminf_{p\to 1}\frac{k^{\frac{1}{\alpha}}VaR_p\left[Y_1\right]}{VaR_p\left[\sum_{i=1}^{k}Y_i\right]}\geq 1 \tag{4.7}$$

and hence

$$\liminf_{p\to 1}\frac{kVaR_p\left[Y_1\right]}{VaR_p\left[\sum_{i=1}^{k}Y_i\right]}>1.$$

If $\alpha=\beta=1$, then by (4.6) and (4.7) one has

$$\lim_{p\to 1}\frac{kVaR_p\left[Y_1\right]}{VaR_p\left[\sum_{i=1}^{k}Y_i\right]}=1. \tag{4.8}$$

This ends the proof of Theorem 4.4.

To give applications of our results we employ extreme value theory techniques. A distribution function F (or the rv X) is said to belong to the Maximum Domain of Attraction (MDA) of the extreme value distribution H if there exist constants $c_n>0, d_n\in R$ such that $c_n^{-1}\left(\left\{\max\left(X_1,\cdots,X_n\right)\right\}-d_n\right)\xrightarrow{d}H$. We write $X\in MDA(H)$ or $F\in MDA(H)$. According to the Fisher-Tippett theorem (see Theorem 3.2.3 in Embrechts et al. (1997)) H belongs to one of the three standard extreme value distributions:

$$\text{Frechet type}:\Phi_\alpha(x)=\begin{cases}0, \text{ if } x\leq 0,\\ \exp\left\{-x^{-\alpha}\right\}, \text{ if } x>0,\end{cases}\quad \alpha>0.$$

$$\text{Weibull type}:\Psi_\alpha(x)=\begin{cases}\exp\left\{-(-x)^{\alpha}\right\}, \text{ if } x\leq 0,\\ 1, \text{ if } x>0,\end{cases}\quad \alpha>0.$$

$$\text{Gumbel type}:\Lambda(x)=\exp\left\{-\mathrm{e}^{-x}\right\}, x\in R.$$

Let x_F denote the right-endpoint of the support of F: $x_F=\inf\left\{x:F(x)=1\right\}$. Then we have the following results (see Embrechts et al. (1997: pp. 132-157)).

- Fréchet case: For some $\alpha>0$, $F\in MDA(\Phi_\alpha)\Leftrightarrow \overline{F}\in R_{-\alpha}^{\infty}$.

Examples are Pareto, Cauchy, Burr, Loggamma and Stable with index $\beta < 2$.

- Weibull case: For some $\alpha > 0$, $F \in MDA(\Psi_\alpha) \Leftrightarrow x_F < \infty$, $\bar{F}(x_F - 1/x) \in R_{-\alpha}^{\infty}$.

  Examples are Uniform and Beta distribution.

- Gumbel case: $F \in MDA(\Lambda_\alpha) \Leftrightarrow x_F \le \infty$ and there exists a positive measurable function *a* such that for $t \in R$

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x + ta(x))}{\bar{F}(x)} = \mathrm{e}^{-t}. \tag{4.9}$$

Examples are Exponential-like, Weibull-like, Gamma, Normal, Lognormal, Benktander-type-I and Benktander-type-II.

Remark 4.2 1) *For* $\alpha > 0$, *if* $X_1, X_2, \cdots, X_k \in MDA(\Psi_\alpha)$, *in view of Weibull case above they are all have finite supports. It follows from Theorem* 4.1, VaR_p *is subadditive for p is sufficiently close to* 1.

