Journal of Electromagnetic Analysis and Applications, 2018, 10, 131-142
http://www.scirp.org/journal/jemaa
ISSN Online: 1942-0749
ISSN Print: 1942-0730

How Does Arise the Force on a Dielectric Slab in a Parallel Plate Capacitor?

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How to cite this paper: Campos-Flores, I., Jiménez-Ramírez, J.-L. and Roa-Neri, J.-A.-E. (2018) How Does Arise the Force on a Dielectric Slab in a Parallel Plate Capacitor? *Journal of Electromagnetic Analysis and Applications*, **10**, 131-142.
https://doi.org/10.4236/jemaa.2018.107010

Received: July 4, 2018
Accepted: July 27, 2018
Published: July 30, 2018

Abstract

We address the problem of the force exerted on a dielectric slab partially introduced into a charged parallel plate capacitor. This elementary problem is usually solved calculating this force as the gradient of an energy and attributing its origin to the action of the fringing field outside the capacitor on the dipoles of the dielectric slab. By applying Maxwell's theory of electromagnetic stresses, we show that this force acts at the interface dielectric-vacuum and originates from the action of these stresses. This approach permits to obtain the force as a volume integration of a force density, or as a surface integral of a stress tensor. This force density and the stress tensor are part of a momentum balance equation derived from Maxwell's equations.

Keywords

Classical Electromagnetism, Maxwell Equations, Electrostatics, Maxwell Balance Equation, Maxwell Stress Tensor

1. Introduction

One of the simplest interactions of electromagnetic fields with matter is the interaction of the electrostatic field of a parallel plate capacitor with a dielectric slab partially inserted in it (see Figure 1). This interaction manifests as a pulling of the slab into the capacitor. How does this force arise? Our aim is to trace the origin of this force, establishing that it arises from Maxwell's stresses, rather than from the action of the fringing field on the dipoles. This is done with a force density that does not appear in the usual approaches.

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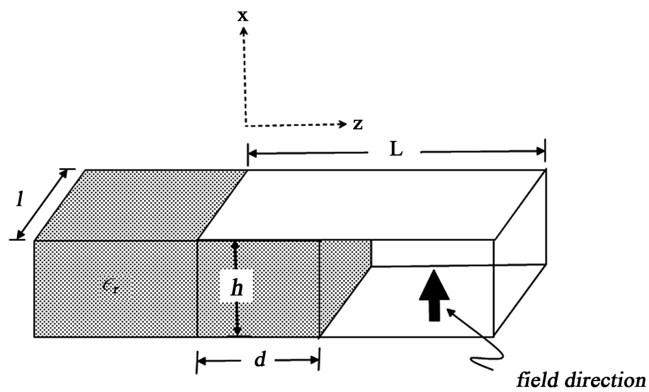

Figure 1. Parallel-plate capacitor with a dielectric slab partially inserted.

This phenomenon is discussed in elementary and intermediate texts on electromagnetism, for example [1] [2] [3], and articles [4] [5] [6] [7]. We review some of these calculations and analyze their consistency. The generally accepted interpretation is that this force results from the interaction of the fringing field with the dipoles of the dielectric [1] [2] [3] [4] [5] [7]. However, calculating the force with this approach is practically impossible since the fringing field cannot be calculated. Furthermore, in the usual calculations [1] [2] [3] [4] [5] [7], the fringing field is neglected, and the uniform field at the interface is used, contradicting the view that it is the fringing field the origin of the force.

In this work, we calculate the force, first in the usual way as a gradient of an energy density but taking into account the discontinuity of the permittivity at the interface dielectric-vacuum. This calculation shows that the force acts at the interface. We then calculate the force with a volume integration of a force density and a surface integration of the Maxwell electrostatic stress tensor. Our results show that this force arises from Maxwell's electrostatic stress, and not from the fringing field.

2. Where Does the Force Acts?

There are several ways of calculating the force, usually as a gradient of an energy according to how the energy is expressed. An elementary treatment is to compare the energies before and after the introduction of the dielectric slab. At constant field, that is, at constant potential difference, the energy is greater after the dielectric is introduced, and the difference in energies is interpreted as the work done on the dielectric.

