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Electromagnetic Modeling of Metallic Elliptical Plates

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ABSTRACT

This paper presents the evaluation of the capacitance of an isolated elliptical plate and two parallel elliptical plates. Integral equations are formed by relating the previously unknown charges on the elliptical plates and the potential on the metallic plates. The integral equations are solved by applying the *method of moments* based on the pulse function and point matching. The elements of the matrix in the method of moments are found by dividing the structure into triangular subsections. The matrix equation is solved in order to compute the unknown charges on each subsection. Numerical results on the capacitance as a function of the geometrical parameters of the ellipse are presented.

Keywords: Elliptical Metallic Plate; Capacitance; Pulse Function; Point Matching

1. Introduction

The evaluation of the capacitance of various structures having different geometrical shapes is of importance to study the behavior of electrostatic charge build-up on bodies that are isolated in space such as space-craft structures in orbit. The analysis of three-dimensional spherical, paraboloidal and truncated conical surfaces and two-dimensional square, rectangular, circular and annular metallic disks have been examined using the method of moments [1-6]. In these works, the capacitance of the different geometrical structures was obtained by subdividing the structure into uniform rectangular planar subsections and computing the effect of the charge on each subsection on the potential of the others. The use of rectangular subsection in a curved boundary requires a very fine meshing in order to obtain convergence for the data of the capacitance and charge distribution. Meshing with triangular elements on the other hand facilitates the mapping of the geometrical boundaries of arbitrary shape very accurately [7]. The meshing techniques employed in [3] are limited by the large ratio of the area of the biggest elements to smallest elements, which possibly results in an unstable solution.

This paper presents the analysis of a planar elliptical metallic plate isolated in free space as well as two parallel elliptical plates using the method of moments in which triangular sub-areas were used for the solution of integral equations. A closed form expression for the capacitance of a single elliptical conducting plate has been reported by Liang *et al.* [8]. They did not present any data on the capacitance for the case of two parallel elliptical plates of finite size. Thus it is worthwhile to carry out the analysis of elliptical structures using the method of moments with triangular subsections for the geometry under consideration unlike that reported in [1-6].

In order to apply the method of moments, the elliptical plate is divided into a number of triangular elements. The unknown charge distribution on the surface of the elliptical plates appears in the form of an integral equation relating the potential function and charge distribution. This integral equation is solved using the method of moments formulation based on a pulse function and point matching. The methods of finding the diagonal and the non-diagonal elements of the matrix are presented. The capacitance is calculated as a function of eccentricity of the ellipse for a unit semi major axis. The validity of the analysis is justified by comparing the data on the capacitance using the present method with that of the closed form expression in [8].

2. General Analysis

2.1. Single Elliptical Plate

Figure 1 shows a single elliptical plate with a semi major axis *a* and semi minor axis *b*. In order to compute the capacitance of this structure, the plate is subdivided using triangular elements as shown in **Figure 2**. The unknown surface charge density on the plate at any point $r'(x', y', z')$ is denoted by $\sigma(r')$. It is assumed that the surface charge density is constant over each subsection

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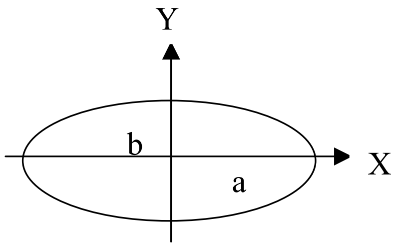

Figure 1. Geometry of a single elliptical plate.

