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Legendre Polynomial and Nonlinear Oscillating Point-Like Charged Particle

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Abstract

In this article we explore the kinematics of a point-like charged particle placed within the interior plane of a charged ring. Analytically we formulate the electric field of the ring along a representative diagonal. Graph of the field as a function of the distance from the center of the ring assists foreseeing oscillating movement of the charged particle. We formulate the equation of motion; this is a nonlinear differential equation. Applying Computer Algebra System (CAS), specifically Mathematica [1] we solve the equation numerically. Utilizing the solution we quantify the kinematic quantities of interest including oscillations period. Although the equation of motion is nonlinear its period is regulated. For better understanding we take an advantage of Mathematica animation features animating the nonlinear oscillations.

Keywords

Charged Ring, Oscillating Charged, Legendre Polynomial, Computer Algebra System, *Mathematica*

1. Motivations and Goals

It is a common practice to derive the electrostatic potential and field of a uniformly charged ring in 3D space [2] [3] [4]. A thorough literature search however reveals the lack of their applied applications; this project fills in the missing link. With applications in mind we derive expressions for the electric field within the ring's plane. This downgrades the dimension of the space from three to two. The ring splits the space to interior and exterior regions each with distinct electrostatic characteristics. Placing a point-like charged particle in these regions exerts a different type of force making the particle behave accordingly. It is the goal of this investigation to objectively explore the nature of the motion in each region quantifying their kinematics. This report consists of three sections. In addition to Motivations and Goals, in Section 2 we present the physics of the problem and provide detailing to the solution. This section also includes the output of the applied CAS. We conclude with closing remarks.

2. Physics of the Problem and Its Solution

Figure 1 figuratively shows the problem at hand. A charged ring of radius R is placed on a horizontal xy-plane. To derive the electrostatic potential of the ring at a point of interest, $p(r,\varphi)$ is placed on the plane of the ring first. We evaluate the potential of a differential charge dq; for exterior points we add (integrate) over the rest of the charge segments over the rim of the ring. This procedure is not true for interior points; more explanation follows later. Quantitatively this is done according to Equation (1).

$$V(r,\varphi) = k\lambda \int \frac{\mathrm{d}q}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{1}$$

here, $k = \frac{1}{4\pi \in_0} = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$ is the electrostatic coupling constant, $\lambda = q/(2\pi R)$ is the charge density of the ring and $V(r,\varphi)$ is the potential at p expressed in polar coordinates. Utilizing Figure 1, we write,

$$\begin{cases} \boldsymbol{r} = r\cos(\varphi)\hat{i} + r\sin(\varphi)\hat{j} \\ \boldsymbol{r}' = r'\cos(\varphi')\hat{i} + r'\sin(\varphi')\hat{j} \end{cases} \tag{2}$$

Applying Equation (2), the denominator of the integrand of Equation (1) is,

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos(\varphi-\varphi')} \tag{3}$$

