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Induced Dipoles in Electromagnetic Media in Motion

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Abstract

We present a non-relativistic approach to the equivalent polarization $\boldsymbol{P}_{eq} = \left(1/c^2\right)\boldsymbol{v}\times\boldsymbol{M}_{eq}$, that appears in a magnetized medium in motion. We apply an analogous method to that used by Panofsky and Phillips to calculate the symmetric effect, the equivalent magnetization that appears in a polarized dielectric in motion, $\boldsymbol{M}_{eq} = \boldsymbol{P}\times\boldsymbol{v}$, This method is based on a particular expression of Maxwell's equations and the application of the convective derivative. These authors argue, however, that the equivalent polarization can be obtained only with a relativistic approach. We show that with the same method, but with a different and equivalent expression of Maxwell's equations, this effect can also be calculated. In this way both effects can be considered relativistic effects to first order in v/c.

Keywords

Moving Media, Polarized Dielectric, Macroscopic Maxwell Equations, Induction of Magnetic Dipoles

1. Introduction

The question of the electrodynamics of moving bodies stimulated the emergence of the special theory of relativity. Therefore, the phenomena of electromagnetism can be seen as relativistic aspects to some order in v/c. One of these phenomena is the induction of magnetic dipoles by a polarized dielectric in motion other is the induction of electric dipoles by a magnetized medium in motion.

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Indeed this effect is related to the homopolar generator, which motivated Einstein to impose the principle of relativity.

In the first case, we have the "non-relativistic" treatment, that is, to first order in v/c, offered by Panofsky and Phillips [1] (P & P in what follows). However, in the second case they argue that only with a relativistic approach can it be obtained with an approximation to first order in v/c. This argument seems justified since the effect goes as v/c^2 in SI units. However, in Gaussian units both effects are of order v/c.

In the present article, we show that the second case can also be derived with the non-relativistic approach used by P & P [1]. This approach is based on a particular expression of the Maxwell equations for stationary media. In order to prove our point, we use the same method of P & P [1], but with a different equivalent expression of Maxwell's equations.

The relevance of choosing a particular and convenient expression of the Maxwell equations can be better appreciated considering different momentum balance equations that can be derived from these different expressions [2]. It must be emphasized that these different forms of the macroscopic Maxwell equations are all equivalent. With one of these forms of expressing Maxwell's equations [2] equations we can get for the motion of a current circuit, which is a typical magnetic dipole moment, the results obtained by Hnizdo [3] and later by Griffiths [4] using relativity. These authors conclude that it is necessary to introduce a new electromagnetic momentum density, the so-called hidden momentum. With our approach, this momentum density appears as a consequence of a balance equation derived from Maxwell's equations expressed in particular form.

2. Panofsky and Phillips's Approach

P & P write Maxwell's equations for stationary media in the form

$$\begin{aligned} &\nabla \cdot \boldsymbol{E} = \left(\frac{1}{\epsilon_0}\right)(\rho - \nabla \cdot \boldsymbol{P}), \\ &\nabla \cdot \boldsymbol{B} = 0, \\ &\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0, \\ &\nabla \times \boldsymbol{B} - \epsilon_0 \mu_0 \partial_t \boldsymbol{E} = \mu_0 \left(\boldsymbol{J} + \partial_t \boldsymbol{E} + \nabla \times \boldsymbol{M}\right). \end{aligned} \tag{1}$$

Then they proceed to extend these equations for media in motion in the non-relativistic limit, that is, media moving with velocity small compared with the velocity of light. From their analysis they conclude that, in this limit, only the Ampere-Maxwell law requires modification. Since the charge density ρ does not change in this limit, the convection current is

$$\boldsymbol{J}_{conv} = \boldsymbol{v}\left(\rho - \nabla \cdot \boldsymbol{P}\right). \tag{2}$$

However, the polarization current is, instead of $\left(\partial \boldsymbol{P}/\partial t\right)$, the material derivative of the polarization, $\left(D\boldsymbol{P}/Dt\right)$, which takes into account the changes due to motion and is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(v \cdot \nabla\right).$$

Using a vector identity where $(\boldsymbol{v}\cdot\nabla)$ appears, and for constant v, the material derivative results

$$\frac{D\boldsymbol{P}}{Dt}=\frac{\partial\boldsymbol{P}}{\partial t}+\nabla\times(\boldsymbol{P}\times\boldsymbol{v})+(\nabla\cdot\boldsymbol{P})\boldsymbol{v}. \tag{3}$$