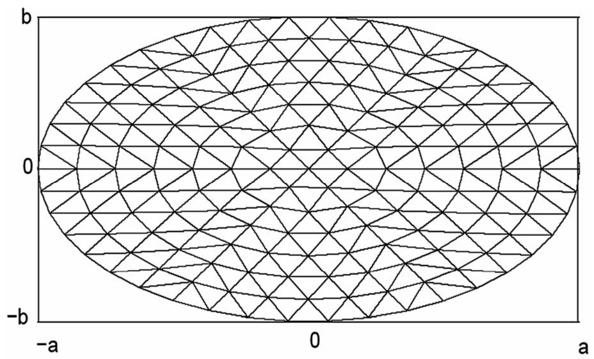

Figure 2. Subdividing an elliptical plate with triangular elements.

so that the charge on each element can be represented by pulse basis functions. The potential at any point $r(x, y, z)$ due to the charges of the plate is expressed as follows

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0} \iint_s \frac{\sigma(r')\,\mathrm{d}s'}{|r - r'|} \tag{1}$$

Assuming that the metallic plate is a perfect conductor, the surface will be an equipotential surface ($\phi = V$ which is the assumed potential of the plate), the above equation forms an integral equation with $\sigma(r')$ being as yet unknown. The surface charge density $\sigma(r')$ can be expressed as [9]

$$\sigma(r') = \sum_{n=1}^{N} \alpha_n f_n \tag{2}$$

where N is the number of triangular elements, α_n are the unknown coefficients and f_n are the pulse basis functions given by

$$f_n = \begin{cases} 1 & \text{on } \Delta s_n \\ 0 & \text{on } \Delta s_m \end{cases} \tag{3}$$

where Δs_m and Δs_n are the areas of the mth and nth triangular elements respectively.

Substituting Equation (2) into Equation (1) and satisfying the resultant equation at the centroid of each triangular element, the following expression is obtained

$$V = \sum_{n=1}^{N} \alpha_n L_{mn}; \quad m = 1, \cdots, N \tag{4}$$

The potential at the centroid of each triangular subsection in terms of an integral can be expressed which can also be reduced to a summation where L_{mn} represents the potential at point m due to charge at the subsection n. The Equation (5) can be written in matrix equation form as follows

$$\begin{bmatrix} L_{11} & L_{12} & \cdots & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{m1} & L_{m2} & \cdots & \cdots & L_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_m \end{bmatrix} \tag{5}$$

where

$$L_{mn} = \frac{\Delta(m)}{4\pi\varepsilon_0 \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}} \tag{6}$$

and $\Delta(m)$ is the area of the m^{th} triangular element.

For $m = n$, this expression for the matrix element is not valid. Thus the diagonal elements have to be treated separately [6]. The diagonal elements are calculated by considering each triangular section to be a circle of equal area. Then the diagonal element is given by [10]

$$L_{mn} = \frac{r_m}{2\varepsilon_o} \tag{7}$$

which corresponds to the potential of a uniformly charged circular disk of radius r_m at the edge of the disk.

The matrix equation can be solved to determine the unknown coefficients as follows

$$[\alpha] = [L]^{-1}[V] \tag{8}$$

The total charge can be computed from

$$Q = \sum_{n=1}^{N} \alpha_n \Delta_n \tag{9}$$

where Δ_n is the area of n^{th} triangular element.

The capacitance can be computed from

$$C = \frac{Q}{V} \tag{10}$$

2.2. Parallel Elliptical Plates

Consider two parallel metal elliptical plates s_1 and s_2 separated by a distance d as shown in **Figure 3**. Let the potential at s_1 be V_{1s} and the potential at s_2 be V_{2s}. The unknown quantities in this case are the charge distributions $\sigma_1(x', y', z')$ and $\sigma_2(x', y', z')$ over the surfaces s_1 and s_2. Let s_1 be discretized by N elements and s_2 from $N + 1$ to $2N$ elements. The discretization is shown in **Figure 4**. Let the total charge be q_1 and q_2 on the surfaces s_1 and s_2 respectively.

Using superposition and the relationship between the

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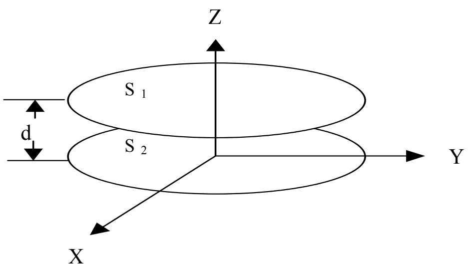

Figure 3. Geometry of parallel elliptical plates.