For the points exterior to the ring, *i.e.* $r > r'$ Equation (3) is written as,

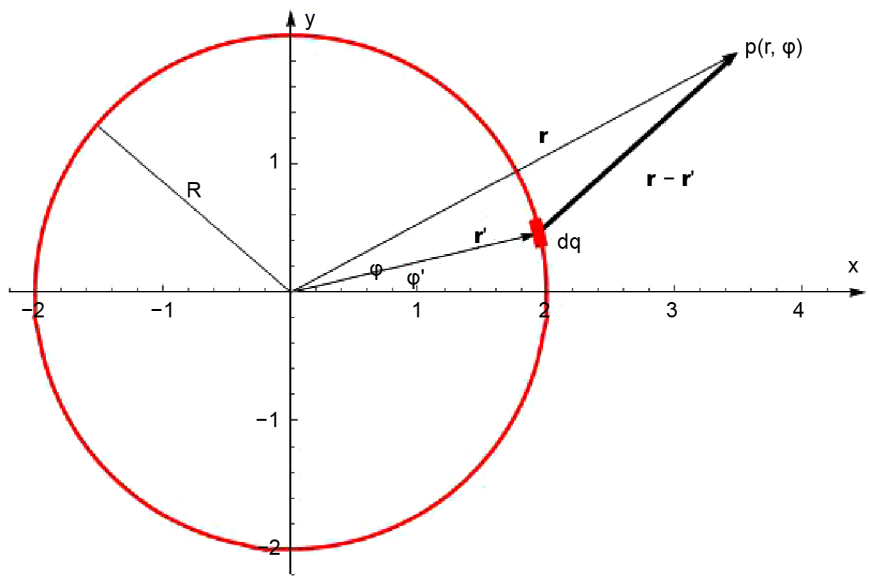

Figure 1. A charged ring of radius R is on a horizontal *xy*-plane. A segment of the ring with differential charge dq located at $\boldsymbol{r}'$, a point of interest, $p(r,\varphi)$ located at $\boldsymbol{r}$ are also shown.

$$\left|\boldsymbol{r}-\boldsymbol{r}'\right| = r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\left(\varphi-\varphi'\right)} \tag{4}$$

and therefore the integrand of Equation (1) according to [2] [3] [4] can be replaced with

$$\frac{1}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|} = \sum_{\ell=0}^{\infty}\frac{\left(r'\right)^{\ell}}{r^{\ell+1}}P_{\ell}\left[\cos\left(\varphi-\varphi'\right)\right] \tag{5}$$

where P_ℓ is the Legendre polynomial of order ℓ.

Equation (5) for interior points is,

$$\frac{1}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|} = \sum_{\ell=0}^{\infty}\frac{r^{\ell}}{\left(r'\right)^{\ell+1}}P_{\ell}\left[\cos\left(\varphi-\varphi'\right)\right] \tag{6}$$

Substituting Equation (5) in Equation (1) and replacing r' with R yields,

$$V\left(r,\varphi\right) = k\lambda\left\{\sum_{\ell=0}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\int_0^{2\pi}P_{\ell}\left[\cos\left(\varphi-\varphi'\right)\right]\mathrm{d}\varphi'\right\} \tag{7}$$

Equation (7) includes the integral of Legendre polynomials of order ℓ. The values of these integrals for the first seven ℓs applying Mathematica are tabulated in **Table 1**.

Table 1 shows only the even values of ℓ give non-vanishing values for Equation (7). More importantly the output of the integration, *i.e.* the second column of **Table 1**, as expected is independent of φ. Meaning, because of the circular symmetry of the ring the derived potential, Equation (7) is independent of the angular position of point p *i.e.* $V\left(r,\varphi\right)$ is the function of r only. Utilizing the relationship between the potential and electric field, namely,

$$\boldsymbol{E}_r = -\partial_r V\left(r\right)\hat{r} \tag{8}$$

We arrive at E_r,

Table 1. The first column is the order of the Legendre polynomial, the second column is the associated integration.

ℓ	$\int_0^{2\pi}P_{\ell}\left[\cos\left(\varphi-\varphi'\right)\right]\mathrm{d}\varphi'$
0	2π
1	0
2	$\frac{1}{2}\pi$
3	0
4	$\frac{9}{32}\pi$
5	0
6	$\frac{25}{128}\pi$
7	0

$$E_r = k\lambda\left\{\{\ell=0,2\pi\},\left\{\ell=2,\frac{\pi}{2}\right\},\left\{\ell=4,\frac{9\pi}{32}\right\},\left\{\ell=6,\frac{25\pi}{128}\right\},\cdots\right\}\sum_{\ell=0,\text{even}}(\ell+1)\frac{R^{\ell+1}}{r^{(\ell+2)}} \quad (9)$$

Now a point-like charged particle, Q, placed in this field would experience a force according to $\boldsymbol{F}=Q\boldsymbol{E}$. The applied force would put the particle of mass m in motion, $\boldsymbol{F}=m\ddot{\boldsymbol{r}}$. Utilizing Equation (9) the equation of motion is,

$$\ddot{x}(t)-\left(\frac{1}{m}kqQ\right)\left(\frac{1}{x(t)^2}+\frac{3}{4}\frac{R^2}{x(t)^4}+\frac{45}{64}\frac{R^4}{x(t)^6}+\cdots\right)=0 \quad (10)$$