The force can also be obtained assuming constant charges on the capacitor. In this case the mechanical work is at expense of the field energy, but is the same as in the case of constant electric field, and therefore the force is the same. If the force is calculated with constant charges the force is

$$f = -\left(\frac{\partial U}{\partial z}\right)_Q . \tag{1}$$

For linear media Maxwell's theory gives an energy density for the field

$$u = \frac{1}{2}\epsilon_0 \epsilon_r E^2, \tag{2}$$

then, for a dielectric slab partially introduced into the capacitor, neglecting fringing effects, the energy is [8]

$$U = \frac{1}{2}\epsilon_0 \epsilon_r E^2 (lh) d + \frac{1}{2}\epsilon_0 E^2 (lh)(L-d). \tag{3}$$

Therefore the force is

$$F_z = \frac{1}{2}(\epsilon_r - 1)\epsilon_0 E^2 (lh). \tag{4}$$

We can see that the force has the structure of a difference of pressures multiplied by the area of the interface. This force is also experimentally supported [9]. But, where does this force acts? This way of calculating the force on the dielectric says nothing about where it acts and how it arises.

It has been proposed [1] [2] [3] [5] [7] that this force arises from the action of the fringing field on dipoles in the dielectric. This seems natural since only a non-uniform electrostatic field can exert a net force on a dipole. Thus, we can use the force density

$$\boldsymbol{f} = (\boldsymbol{P} \cdot \nabla)\boldsymbol{E}, \tag{5}$$

where $\boldsymbol{E}$ is a non-uniform field, as the fringing field.

However, sometimes this force density is replaced by the force density

$$\boldsymbol{f} = -\nabla(\boldsymbol{P} \cdot \boldsymbol{E}), \tag{6}$$

taking them as equivalent. The energy $-\boldsymbol{P} \cdot \boldsymbol{E}$ results from transforming the change in energy in introducing a dielectric into an electrostatic field, which with the positive gradient of energy that is used with a constant field gives the correct force on the dielectric slab. This deserves some attention, since the first, "Equation (5)", is zero in a uniform electrostatic field, while the second, "Equation (6)" may be different from zero in a uniform field. We discuss below the equivalence of these expressions and show that they are not equivalent. Also, by taking into account that the field energy density is discontinuous at the interface, we show that the force acts at the interface. This leads us to consider Maxwell's stresses as the origin of this force.

The familiar force density "Equation (6)" gives the known force if the discontinuity in polarization $\boldsymbol{P}$ is taken into account; this can be seen as follows.

The polarization is

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}, \tag{7}$$

and we must take into account that the susceptibility is discontinuous,

$$\chi_e(z) = \chi_e \Theta(d-z). \tag{8}$$

Θ is Heviside's distribution and d is the position of the interface inside the capacitor. Then

$$\boldsymbol{f} = -\frac{1}{2}\left(\partial_z \chi_e \Theta(d-z) E^2\right)\hat{\boldsymbol{k}}. \tag{9}$$

Making the derivative results in

$$\boldsymbol{f} = -\frac{1}{2}\chi_e E^2 \delta(d-z)\hat{\boldsymbol{k}}, \tag{10}$$

and the usual force can be obtained integrating this force density over the volume of the capacitor. We can see then that the energy method permits to determine where the force is exerted.

It is not clear the relation between "Equation (5)", deduced from the force a non-uniform electric field exerts on an electric dipole, and "Equation (6)", which may be applied in the case of a uniform electrostatic field. Sometimes [3] [10] these force densities "Equation (5)" and "Equation (6)" are taken as equivalent. However, if we take into account the identity

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\cdot\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}), \tag{11}$$

"Equation (6)" takes the form

$$\boldsymbol{f} = -\frac{1}{2}\left\{(\boldsymbol{P}\cdot\nabla)\boldsymbol{E} + (\boldsymbol{E}\cdot\nabla)\boldsymbol{P} + \boldsymbol{P}\times(\nabla\times\boldsymbol{E}) + \boldsymbol{E}\times(\nabla\times\boldsymbol{P})\right\}. \tag{12}$$

We observe that in the case of a uniform electrostatic field the three first terms on the right of "Equation (12)" are zero, and therefore the relation between "Equation (5)" and "Equation (6)" is rather given by

$$\boldsymbol{f} = -\frac{1}{2}\boldsymbol{E}\times(\nabla\times\boldsymbol{P}). \tag{13}$$

The force density "Equation (13)" is therefore equivalent to the force density Equation (6), and must give the correct result when the discontinuity in the polarization is taken into account. We can show this as follows.