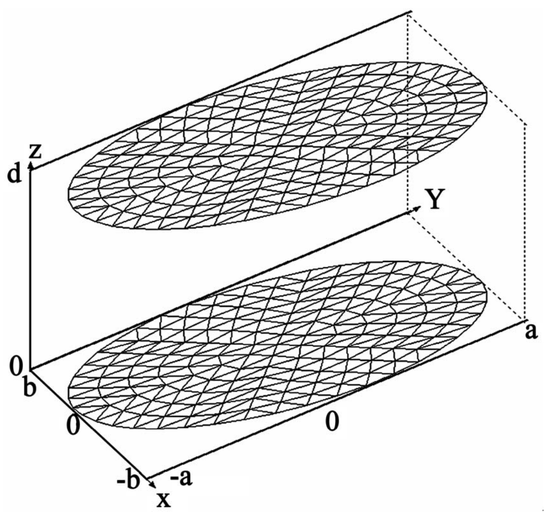

Figure 4. Subdividing parallel elliptical plates with triangular Elements.

charge and the potential [8], we have

$$q_1 = c_{11}V_{1s} + c_{12}V_{2s} \tag{11}$$

$$q_2 = c_{21}V_{1s} + c_{22}V_{2s} \tag{12}$$

where

$$q_1 = \iint \sigma_{s1}(x,y,z)\mathrm{d}s_1$$

$$q_2 = \iint \sigma_{s2}(x,y,z)\mathrm{d}s_2$$

Equations (11) and (12) can be expressed in the matrix equation form as

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} V_{1s} \\ V_{2s} \end{bmatrix} \tag{13}$$

If V_1 is the potential on s_1 due to q_1 and V_2 is the potential on s_1 due to q_2, then the potential

$$\begin{aligned} V_{1s} &= V_1 + V_2 \\ &= \frac{1}{4\pi\varepsilon_0}\int_{s_1} \frac{\sigma_{s1}(x',y',z')\mathrm{d}s_1}{\sqrt{(x-x')^2+(y-y')^2}} \\ &+ \frac{1}{4\pi\varepsilon_0}\int_{s_1} \frac{\sigma_{s2}(x',y',z')\mathrm{d}s_2}{\sqrt{(x-x')^2+(y-y')^2+\mathrm{d}^2}} \end{aligned} \tag{14}$$

If V_3 is the potential on s_2 due to q_1 and V_4 is the potential on s_2 due to q_2, then the potential of s_2 is

$$\begin{aligned} V_{2s} &= V_3 + V_4 \\ &= \frac{1}{4\pi\varepsilon_0}\int_{s_1} \frac{\sigma_{s1}(x',y',z')\mathrm{d}s_1}{\sqrt{(x-x')^2+(y-y')^2+\mathrm{d}^2}} \\ &+ \frac{1}{4\pi\varepsilon_0}\int_{s_1} \frac{\sigma_{s2}(x',y',z')\mathrm{d}s_2}{\sqrt{(x-x')^2+(y-y')^2}} \end{aligned} \tag{15}$$

Equations (14) and (15) are two integral equations which can be solved using the same procedure that was used for the isolated ellipse case as given in section A. applying the pulse function and point matching techniques, the following sets of equations are obtained

$$\left.\begin{aligned} V_{1s} &= \sum_{n=1}^{N}\alpha_{1n}l_{mn}^{11} + \sum_{n=1+N}^{2N}\alpha_{2n}l_{mn}^{12} \\ V_{2s} &= \sum_{n=1}^{N}\alpha_{1n}l_{mn}^{21} + \sum_{n=1+N}^{2N}\alpha_{2n}l_{mn}^{22} \end{aligned}\right\} \tag{16}$$