Here, for the sake of clarity we labeled the radial direction as *x*. Equation (10) is a second order nonlinear differential equation. Nonlinearity stems from the electric field. The field has a diminishing distance dependence character. Therefore its impact diminishes as the particle gets pushed away from the ring. With the exception of the first term of the second parentheses the rest of the terms depend on the ring size, *R*. Rather than assigning numeric values to the needed parameters, {*k*, *q*, *Q*, *m*}, we set the grouped coefficient, (1/*m kqQ*) = 1. This helps focusing on the generic feature of the motion. We set the ring size to unity, *R* = 1.0. Applying Mathematica with a set of meaningful initial conditions e.g. $x[0]=1.2$ and $x'[0]=0$, we solve the equation numerically. The solution is shown in **Figure 2**.

Plot (a) shows the impact of the force vs. time. The impact of the force is limited, meaning, because the particle gets pushed away from the ring and because the field diminishes for large distances, after the initial push the particle cruises at constant speed, this is depicted in plot (b). The time axis of plot (b) intentionally is stretched to 20, so that the plateau shows the cruising character. Plot (c) shows the small value of the acceleration for time beyond 10. For $t > 10$ the particle has no acceleration.

Figure 3 is the animation profile of the problem. If the manuscript was prepared utilizing Mathematica sliding the slider would have put the charge in motion, thus enhancing the understanding of the impact of the nonlinear force on the rectilinear movement of the particle; MSW is incapable of animation. However, an interested reader may contact the author to receive a free copy of Mathematica animation code.

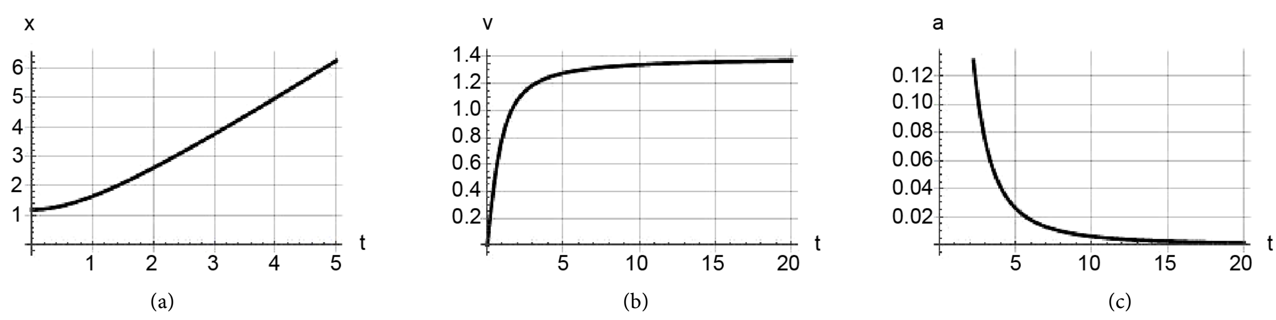

Figure 2. Plots (a), (b), (c) are the distance, speed and acceleration of the particle vs. time, respectively.

For interior points as mentioned in the beginning of the section differential fields *algebraically* are additive. This is in contrast to exterior points where the fields arithmetically are additive. For the former, the algebraic sum of the differential fields results a net field with alternating orientation. For instance, if the loose particle is placed along the horizontal axis close to the right rim of the ring, the field would orient to the left, when it passes the center of the ring it reverses direction. Intuitively these reversal fields are the cause of the oscillations. Therefore, kinematics of the particle as it moves within the ring is quite different from those at the exterior.

Applying Equation (6)-(9) and Table 1 we arrive at interior field,

$$E_r = -kq\left(\frac{1}{2R^3}r + \frac{9}{16R^5}r^3 + \cdots\right) \tag{11}$$

its plot is shown in Figure 4.