We have the constitutive relation

$$\boldsymbol{P} = \epsilon_0\chi_e\boldsymbol{E}. \tag{14}$$

Since the polarization $\boldsymbol{P}$ is in the same direction as $\boldsymbol{E}$ and is different from zero only where there is dielectric, we have that the electric susceptibility is

$$\chi_e(z) = \chi_e\Theta(d-z), \tag{15}$$

where Θ is the usual Heaviside distribution. Then the polarization is

$$\boldsymbol{P}(z) = P_x\Theta(d-z)\hat{\boldsymbol{i}}, \tag{16}$$

where $\hat{\boldsymbol{i}}$ is a unit vector in the x direction.

Therefore, the force density "Equation (13)" results

$$\boldsymbol{f} = -\frac{1}{2}E\left(\partial_z P_x(z)\right)\hat{\boldsymbol{k}} \tag{17}$$

($\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are the usual Cartesian unit vectors).

With the susceptibility relation (15) we have

$$P_x(x) = \epsilon_0\chi_e\Theta(d-z)E, \tag{18}$$

and "Equation (13)" becomes

$$\boldsymbol{f} = -\frac{1}{2}\epsilon_0\chi_e E^2\delta(d-z)\hat{\boldsymbol{k}}. \tag{19}$$

We note that this force density is orthogonal to the interface. Integrating this force density over any volume around the interface results in the usual force. See Figure 2.

Thus,

$$\boldsymbol{F} = \int_{\text{vol around interface}} \boldsymbol{f}\, dS\mathrm{d}z. \tag{20}$$

Since the volume element is $lh\mathrm{d}z$, (see **Figure 2**),

$$\boldsymbol{F} = \int_{\text{vol around interface}} \boldsymbol{f}\, lh\mathrm{d}z. \tag{21}$$

Then,

$$\boldsymbol{F} = \int_{\text{vol around interface}} -\frac{1}{2}\epsilon_0 \chi_e E^2 \hat{\boldsymbol{k}} lh\delta(d-z)\mathrm{d}z, \tag{22}$$

and the force on the dielectric slab, of cross section lh, is

$$\boldsymbol{F} = \frac{1}{2}\epsilon_0 \chi_e lhE^2 \hat{\boldsymbol{k}}. \tag{23}$$

Since this is the known result we can conclude that the strange force density $-\frac{1}{2}\boldsymbol{E}\times(\nabla\times\boldsymbol{P})$ makes sense and can be used even with uniform electrostatic fields.

It is interesting to consider the following analogy. Usually the magnetic Lorentz force density

$$\boldsymbol{f} = \boldsymbol{J}\times\boldsymbol{B}, \tag{24}$$

can be applied to a magnetic medium with the current

$$\boldsymbol{J} = \boldsymbol{J}_M = \nabla\times\boldsymbol{M}, \tag{25}$$

obtaining

$$\boldsymbol{f} = -\boldsymbol{B}\times(\nabla\times\boldsymbol{M}). \tag{26}$$

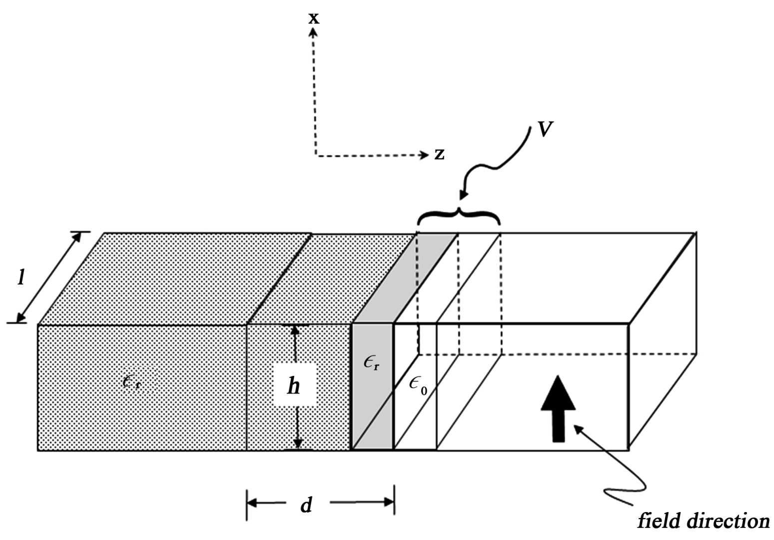

Figure 2. Volume V for the volume integration of the force density. It includes the interface.