where

$$l_{mn}^{11} = \frac{\Delta(m_1)}{4\pi\varepsilon_0\sqrt{(x_m-x_n)^2+(y_m-y_n)^2}} \tag{17}$$

$$l_{mn}^{21} = \frac{\Delta(m_2)}{4\pi\varepsilon_0\sqrt{(x_m-x_n)^2+(y_m-y_n)^2+\mathrm{d}^2}} \tag{18}$$

$$l_{mn}^{12} = \frac{\Delta(m_1)}{4\pi\varepsilon_0\sqrt{(x_m-x_n)^2+(y_m-y_n)^2+\mathrm{d}^2}} \tag{19}$$

$$l_{mn}^{22} = \frac{\Delta(m_2)}{4\pi\varepsilon_0\sqrt{(x_m-x_n)^2+(y_m-y_n)^2}} \tag{20}$$

In Equations (17)-(20), $\Delta(m_1)$ is the area of the *m*th triangular element of s_1 and $\Delta(m_2)$ is the area of the *m*th triangular element of s_2. The diagonal elements are similarly obtained by assuming that the triangular element is replaced with a circle with the equivalent area and the potential at the edge of a circle is computed. The diagonal element is given by [9]

$$l_{nn} = \frac{1}{2\varepsilon_0}\left[\sqrt{r^2-\mathrm{d}^2} - \mathrm{d}\right] \tag{21}$$

where *r* is the radius of the equivalent circle.

The above set of simultaneous equations can be written in matrix equation form as follows

$$\begin{bmatrix} [l_{mn}^{11}] & [l_{mn}^{12}] \\ [l_{mn}^{21}] & [l_{mn}^{22}] \end{bmatrix} \begin{bmatrix} [\alpha_{1n}] \\ [\alpha_{2n}] \end{bmatrix} = \begin{bmatrix} V_{1s} \\ V_{2s} \end{bmatrix} \tag{22}$$

The solution of Equation (22) is obtained as

$$\begin{bmatrix}[\alpha_{1n}]\\ [\alpha_{2n}]\end{bmatrix} = \begin{bmatrix}[\xi_{mn}^{11}] & [\xi_{mn}^{12}]\\ [\xi_{mn}^{21}] & [\xi_{mn}^{22}]\end{bmatrix}\begin{bmatrix}V_{1s}\\ V_{2s}\end{bmatrix} \tag{23}$$

where ξ_{mn} are the elements of inverse of the square matrix of Equation (22). Comparing Equations (13) and (23), the elements of the capacitance matrix in (13) are obtained as the sum of the elements of each sub matrix of Equation (23).The capacitance of the structure of **Figure 3** is determined by [9]

$$C = \frac{c_{11}c_{22} - c_{12}^2}{c_{11} + c_{22} + 2c_{12}} \tag{24}$$

3. Numerical Analysis

Using Equations (6)-(8), the matrix elements and the unknown coefficients are calculated assuming that the potential of the plate is equal to 1. Using Equations (8) and (9), the capacitance of an elliptical isolated disk with eccentricity e = 0.85 has been calculated. The convergence of the computed numerical values is illustrated in shown in **Figure 5**. The converged value of the capacitance is 49.873 pF. The computed capacitance as a function of eccentricity is presented in **Figure 6** and the numerical value of the capacitance is presented in **Table 1**. The numerical results can be compared with a closed form solution for a circular disk (eccentricity=0) in **Table 1** [8].