2) For $\alpha > 0$, if $X_1, X_2, \cdots, X_k \in MDA(\Phi_\alpha)$ and are identically distributed, $(-X_1, -X_2, \cdots, -X_k)$ has an Archimedean copula with generator ψ, which is regularly varying at 0 with index $-\beta < 0$. We apply (2.2) in Alink et al. (2004) and Definition 4.1 to obtain

$$\begin{aligned}
\lim_{z\to\infty} \frac{P\left(\sum_{i=1}^k X_i > z\right)}{P\left(X_1 > \frac{z}{k}\right)} &= \lim_{z\to\infty} \frac{P\left(\sum_{i=1}^k X_i > z\right)}{P(X_1 > z)} \frac{P(X_1 > z)}{P\left(X_1 > \frac{z}{k}\right)} \\
&= q_k(\beta, \alpha) \lim_{z\to\infty} \frac{P(X_1 > z)}{P\left(X_1 > \frac{z}{k}\right)} \\
&= \lim_{z\to\infty} \frac{P\left(X_1 > z\left(q_k(\beta,\alpha)\right)^{-1/\alpha}\right)}{P\left(X_1 > \frac{z}{k}\right)} \\
&= k^{-\alpha} q_k(\beta, \alpha) \begin{cases} <1, & \text{if } \alpha > 1, \\ =1, & \text{if } \alpha = 1, \\ >1, & \text{if } \alpha < 1, \end{cases}
\end{aligned}$$

where in the last step we have used Lemma 3.1(d) in Embrechts et al. (2009) which states that

$$\min\left\{k^\alpha, k\right\} \le q_k(\beta, \alpha) \le \max\left\{k^\alpha, k\right\}.$$

This, together with Theorem 4.2 we recover the result Theorem 2.5 in Embrechts et al. (2009).

3) If $X_1, X_2, \cdots, X_k \in MDA(\Lambda_\alpha)$ have common distribution *F*, $(-X_1, -X_2, \cdots, -X_k)$ has an Archimedean copula with generator ψ, which is regularly varying at 0 with index $-\beta < 0$. We apply (2.6) in Alink et al. (2004) to obtain

$$\lim_{z\to\infty} \frac{P\left(\sum_{i=1}^k X_i > z\right)}{P\left(X_1 > \frac{z}{k}\right)} = \mathrm{e}^{-\frac{1}{k} q_k^G(\beta)},$$

where

$$q_k^G(\beta)=\int_{\sum_{i=1}^k x_i\le 1}\frac{\mathrm{d}^k}{\mathrm{d}x_1\cdots\mathrm{d}x_k}\left(\sum_{i=1}^k \mathrm{e}^{-\beta x_i}\right)^{-1/\beta}\mathrm{d}x_1\cdots\mathrm{d}x_k.$$

The constant $q_k^G(\beta)\le \mathrm{e}^{\frac{1}{k}}$ is strictly increasing in β with

$$\lim_{\beta\to 0} q_k^G(\beta)=0,\ \lim_{\beta\to\infty} q_k^G(\beta)=\mathrm{e}^{\frac{1}{k}}.$$

For more details, see Alink et al. (2004) for the case $k=2$ and Chen et al. (2012) for general case. Thus by Theorem 4.2,

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^k X_i\right]}{\sum_{i=1}^k VaR_p\left[X_i\right]}\le 1.$$

In particular, when $\alpha\to\infty$,

$$\lim_{p\to 1}\frac{VaR_p\left[\sum_{i=1}^k X_i\right]}{\sum_{i=1}^k VaR_p\left[X_i\right]}=1$$

5. Tail-Subadditivity for Distortion Risk Measures

The tail-subadditivity property for GlueVaR risk measures were initially defined by Belles-Sampera et al. (2014a) and the milder condition of subadditivity in the tail region is investigated. Furthermore, they verified that a GlueVaR risk measure is tail-subadditive if its associated distortion function $k_{\beta,\alpha}^{h_1,h_2}(u)$ is concave in $[0,1-\alpha)$, where parameters α is confidence level and β is an extra confidence level such that $0\le\alpha\le\beta\le 1$ and,

$$k_{\beta,\alpha}^{h_1,h_2}(u)=\begin{cases}\dfrac{h_1}{1-\beta}u, & \text{if } 0\le u<1-\beta,\\ h_1+\dfrac{h_2-h_1}{\beta-\alpha}(u-1+\beta), & \text{if } 1-\beta\le u<1-\alpha,\\ 1, & \text{if } 1-\alpha\le u\le 1,\end{cases}$$