In this way Maxwell's equations for media moving slowly can be written in the form

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E}=\left(\frac{1}{\epsilon_0}\right)(\rho-\nabla\cdot\boldsymbol{P}),\\
&\nabla\cdot\boldsymbol{B}=0,\\
&\nabla\times\boldsymbol{E}+\partial_t\boldsymbol{B}=0,\\
&\nabla\times\boldsymbol{B}-\epsilon_0\mu_0\partial_t\boldsymbol{E}=\mu_0\left(\boldsymbol{J}+\boldsymbol{v}(\rho-\nabla\cdot\boldsymbol{P})+\partial\boldsymbol{P}_t+\nabla\times(\boldsymbol{P}\times\boldsymbol{v})+(\nabla\cdot\boldsymbol{P})\boldsymbol{v}+\nabla\times\boldsymbol{M}\right).
\end{aligned} \tag{4}$$

For a non-magnetic medium the Ampére-Maxwell law reduces to

$$\nabla\times\boldsymbol{B}-\epsilon_0\mu_0\partial_t\boldsymbol{E}=\mu_0\left(\boldsymbol{J}+\rho\boldsymbol{v}+\frac{\partial\boldsymbol{P}}{\partial t}+\nabla\times(\boldsymbol{P}\times\boldsymbol{v})\right), \tag{5}$$

which expressed in terms of $\boldsymbol{D}$ becomes

$$\nabla\times\boldsymbol{B}=\mu_0\left(\boldsymbol{J}+\rho\boldsymbol{v}+\frac{\partial\boldsymbol{D}}{\partial t}+\nabla\times(\boldsymbol{P}\times\boldsymbol{v})\right). \tag{6}$$

Rearranging terms and in absence of free charge and current densities they finally obtain

$$\nabla\times\left[\boldsymbol{B}-\mu_0(\boldsymbol{P}\times\boldsymbol{v})\right]=\mu_0\left(\frac{\partial\boldsymbol{D}}{\partial t}\right), \tag{7}$$

concluding that [1] "…from the macroscopic point of view a moving polarizable dielectric is equivalent to a magnetized material of magnetized moment

$$\boldsymbol{M}_{eq}=\boldsymbol{P}\times\boldsymbol{v}''. \tag{8}$$

Their conclusion is based, though they do not say, in the identification of the constitutive relation

$$\boldsymbol{H}=\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{P}\times\boldsymbol{v}. \tag{9}$$

In considering the analogous case of the motion of a magnetized medium they affirm that the problem with induced or permanent magnetic dipoles is more complicated and requires the special theory of relativity for its treatment. They add that [1] "…the source of the electrostatic field will not become fully clear until the equations for moving media are modified to include permeable media. Unfortunately, this modification cannot be made in a reasonable way without introducing relativistic considerations." With these relativistic considerations, they show that if an observer moves with velocity v relative to a medium with magnetization $\boldsymbol{M}$, he will observe an equivalent electric moment given by

$$\boldsymbol{P}_{eq}=\frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{M}, \tag{10}$$

and therefore, the source of the electrostatic field is given by

$$\nabla \cdot \boldsymbol{E} = -\frac{1}{\epsilon_0} \nabla \cdot \boldsymbol{P}_{eq}. \tag{11}$$

That is,

$$\nabla \cdot \boldsymbol{E} = -\frac{1}{\epsilon_0} \frac{1}{c^2} \boldsymbol{v} \cdot \nabla \times \boldsymbol{M}, \tag{12}$$

and introducing the magnetization current

$$\boldsymbol{J}_{mag} = \nabla \times \boldsymbol{M} \tag{13}$$

we can write “Equation (12)” as

$$\nabla \cdot \boldsymbol{E} = -\frac{1}{\epsilon_0} \frac{1}{c^2} \boldsymbol{v} \cdot \boldsymbol{J}_{mag}, \tag{14}$$

We have in this way a new induced charge density resulting from the induced magnetization. As advanced by these authors, they obtain Equation (10) when they develop the covariant formulation of electrodynamics. What we intend in the present work is to show that the structure of macroscopic electromagnetism permits to derive this polarization of a magnetic medium in motion using the same method used by P & P.

It is also noteworthy that the results obtained by P & P [1] depend on the particular form in which Maxwell’s equations are expressed in terms of fields $\boldsymbol{E}$, $\boldsymbol{B}$, and polarizations $\boldsymbol{P}$ and $\boldsymbol{M}$. If the analysis is made with the most common expression of Maxwell’s equations in terms of fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$, it is not clear how to proceed. Also, it is apparent the asymmetry between electric and magnetic polarizations resulting from Maxwell’s equations expressed as in “Equation (1)”.