We can then establish the analogy

$$\boldsymbol{B} \to \boldsymbol{E} \tag{27}$$

and

$$\nabla \times \boldsymbol{M} \to \nabla \times \boldsymbol{P}. \tag{28}$$

That is, if $\nabla \times \boldsymbol{P}$ is regarded as a kind of polarization current $\boldsymbol{J}_{\text{newP}}$ analogous to the magnetization current $\nabla \times \boldsymbol{M}$, the force density "Equation (13)" can be seen as a Lorentz type force $\boldsymbol{f} = -\boldsymbol{E} \times \boldsymbol{J}_{\text{newP}}$ or

$$\boldsymbol{f} = \boldsymbol{J}_{\text{newP}} \times \boldsymbol{E}. \tag{29}$$

This unusual force density may be accepted if we remember that the force density involving the polarization current $\boldsymbol{J}_P = \partial_t \boldsymbol{P}$ is well known and accepted. This symmetry is exhibited in the force densities

$$\boldsymbol{f}_{m1} = \boldsymbol{J}_P \times \boldsymbol{B}, \tag{30}$$

$$\boldsymbol{f}_{m2} = \boldsymbol{J}_M \times \boldsymbol{B}, \tag{31}$$

and

$$\boldsymbol{f}_{\text{newP}} = \boldsymbol{J}_{\text{newP}} \times \boldsymbol{E}, \tag{32}$$

where $\boldsymbol{J}_{\text{newP}} = -\frac{1}{2} \nabla \times \boldsymbol{P}$. The two first densities are well known and accepted, not so the third.

It is convenient to analyze the force density "Equation (13)" from a more general perspective. We begin by noting that "Equation (29)" can be rewritten as

$$\boldsymbol{f} = -\frac{1}{2} \epsilon_0 E^2 \nabla \chi_e(z). \tag{33}$$

This force density is well known and is part of the force density obtained by Helmholtz at the end of the 19th century. Helmholtz found, for fluids, the force density [11] [12] [13]

$$\boldsymbol{f}_{\text{Hfluids}} = \rho \boldsymbol{E} - \frac{1}{2} \epsilon_0 E^2 \nabla \epsilon_r + \frac{1}{2} \epsilon_0 \nabla \left(E^2 \rho_m \frac{\partial \epsilon_r}{\partial \rho_m} \right), \tag{34}$$

which was later generalized for solids [11] [13] [14] as

$$\boldsymbol{f}_{\text{Hfluids}} = \rho \boldsymbol{E} - \frac{1}{2} \epsilon_0 E^2 \nabla \epsilon_r + \nabla \cdot \boldsymbol{T}_{\text{solids}}, \tag{35}$$

where $\boldsymbol{T}_{\text{solids}}$ is a tensor that contains the mechanical information of the solid. Its particular form is not necessary here, but it is convenient to note that "Equation (35)" reduces to "Equation (34)" (if the solid is rigid and there are not free charges), obtaining

$$\boldsymbol{f}_{\text{Hfluids}} = \frac{1}{2} \epsilon_0 E^2 \nabla \epsilon_r, \tag{36}$$

which we will call Helmholtz force density, though it is a part of the usual one.

3. Force Density and Its Corresponding Momentum Balance

The force density "Equation (13)" is unfamiliar, but it solves the problem of the force on a dielectric inside a uniform electrostatic field. However, it was obtained just applying a dyadic identity to the gradient of the energy density "Equation (6)". In what follows we consider this force density in the context of a balance equation for electromagnetic momentum obtained from the Maxwell equations.