where h_1 and h_2 are two distorted survival probabilities at levels $1-\beta$ and $1-\alpha$, respectively. Here $0\le h_1\le h_2\le 1$. We note, however, from their proof to Theorem 6.1 that the result will hold for any distortion function that is concave in $[0,1-\alpha)$, not restricted to $k_{\beta,\alpha}^{h_1,h_2}(u)$. In this section we state the corresponding result without proof. As in Belles-Sampera et al. (2014a), for a given confidence level α, the tail region of a random variable Z is defined as $Q_{\alpha,Z}=\{w\,|\,Z(w)>s_\alpha\}\subseteq\Omega$, where $s_\alpha=\inf\{z\,|\,\bar{F}_Z(z)\le 1-\alpha\}$ is the α-quantile. For simplicity, we use the notation $S_Z(z):=\bar{F}_Z(z)$.

Theorem 5.1 *For a confidence level $\alpha\in[0,1]$ and two risks X,Y defined on the same probability space. If $Q_{\alpha,X}\cap Q_{\alpha,Y}\cap Q_{\alpha,X+Y}\neq\varnothing$ and g is a concave distortion function in $[0,1-\alpha]$, then the distortion risk measure ρ_g is tail-subadditive. That is*

$$\begin{aligned}&\int_{0\wedge m_\alpha}^{0}\left[g\left(S_{X+Y}(z)\right)-1\right]\mathrm{d}z+\int_{0\vee m_\alpha}^{\infty}g\left(S_{X+Y}(z)\right)\mathrm{d}z\\&\le\int_{0\wedge m_\alpha}^{0}\left[g\left(S_X(z)\right)-1\right]\mathrm{d}z+\int_{0\vee m_\alpha}^{\infty}g\left(S_X(z)\right)\mathrm{d}z\\&\quad+\int_{0\wedge m_\alpha}^{0}\left[g\left(S_Y(z)\right)-1\right]\mathrm{d}z+\int_{0\vee m_\alpha}^{\infty}g\left(S_Y(z)\right)\mathrm{d}z,\end{aligned}$$

where $m_\alpha=\sup\left\{s_\alpha(X),s_\alpha(Y),s_\alpha(X+Y)\right\}$.

Example 5.1 Consider the distortion functions associated with the Gumbel-Hougaard copulas (cf. Section 3.3):

$$g_{\alpha,v}(u)=\frac{1}{v}\exp\left\{-\left(\left(-\ln u\right)^\alpha+\left(-\ln v\right)^\alpha\right)^{1/\alpha}\right\}.$$

If $\alpha>2$, then $g_{\alpha,v}(u)$ is concave on $\left(0,\mathrm{e}^{2-\alpha}\right)$ and convex on $\left(\mathrm{e}^{2-\alpha},1\right]$. Thus the distortion risk measure $\rho_{g_{\alpha,v}(u)}$ is tail-subadditive.

6. Conclusion

In this paper, we give three methods to construct new class of distortion functions and distortion risk measures and then we investigate the tail asymptotics of distortion risk measures for the sum of possibly dependent risks with emphasis on VaR. We study the concept of tail subadditivity for distortion risk measures and give sufficient conditions for a distortion risk measure to be tail subadditive. The multivariate tail distortion risk measure and more applications of the risk measure will be investigated in the coming research.