To follow P & P’s approach we need, therefore, to write Maxwell’s equations in a form where $\partial \boldsymbol{M}/\partial t$ appears and substitute it with the convective derivative. This can be done only in Faraday’s law. Since we are considering non-polarizable media, we can change in Faraday’s law $\boldsymbol{E}$ with $\boldsymbol{D}$ given the constitutive relation

$$\frac{\boldsymbol{D}}{\epsilon_0} = \boldsymbol{E}. \tag{15}$$

3. A Non-Relativistic Approach

Since there are several ways in which the Maxwell equations can be written, one can ask if there is a particular form of expressing the Maxwell equations such that, following the method of P & P [1], result “Equation (10)” can be obtained without recurring to relativistic arguments. We can show that this can be done as follows.

We begin by noting that Faraday’s law can be expressed in terms of the field $\boldsymbol{H}$, magnetization $\boldsymbol{M}$ and electric displacement $\boldsymbol{D}$. In this way Faraday’s law can be written as

$$\nabla \times \frac{1}{\epsilon_0} \boldsymbol{D} + \mu_0 \partial_t \boldsymbol{H} = -\mu_0 \partial_t \boldsymbol{M}. \tag{16}$$

It is important to note that the constitutive relation

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \tag{17}$$

separates the material part from the field $\boldsymbol{B}$ part, and therefore the convective derivative is applied only to $\boldsymbol{M}$. This is analogous to of P & P's approach, who separate the polarization $\boldsymbol{P}$ from the field $\boldsymbol{E}$, and the convective derivative is applied only to $\boldsymbol{P}$

Following the method of P & P, we substitute $\partial \boldsymbol{M}/\partial t$ with the convective derivative, applying it only to matter, and not to the field $\boldsymbol{B}/\mu_0$,

$$\frac{D\boldsymbol{M}}{Dt} = \frac{\partial \boldsymbol{M}}{\partial t} + \nabla \times (\boldsymbol{M} \times \boldsymbol{v}) + (\nabla \cdot \boldsymbol{M})\boldsymbol{v}, \tag{18}$$

resulting

$$\nabla \times \frac{1}{\epsilon_0}\boldsymbol{D} + \mu_0 \partial_t \boldsymbol{H} = -\mu_0\left(\frac{\partial \boldsymbol{M}}{\partial t} + \nabla \times (\boldsymbol{M} \times \boldsymbol{v}) + (\nabla \cdot \boldsymbol{M})\boldsymbol{v}\right). \tag{19}$$

It is better to express this equation as Ampere's Law "Equation (6)"

$$\nabla \times \boldsymbol{D} + \epsilon_0\mu_0 \partial_t \boldsymbol{H} = -\epsilon_0\mu_0\left(\frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times (\boldsymbol{M} \times \boldsymbol{v}) + (\nabla \cdot \boldsymbol{M})\boldsymbol{v}\right). \tag{20}$$

Then

$$\nabla \times \left(\boldsymbol{D} + \epsilon_0\mu_0 (\boldsymbol{M} \times \boldsymbol{v})\right) = -\epsilon_0\mu_0\left(\frac{\partial \boldsymbol{H}}{\partial t} + \frac{\partial \boldsymbol{M}}{\partial t} + (\nabla \cdot \boldsymbol{M})\boldsymbol{v}\right). \tag{21}$$

Now, taking into account the constitutive relation analogous to "Equation (14)",

$$\epsilon_0 \boldsymbol{E} = \boldsymbol{D} - \boldsymbol{P}, \tag{22}$$

we identify in "Equation (21)"

$$\epsilon_0 \boldsymbol{E} = \boldsymbol{D} + \epsilon_0\mu_0 (\boldsymbol{M} \times \boldsymbol{v}), \tag{23}$$

and since

$$\epsilon_0\mu_0 = \frac{1}{c^2}, \tag{24}$$

then we can see that

$$\boldsymbol{P}_{eq} = -\frac{1}{c^2}\boldsymbol{M} \times \boldsymbol{v}. \tag{25}$$

This is the result obtained by P & P [1], "Equation (10)", with relativistic arguments.