Using Equations (17)-(23), the matrix elements and the unknown coefficients are calculated assuming the upper plate is at potential $V_{s1} = +1$ while the potential on the lower plate is $V_{s2} = -1$ with various values of eccentricity and separation distance between the plates. Using Equation (24), the capacitance of parallel elliptical plates separated by a distance d = 1 and e = 0.85 has been calculated to check the convergence of the data on capacitance. The convergence of the data on the capacitance has been checked by increasing the number of subsections and the converged value of the capacitance is 39.263 pF. The convergence data is shown in **Figure 7**. The capacitance values for various eccentricities are presented in **Figure 8**. The numerical value of the capacitance for two parallel circular disk (eccentricity = 0) with d = 1 is 63.378 pF as shown in **Figure 8**. The numerical value of the capacitance is computed as a function of distance and is presented in **Table 2**. **Figure 9** shows the variation of the capacitance normalized by the closed from of the simplified expression εA/d (C_0), where A is the area of the ellipse as a function of A/d.

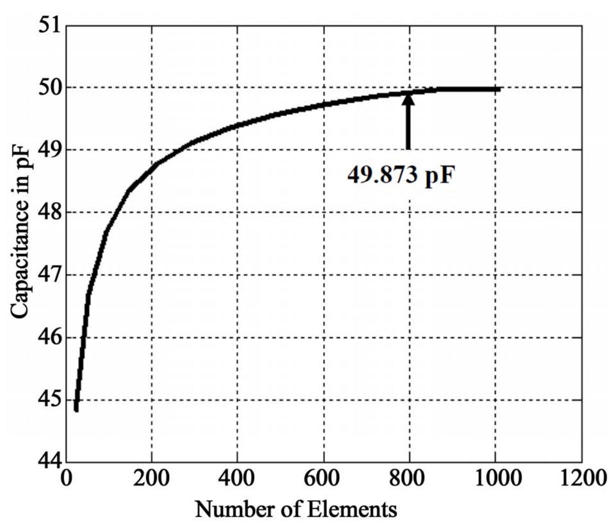

Figure 5. Convergence of data on capacitance for an isolated elliptical plate.

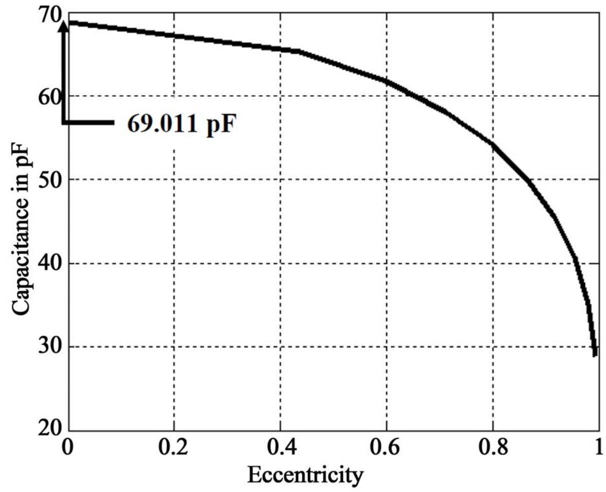

Figure 6. Capacitance of a metallic isolated elliptical plate as function of eccentricity of unit semi major axis.

Table 1. Numerical Value of Capacitance Data with Function of Eccentricity.

Eccentricity (e)	Capacitance (pF)
0.85	49.873
0.8	54.723
0.6	63.846
0.4	66.022
0.2	67.004
0	69.011 [8]

4. Discussion and Conclusions

This investigation demonstrates that convergence can be obtained for an elliptical plate for a finite number of elements which is approximately 725 elements for an isolated elliptical plate. With this number, the convergence achieved is at the fourth decimal place. The capacitance of an isolated elliptical plate decreases mono-

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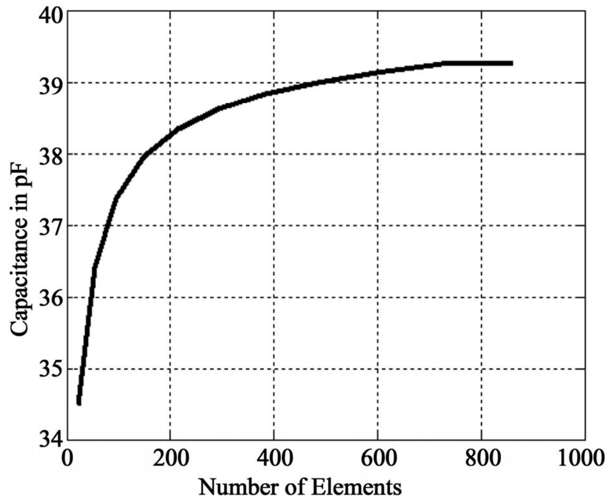

Figure 7. Convergence of parallel elliptical plates (d = 1).