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Author Contributions

Two authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

References

Acerbi, C. (2002). Spectral Measures of Risk: A Coherent Representation of Subjective Risk Aversion. *Journal of Banking Finance, 26,* 1505-1518. https://doi.org/10.1016/S0378-4266(02)00281-9

Alink, S., Löwe, M., & Wührich, M. V. (2004). Diversification of Aggregate Dependent Risks. *Insurance Mathematics & Economics, 35,* 77-95. https://doi.org/10.1016/j.insmatheco.2004.05.001

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1997). Thinking Coherently. *Risk, 10,* 68-71.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance, 9,* 203-228. https://doi.org/10.1111/1467-9965.00068

Bannör, K. F., & Scherer, M. (2014). On the Calibration of Distortion Risk Measures to Bid-Ask Prices. *Quantitative Finance, 14,* 1217-1228. https://doi.org/10.1080/14697688.2014.887220

Belles-Sampera, J., Guillén, M., & Santolino, M. (2014a). Beyond Value-at-Risk: GlueVaR Distortion Risk Measures. *Risk Analysis, 34,* 121-134.

Belles-Sampera, J., Guillén, M., & Santolino, M. (2014b). GlueVaR Risk Measures in Capital Allocation Applications. *Insurance: Mathematics and Economics, 58,* 132-137.

Bingham, N. H., Goldie, C. M., & Teugels, J. L. (1987). *Regular Variation*. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511721434

Chen, D., Mao, T., Pan, X., & Hu, T. Z. (2012). Extreme Value Behavior of Aggregate Dependent Risks. *Insurance: Mathematics and Economics, 50,* 99-108.

Cherny, A. S. (2006). Weighted VaR and Its Properties. *Finance and Stochastics, 10,* 367-393. https://doi.org/10.1007/s00780-006-0009-1

Cherubini, U., & Mulinacci, S. (2014). Contagion-Based Distortion Risk Measures. *Applied Mathematics Letters, 27,* 85-89. https://doi.org/10.1016/j.aml.2013.07.007

Daníelsson, J., Jorgensen, B. N., Samorodnitsky, G., Sarmad, M., & Vries, C. G. (2013). Fat Tails, VaR and Subadditivity. *Journal of Econometrics, 172,* 283-291. https://doi.org/10.1016/j.jeconom.2012.08.011

Davis, R. A., & Resnick, S. I. (1996). Limit Theory for Bilinear Processes with Heavy-Tailed Noise. *The Annals of Applied Probability, 6,* 1191-1210. https://doi.org/10.1214/aoap/1035463328

Denneberg, D. (1994). *Non-Additive Measure and Integral, Theory and Decision Library* (Vol. 27). Dordrecht: Kluwer Academic Publilshers. https://doi.org/10.1007/978-94-017-2434-0

Denuit, M., Dhaene, J., Goovaerts, M., & Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Hoboken, NJ: John Wiley & Sons, Ltd. https://doi.org/10.1002/0470016450

Dhaene, J., Kukush, A., Linders, D., & Tang, Q. (2012). Remarks on Quantiles and Distortion Risk Measures. *European Actuarial Journal, 2,* 319-328. https://doi.org/10.1007/s13385-012-0058-0

Dhaene, J., Vanduffel, S., Tang, Q., Goovaerts, M., Kaas, R., & Vyncke, D. (2006). Risk Measures and Comonotonicity: A Review. *Stochastic Models, 22,* 573-606. https://doi.org/10.1080/15326340600878016

Dolati, A., & Nezhad, A. D. (2014). Some Results on Convexity and Concavity of Multivariate Copulas. *Iranian Journal of Mathematical Sciences and Informatics, 9,* 87-100.

Embrechts, P., Klüppelberg, C., & Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Berlin: Springer-Verlag.

Embrechts, P., Nešehová, J., & Wührich, M. V. (2009). Additivity Properties for Value-at-Risk under Archimedean Dependence and Heavy-Tailedness. *Insurance: Mathematics and Economics, 44,* 164-169.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* (Vol. 2, 2nd ed.). New York, NY: Wiley.

Hardy, M. R., (2006). *An Introduction to Risk Measures for Actuarial Applications*. Schaumburg, IL: Society of Actuaries.