As has been pointed elsewhere [2], it is possible to write Maxwell's equations in different but equivalent forms, depending on what fields and polarizations are used to express them. We have thus the particular form, "Equation (1)", used by P & P [1] to obtain their result. However, an equivalent form of Maxwell's equations is

$$\nabla \cdot \boldsymbol{E} = \left(\frac{1}{\epsilon_0}\right)(\rho - \nabla \cdot \boldsymbol{P}),$$

$$\begin{aligned}
&\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}, \\
&\nabla \times \boldsymbol{E} + \mu_0 \partial_t \boldsymbol{H} = -\mu_0 \partial_t \boldsymbol{M}, \\
&\nabla \times \boldsymbol{H} - \epsilon_0 \partial_t \boldsymbol{E} = \boldsymbol{J} + \partial_t \boldsymbol{P}.
\end{aligned} \tag{26}$$

This form of Maxwell's equations exhibits more symmetry than that used by P & P [1] and, as we will show, allows to display the symmetry between the electric and magnetic induced polarizations appearing in electromagnetic media in motion.

Maxwell's equation can also be written expressing directly Faraday's law in terms of the electric displacement $\boldsymbol{D}$, in "Equation (26)", as

$$\begin{aligned}
&\nabla \cdot \boldsymbol{E} = \left(\frac{1}{\epsilon_0}\right)(\rho - \nabla \cdot \boldsymbol{P}), \\
&\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}, \\
&\nabla \times \boldsymbol{D} + \epsilon_0\, \mu_0 \partial_t \boldsymbol{H} = -\mu_0 \partial_t \boldsymbol{M} + \nabla \times \boldsymbol{P}, \\
&\nabla \times \boldsymbol{H} - \epsilon_0 \partial_t \boldsymbol{E} = \boldsymbol{J} + \partial_t \boldsymbol{P}.
\end{aligned} \tag{27}$$

Indeed, there are several ways of writing Maxwell's equations, all equivalent, depending on the fields and polarizations used [2].

4. Symmetry between Induced Electric and Magnetic Dipoles in Media in Motion

We can now use these results in the first two of "Equation (26)", that is, the electric and magnetic Gauss laws. If we use "Equation (8)"

$$\nabla \cdot \boldsymbol{E} = \left(\frac{1}{\epsilon_0}\right)\left(\rho - \nabla \cdot \left(\boldsymbol{P} + \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{M}\right)\right), \tag{28}$$

$$\nabla \cdot \boldsymbol{H} = -\nabla \cdot (\boldsymbol{M} - \boldsymbol{v} \times \boldsymbol{P}), \tag{29}$$

which with the identities

$$\nabla \cdot (\boldsymbol{v} \times \boldsymbol{M}) = -\boldsymbol{v} \cdot (\nabla \times \boldsymbol{M}), \tag{30}$$

$$\nabla \cdot (\boldsymbol{v} \times \boldsymbol{P}) = -\boldsymbol{v} \cdot (\nabla \times \boldsymbol{P}), \tag{31}$$

can be transformed into the form

$$\nabla \cdot \boldsymbol{E} = \left(\frac{1}{\epsilon_0}\right)\left(\rho - \nabla \cdot \boldsymbol{P} - \boldsymbol{v} \cdot \frac{1}{c^2}(\nabla \times \boldsymbol{M})\right), \tag{32}$$

$$\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M} - \boldsymbol{v} \cdot (\nabla \times \boldsymbol{P}). \tag{33}$$

We observe that besides the "magnetic charge density", $-\nabla \cdot \boldsymbol{M}$, there appear another contribution to the source of $\boldsymbol{H}$, $\boldsymbol{v} \cdot (\nabla \times \boldsymbol{P})$, analogous to the source of the electric field given in "Equation (12)", $-\boldsymbol{v} \cdot (\nabla \times \boldsymbol{M})$, and already considered by P & P. Thus, we find that the term $\nabla \times \boldsymbol{P}$ is analogous to $\nabla \times \boldsymbol{M}$, the magnetization current.

We can therefore define a new electric current density

$$\boldsymbol{J}_{new\ elec} = \nabla \times \boldsymbol{P}, \tag{34}$$

analogous to the magnetization current $\nabla \times \boldsymbol{M}$.

With definition, "Equation (34)", the equations for the sources of $\boldsymbol{E}$ and $\boldsymbol{H}$, "Equation (32)" and "Equation (33)", can be written in the form

$$\nabla\cdot\boldsymbol{E}=\left(\frac{1}{\epsilon_0}\right)\left(\rho-\nabla\cdot\boldsymbol{P}-\boldsymbol{v}\cdot\frac{1}{c^2}\boldsymbol{J}_{mag}\right), \tag{35}$$

$$\nabla\cdot\boldsymbol{H}=-\nabla\cdot\boldsymbol{M}-\boldsymbol{v}\cdot\boldsymbol{J}_{new\ elec}. \tag{36}$$

Now the symmetry between the electric and magnetic cases is evident.