We have that the Maxwell equations can be written in its most usual form as

$$\begin{aligned} &\nabla \cdot \boldsymbol{D} = \rho, \\ &\nabla \cdot \boldsymbol{B} = 0, \\ &\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0, \\ &\nabla \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{J}, \end{aligned} \tag{37}$$

From these the following balance equation can be obtained using vector and tensor identities [15] [16],

$$\begin{aligned} &\nabla \cdot \left\{ \left(\boldsymbol{DE} + \boldsymbol{BH} \right) - \frac{1}{2} \boldsymbol{I} \left(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H} \right) \right\} - \partial_t \left(\boldsymbol{D} \times \boldsymbol{B} \right) \\ &= \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} + \frac{1}{2} \left[\left(\nabla \boldsymbol{E} \right) \cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D} \right) \cdot \boldsymbol{E} + \left(\nabla \boldsymbol{H} \right) \cdot \boldsymbol{B} - \left(\nabla \boldsymbol{B} \right) \cdot \boldsymbol{H} \right]. \end{aligned} \tag{38}$$

Here the fields are the total fields solution of the Maxwell equations and therefore the usual Lorentz force density is a limit case of the above balance equations.

Having a balance equation permits to obtain the force either with a volume integration of the force density or, using Gauss's theorem, with a surface integration of the stress tensor. Other force densities arise as particular cases of the balance equation. For example, for the electrostatic case and in the absence of free charges this balance equation reduces to

$$\nabla \cdot \left\{ \boldsymbol{DE} - \frac{1}{2} \boldsymbol{I} \left(\boldsymbol{D} \cdot \boldsymbol{E} \right) \right\} = \frac{1}{2} \left[\left(\nabla \boldsymbol{E} \right) \cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D} \right) \cdot \boldsymbol{E} \right]. \tag{39}$$

Using now the constitutive relation

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \boldsymbol{E}, \tag{40}$$

where $\varepsilon_r = 1 + \chi_e$, transforms the right-hand term into

$$\frac{1}{2} \left[\left(\nabla \boldsymbol{E} \right) \cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D} \right) \cdot \boldsymbol{E} \right] = -\frac{1}{2} \epsilon_0 E^2 \nabla \chi_e \left(z \right). \tag{41}$$

This term is the force density appearing in "Equation (33)", which solves the problem of the force on a dielectric in a uniform electrostatic field and is part of the Helmholtz force density. It can be shown [16] that this balance contains the whole Helmholtz force density.

4. Nature of the Force on the Dielectric

As we have established the force on the dielectric slab acts at the interface, where a discontinuity in the polarization, and therefore in the energy, occurs. This is to be expected, since only at the interface the gradient of the energy is different from zero.

We have then the question posed by Margulies: [5]

"How does the electric field, which is directed vertically, ··· produce a transverse force ··· on the dielectric?"

To answer this question, we apply a Maxwellian approach. Maxwell wrote [17]:

"If we further admit that every part of a dielectric medium through which electric induction is taking place there is a tension, like that of a rope, in the direction of the lines of force, and a pressure in all directions at right angles to the lines of force, we may account for all the mechanical actions which take place between electrified bodies."

This qualitative description is made quantitative with the Maxwell electrostatic stress tensor,

$$\vec{\vec{T}} = \left[\boldsymbol{DE} - \frac{1}{2} \vec{\vec{I}} \left(\boldsymbol{D} \cdot \boldsymbol{E} \right) \right]. \tag{42}$$

We apply it to the interface, where we have seen that the force acts. We can therefore obtain a better understanding of the force acting on the dielectric performing a surface integral of the stress tensor. It is shown in advanced texts on electromagnetism [11] [12] that the stress tensor can be decomposed in directions along and traverse to the field $\boldsymbol{E}$. These components give a tension along field $\boldsymbol{E}$ and a compression orthogonal to it of magnitude, see **Figure 3**.

$$t = \frac{1}{2} \epsilon_0 \epsilon_r E^2, \tag{43}$$

Thus, these components of stress are positive along $\boldsymbol{E}$ and negative in a plane orthogonal to it. The tension is equilibrated by mechanical forces on the plates of the capacitor, while the compression is zero in vacuum and in the dielectric, but not so at the interface. This accounts as a pressure difference at the interface. This can be seen as follows. Since we have a cylindrical geometry around a line of uniform electrostatic field and only the components orthogonal to the interface contribute to the force, we can project the compression on a surface element $h\mathrm{d}y$ of the interface, see **Figure 4**.