He, X. D., Jin, H., & Zhou, X. Y. (2015). Dynamic Portfolio Choice When Risk Is Measured by Weighted VaR. *Mathematics of Operations Research, 40,* 773-796. https://doi.org/10.1287/moor.2014.0695

Hua, L., & Joe, H. (2012). Tail Comonotonicity: Properties, Constructions, and Asymptotic Additivity of Risk Measures. *Insurance: Mathematics and Economics, 51,* 492-503.

Jang, J., & Jho, J. H. (2011). Asymptotic Super (Sub) Additivity of Value-at-Risk of Regularly Varying Dependent Variables. *Journal of Risk Management, 22,* 181-202. https://doi.org/10.21480/tjrm.22.1.201106.008

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. London: Chapman & Hall.

Kriele, M., & Wolf, J. (2014). *Value-Oriented Risk Management of Insurance Companies*. London: Springer-Verlag.

Lv, W., Pan, X., & Hu, T. (2013). Asymptotics of the Risk Concentration Based on the Tail Distortion Risk Measure. *Statistics & Probability Letters, 83,* 2703-2710. https://doi.org/10.1016/j.spl.2013.09.006

Mao, T., & Hu, T. (2013). Second-Order Properties of Risk Concentrations without the Condition of Asymptotic Smoothness. *Extremes, 16,* 383-405. https://doi.org/10.1007/s10687-012-0164-z

Mao, T., Lv, W., & Hu, T. (2012). Second-Order Expansions of the Risk Concentration Based on CTE. *Insurance: Mathematics & Economics, 51,* 449-456. https://doi.org/10.1016/j.insmatheco.2012.07.002

Nelsen, R. B. (1999). *An Introduction to Copulas*. New York, NY: Springer-Verlag.

Tsukahara, H. (2009). One-Parameter Families of Distortion Risk Measures. *Mathematical Finance, 19,* 691-705. https://doi.org/10.1111/j.1467-9965.2009.00385.x

Wang, S. S. (1996). Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin, 26,* 71-92. https://doi.org/10.2143/AST.26.1.563234

Wang, S. S. (2000). A Class of Distortion Operators for Pricing Financial and Insurance Risks. *Journal of Risk and Insurance, 67,* 15-36. https://doi.org/10.2307/253675

Wang, S., & Dhaene, J. (1998). Comonotonicity, Correlation Order and Premium Principles. *Insurance: Mathematics and Economics, 22,* 235-242.

Wang, S., & Young, V. R. (1998). Ordering Risks. Expected Utility Theory versus Yaari Dual Theory of Risk. *Insurance: Mathematics Economics, 22,* 145-161.

Wei, P. (2017). Risk Management with Weighted VaR. *Mathematical Finance,* 1-41.

Wirch, J. L., & Hardy, M. R. (1999). A Synthesis of Risk Measures for Capital Adequacy. *Insurance: Mathematics and Economics, 25,* 337-347.

Yaari, M. E. (1987). The Dual Theory of Choice under Risk. *Econometrica, 55,* 95-115. https://doi.org/10.2307/1911158

Yang, F. (2015). First- and Second-Order Asymptotics for the Tail Distortion Risk Measure of Extreme Risks. *Communications in Statistics-Theory and Methods, 44,* 520-532. https://doi.org/10.1080/03610926.2012.751116

Yang, J. P., Cheng, S., & Zhang, L. H. (2006). Bivariate Copula Decomposition in Terms of Comonotonicity, Countermonotonicity and Independence. *Insurance: Mathematics and Economics, 39,* 267-284. https://doi.org/10.1016/j.insmatheco.2006.02.015

Zhang, Y., Shen, X., & Weng, C. (2009). Approximation of the Tail Probability of Randomly Weighted Sums and Applications. *Stochastic Processes and their Applications, 119,* 655-675. https://doi.org/10.1016/j.spa.2008.03.004

Zhu, L., & Li, H. (2012). Tail Distortion Risk and Its Asymptotic Analysis. *Insurance: Mathematics & Economics, 51,* 115-121. https://doi.org/10.1016/j.insmatheco.2012.03.010