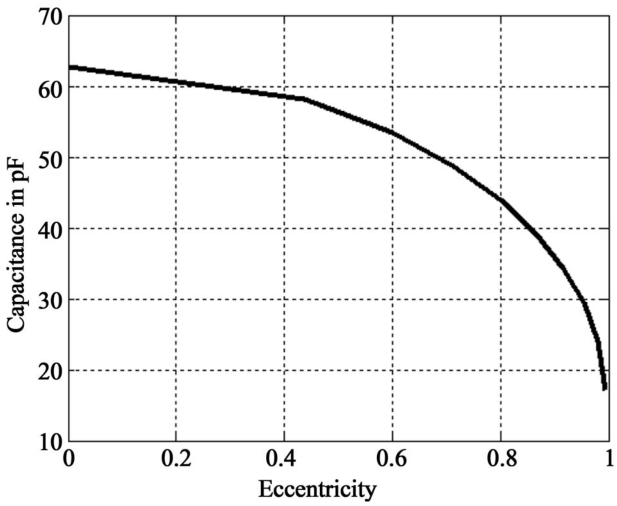

Figure 8. Capacitance of parallel elliptical plates as function of eccentricity of unit semi major axis.

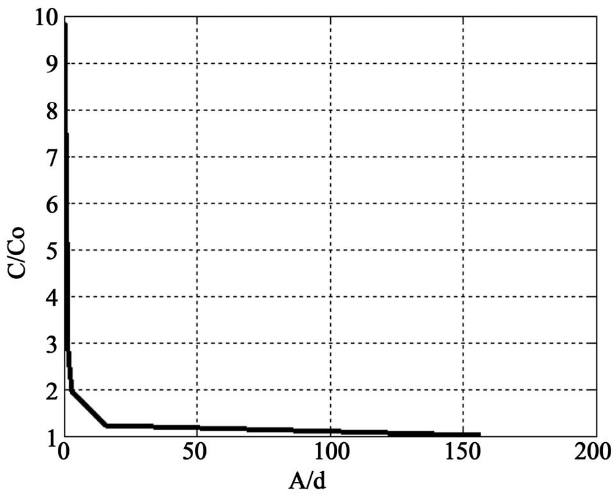

Figure 9. Capacitance of parallel elliptical plates as function of A/d.

Table 2. Capacitance data for the structure of Figure 4 as a function of distance.

Distance d	Capacitance (pF)
1	39.263
1.5	34.519
2	32.073
4	28.535
6	27.422

tonically with an increase in eccentricity. If the eccentricity = 0 which is the case of a circle, the results can be compared with an analytical result.

When the eccentricity = 0, which is a case of a circle, the result obtained by this technique is 6.90e-011 F which is very close to the approximate closed form expression of capacitance $8\varepsilon_0$ for a circular disk of unit radius [11]. **Figure 7** shows that the convergence of the parallel ellipse occurs at 863 elements at the third decimal point. **Figure 8** shows how the capacitance of a parallel elliptical plate decreases monotonically with an increase in eccentricity. The data presented in **Table 2** depicts that there is a decrease in the capacitance with an increase in the distance between the metallic elliptical plates. For the further validation, **Figure 9** shows that C/C_0 decreases as A/d increases, for very large A/d, which corresponds to the fringeless case, the ratio of C/C_0 is very close to unity.

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