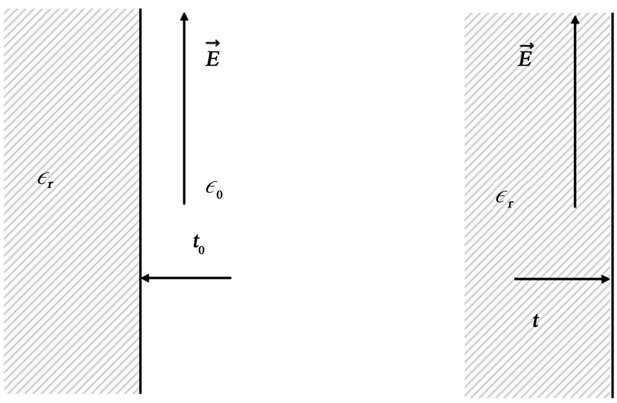

Figure 3. Compression t_0 and t at the interface in the case that the electrostatic field line is parallel to a surface.

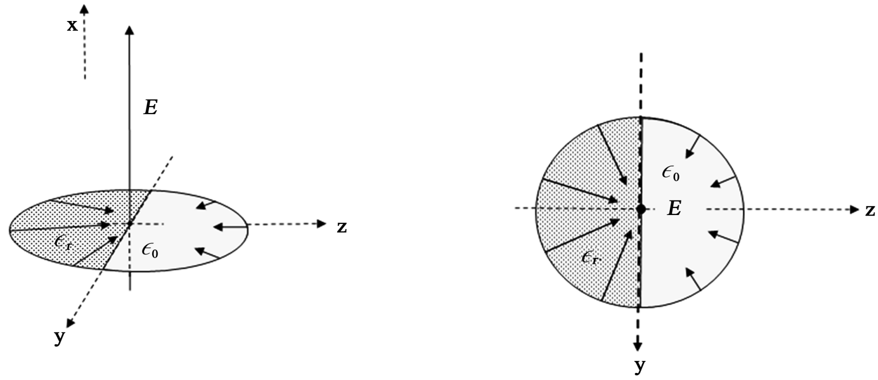

Figure 4. The interface surface is divided into elements $h\mathrm{d}y$. At the centers of these elements there is a field line around which there is a cylinder of radius $\frac{1}{2}\mathrm{d}y$ and height h. The compression components of the stress tensor can the projected on this surface resulting a pressure $p=\frac{1}{2}\epsilon_0\chi_e E^2$.

The result is a difference of compressions at the interface, equivalent to a pressure from the dielectric towards vacuum, equal to $p=t-t_0$ and the result is

$$p=\frac{1}{2}\epsilon_0\chi_e E^2. \tag{44}$$

We have then that the surface integral of the Maxwell stresses at the interface gives the usual result, verifying the particularized balance equation "Equation (39)".

Given the simple geometry of the problem, we can also obtain the force as follows. The force can be obtained by means of a surface integration of the stress tensor,

$$\boldsymbol{F}=\oint_\sigma \mathrm{d}\boldsymbol{S}\cdot\vec{\boldsymbol{T}}, \tag{45}$$

where σ is a closed surface, in this case a surface around the interface as indicated in **Figure 2**. The Maxwell electrostatic stress tensor is given in "Equation (42)"

Since at the interface the electric field is in the x-direction, the constitutive relation "(40)" can be written as

$$\boldsymbol{D}=\epsilon_0\epsilon_r E\hat{\boldsymbol{i}}. \tag{46}$$

Therefore, the stress tensor is

$$\vec{\boldsymbol{T}}=\epsilon_0\epsilon_r E^2\left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}-\frac{1}{2}\vec{\boldsymbol{I}}\right], \tag{47}$$

with

$$\vec{\boldsymbol{I}}=\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}. \tag{48}$$