As discussed, the electric field for points along the x-axis close to the rim of the ring is the strongest, orienting toward the origin. At the center of the ring due to circular symmetry of the ring and cancellation of the fields its value is zero. On the other side of the origin the field gradually becomes stronger orienting along the opposite direction.

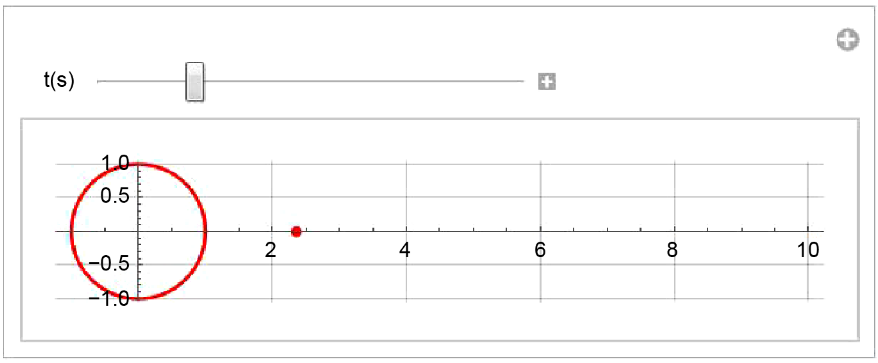

Figure 3. This figure shows the animation profile of the problem at hand. A loose point-like charged particle is placed outside the charged ring.

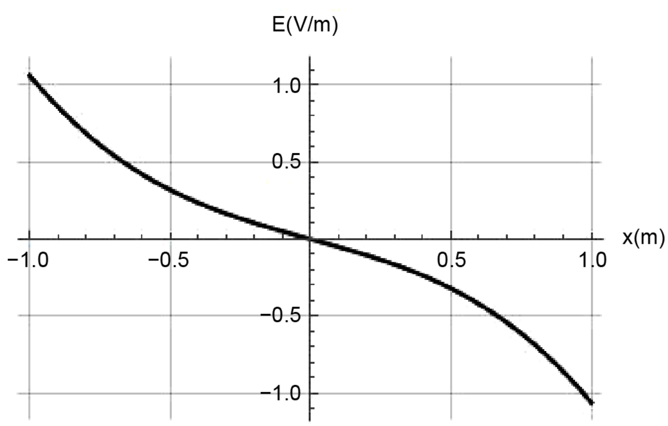

Figure 4. Electric field E(V/m) along the horizontal diagonal of the ring vs. distance is shown. Ring size is $R = 1.0$ m.

Utilizing Equation (11) the equation of motion is,

$$\ddot{x}(t)+\left(\frac{1}{m}kqQ\right)\left(\frac{1}{2R^3}x(t)+\frac{9}{16R^5}x(t)^3+\cdots\right)=0 \tag{12}$$

As in the previous case for a ring size of one, we set the composite coefficient of the first parentheses to unit value. Applying Mathematica ND Solve we solve the equation numerically. We set a meaningful initial conditions, namely, $x[0]=0.7$, with $x'[0]=0$. Utilizing the solution its associated kinematics are shown in Figure 5.

As intuitively predicted plot (a) shows the oscillating particle. Plot (b) is the speed of the particle. A trained eye recognizes the impact of the nonlinearity of the force on the acceleration, plot (c).

Utilizing the solution of Equation (12) Figure 6 displays the animation profile of the nonlinear oscillation of the particle.