The relevance of choosing conveniently the fields and polarizations in writing Maxwell's equations is better appreciated in the various balance equations that can be deduced from these different ways of writing these equations. Thus, from "Equation (1)" we can deduce the balance equation [2]

$$\begin{aligned}&\nabla\cdot\left(\epsilon_0\boldsymbol{EE}+\epsilon_0\boldsymbol{BB}-\frac{1}{2}\boldsymbol{I}\left(\epsilon_0E^2+\mu_0B^2\right)\right)-\epsilon_0\partial_t\left(\boldsymbol{E}\times\boldsymbol{B}\right)\\&=\left(\rho-\nabla\cdot\boldsymbol{P}\right)\boldsymbol{E}+\left(\boldsymbol{J}+\partial_t\boldsymbol{P}+c\nabla\times\boldsymbol{M}\right)\times\boldsymbol{B}.\end{aligned} \tag{37}$$

On the other hand, from "Equation (26)" we can deduce the balance equation [2]

$$\begin{aligned}&\nabla\cdot\left\{\epsilon_0\boldsymbol{EE}+\mu_0\boldsymbol{BB}-\frac{1}{2}\left(\epsilon_0E^2+\mu_0H^2\right)\right\}-\epsilon_0\mu_0\partial_t\left(\boldsymbol{E}\times\boldsymbol{H}\right)\\&=\rho\boldsymbol{E}+\mu_0\boldsymbol{J}\times\boldsymbol{H}-\left(\nabla\cdot\boldsymbol{P}\right)\boldsymbol{E}-\mu_0\left(\nabla\cdot\boldsymbol{M}\right)\boldsymbol{H}+\mu_0\left(\partial_t\boldsymbol{P}\right)\times\boldsymbol{H}-\mu_0\epsilon_0\left(\partial_t\boldsymbol{M}\right)\times\boldsymbol{E}.\end{aligned} \tag{38}$$

They look very different, but both are consequences of the Maxwell equations expressed in different ways in terms of the fields and polarizations. Let us note that in Equation (38) there appear a new force density,

$$\boldsymbol{f}_{new\ hidden\ mom}=-\epsilon_0\mu_0\left(\partial_t\boldsymbol{M}\right)\times\boldsymbol{E}. \tag{39}$$

Some years ago, Hnizdo [3] and then Griffiths [4] studied the problem of a magnetic dipole in motion, modelled as a rectangular current loop, and obtained with relativistic arguments the induced electric dipole moment, "Equation (25)". However, they have to consider a new force density, associated to the so-called hidden electromagnetic momentum, discussed since long ago by Shockley and James [5], Costa de Beauregard [6], Coleman and van Vleck [7]. It must be emphasized that this "hidden electromagnetic momentum" is already contained in the momentum balance equation, "Equation (38)", where it appears naturally.

5. Conclusions

We have shown that the non-relativistic method applied by P & P [1] to show that a polarized medium in motion is equivalent to a magnetized medium can also be used to show that a magnetized medium in motion is equivalent to a polarized medium. This is done by writing Maxwell's equations in a particular form, equivalent but different to the form used by P & P [1].

Thus, we find that different expressions of Maxwell's equations in terms of the fields and polarizations can be used for different purposes. In this case, Maxwell's equations expressed as in "Equation (1)" are convenient to show to first order in v/c, that a polarized medium in motion is equivalent to a magnetized medium, while written as in "Equation (25)" are convenient to show to first order in v/c, that a magnetized medium in motion is equivalent to a polarized medium.

Following this approach of writing Maxwell's equations in different forms, permits also to show how more symmetry can be displayed, between electric and magnetic phenomena.

Finally, we remark that in one of the momentum balance equations that can be derived from Maxwell's equations [2], there appear the so-called hidden momentum, introduced by, Schocley and James [5], Costa de Beauregard [6], and Coleman and Van Vleck [7] and considered by Hnizdo [3], and Griffiths [4] necessary to study the motion of a magnetic dipole modelled by a loop of current. Indeed, there are other expressions of Maxwell's equations from which balance equations can be derived, appearing interesting force densities and stress tensors.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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