The surface around the interface is formed by a parallelepiped with planes close to the interface, and a "ribbon" around it. Given the symmetry, only the contributions of these planes are different from zero, with directions $\hat{\boldsymbol{k}}$ and $-\hat{\boldsymbol{k}}$. Then, substituting "Equation (48)" in "Equation (47)" results in

$$\boldsymbol{F} = \int_{\sigma\,\text{medium}} \left(-\hat{\boldsymbol{k}}\right) \mathrm{d}S \cdot \epsilon_0 \epsilon_r E^2 \left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}} - \frac{1}{2}\vec{\boldsymbol{I}}\right] - \int_{\sigma\,\text{medium}} \hat{\boldsymbol{k}} \mathrm{d}S \cdot \epsilon_0 \epsilon_r E^2 \left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}} - \frac{1}{2}\vec{\boldsymbol{I}}\right], \tag{49}$$

so that

$$\boldsymbol{F} = \frac{1}{2}\epsilon_0 E^2 \left(\epsilon_r - 1\right) \hat{\boldsymbol{k}} \mathrm{d}S, \tag{50}$$

and

$$\int \hat{\boldsymbol{k}} \mathrm{d}S = \hat{\boldsymbol{k}} l h. \tag{51}$$

Then expressing the relative permittivity in terms of the susceptibility leads to

$$\boldsymbol{F} = \frac{1}{2}\epsilon_0 \chi_e L h \hat{\boldsymbol{k}}, \tag{52}$$

which is the known result.

We now show that the force density in this balance equation contains the unfamiliar force density "Equation (13)", whose volume integration gives the usual force on the dielectric.

We begin with the dyadic identity

$$\left(\nabla \boldsymbol{v}\right) \cdot \boldsymbol{u} = \boldsymbol{u} \times \left(\nabla \times \boldsymbol{v}\right) + \left(\boldsymbol{u} \cdot \nabla\right) \boldsymbol{v}, \tag{53}$$

which permits to write the right member of the balance equation, "Equation (38)", as

$$\begin{aligned} &\frac{1}{2}\left[\left(\nabla \boldsymbol{E}\right) \cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D}\right) \cdot \boldsymbol{E}\right] \\ &= \frac{1}{2}\left[\boldsymbol{P} \times \left(\nabla \times \boldsymbol{E}\right) - \left(\boldsymbol{E} \cdot \nabla\right) \boldsymbol{P} - \boldsymbol{E} \times \left(\nabla \times \boldsymbol{P}\right) + \left(\boldsymbol{P} \cdot \nabla\right) \boldsymbol{E}\right]. \end{aligned} \tag{54}$$

Since we are dealing with an electrostatic field we have that

$$\nabla \times \boldsymbol{E} = 0. \tag{55}$$

Also, we have that the field $\boldsymbol{E}$ is uniform and therefore the polarization is uniform in a linear medium as is usually assumed. Then

$$\left(\boldsymbol{E} \cdot \nabla\right) \boldsymbol{P} = 0, \tag{56}$$

since

$$E_x \partial_x \boldsymbol{P}\left(z\right) = 0, \tag{57}$$

and

$$\left(\boldsymbol{P} \cdot \nabla\right) \boldsymbol{E} = 0, \tag{58}$$

since the electric field $\boldsymbol{E}$ is uniform and continuous at the interface, and we get the particular force density

$$\frac{1}{2}\left[\left(\nabla \boldsymbol{E}\right) \cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D}\right) \cdot \boldsymbol{E}\right] = -\frac{1}{2}\boldsymbol{E} \times \left(\nabla \times \boldsymbol{P}\right), \tag{59}$$

Therefore, in this case the balance equation results

$$\nabla \cdot \left\{ \boldsymbol{D}\boldsymbol{E} - \frac{1}{2}\boldsymbol{I}\left(\boldsymbol{D}\cdot\boldsymbol{E}\right) \right\} = -\frac{1}{2}\boldsymbol{E}\times\left(\nabla\times\boldsymbol{P}\right), \tag{59}$$

with which the force on the dielectric can be obtained with a volume integration of the force density, or with a surface integral of the stress tensor, as we have shown.

5. Conclusions

We have shown that the usual method with which the force on a dielectric slab partially introduced into a parallel plate capacitor is calculated, that is, with the gradient of an energy density, establishes firmly, taking into account the discontinuity in the energy, that the force acts at the interface.

With the method based on a momentum balance equation, which involves a volume integration of a force density, or the surface integration of the Maxwell electrostatic stress tensor, the usual result for the force is obtained.

Though the force density is unfamiliar, it is firmly sustained on a balance equation derived from Maxwell's equations with linear media. Therefore, our results are well founded on Maxwell's equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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