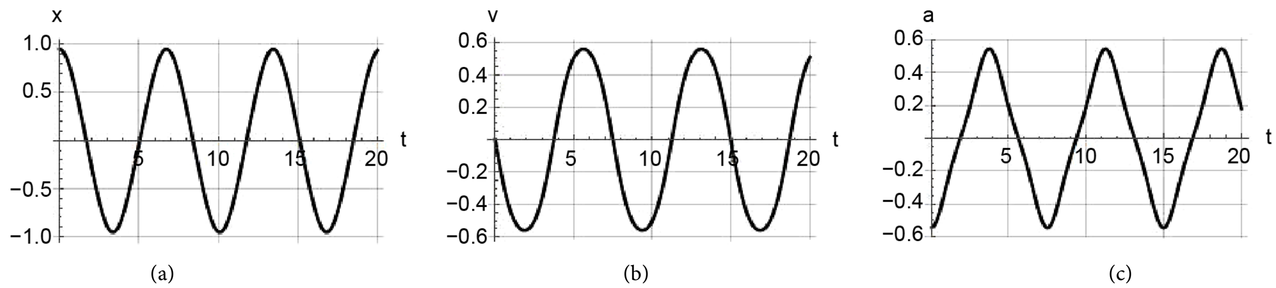

Figure 5. Plots (a), (b), (c) are the distance, speed and acceleration of the particle vs. time, respectively.

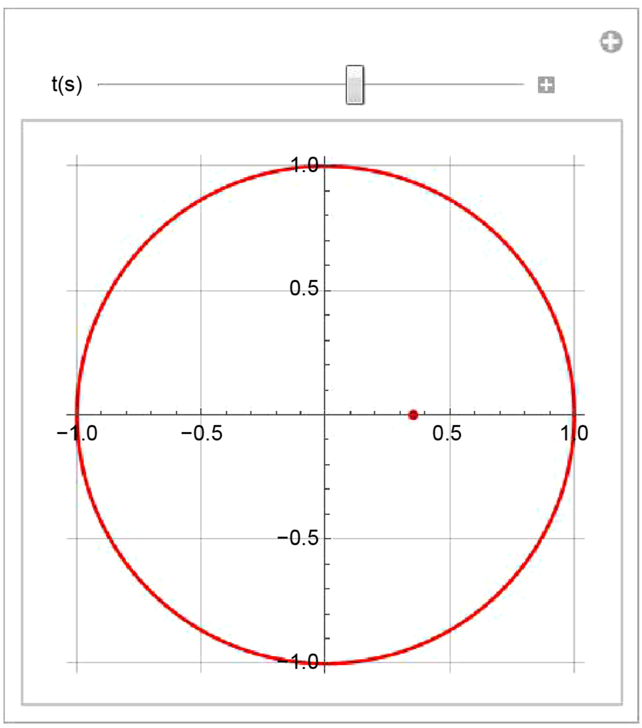

Figure 6. This figure shows the animation profile of the problem at hand. A loose point-like charged particle is placed inside the charged ring.

Figure 6 is the animation profile of the problem. Similar to the previous statement if the manuscript was prepared utilizing Mathematica sliding the slider would have put the charge in an oscillatory mode. The static mode of the oscillations is shown in Figure 5. The interested reader may contact the author to receive a free copy of Mathematica animation code. Animation runtime is 10 s; the program recycles automatically.

3. Conclusions

It is the objective of this investigation to explore applications of the electric field of a charged ring. Two scenarios are considered. First we derived expressions for the fields at points exterior to the ring, and then for the interior points. Intuitively we expect the field at exterior points to die off at distances longer than the size of the ring. For the interior points, the field should flip-flop direction. Quantitatively we confirm both characteristics. As shown for both cases, characteristics of the field keenly relate to Legendre polynomial. Most interesting is the application of this analysis for the interior region. Flipping-flopping field makes the charge to oscillate. The plot of the oscillations enables determining the period. The equation of motion of each region is a nonlinear ODE requiring numeric solution. This report underlines the need of a CAS. Mathematica is applied obtaining numeric solutions and the needed plots. Our analysis also includes animation profiles of the oscillations assisting visual understanding about the impact of the nonlinearity of the forces. Our possible future investigation will include applications of fields of a charged ellipse. The interested reader may contact the author for free copies of the Mathematica animation code. Useful plotting techniques are available in [5] as well as in a newly published reference [6].

A Word about the References

The literature search reveals numerous articles on derivation of electric field of a charged ring—none with applications in mind. The majority of the articles formulate the simplest cases, such as the field along the symmetry axis perpendicular to the plane of the ring through the center. The author listed classic references [2] [3] [4]; an additional Google search returns references that are not directly pertain to the objective of this paper.

Acknowledgements

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