Journal of Applied Mathematics and Physics, 2019, 7, 2564-2578
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

# On Trigonometric Numerical Integrator for Solving First Order Ordinary Differential Equation

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**How to cite this paper:** Obayomi, A.A., Ayinde, S.O. and Ogunmiloro, O.M. (2019) On Trigonometric Numerical Integrator for Solving First Order Ordinary Differential Equation. *Journal of Applied Mathematics and Physics*, **7**, 2564-2578.
https://doi.org/10.4236/jamp.2019.711175

**Received:** September 10, 2019
**Accepted:** November 2, 2019
**Published:** November 5, 2019



## Abstract

In this paper, we used an interpolation function with strong trigonometric components to derive a numerical integrator that can be used for solving first order initial value problems in ordinary differential equation. This numerical integrator has been tested for desirable qualities like stability, convergence and consistency. The discrete models have been used for a numerical experiment which makes us conclude that the schemes are suitable for the solution of first order ordinary differential equation.

## Keywords

Numerical Integrator, Ordinary Differential Equation, Initial Value Problems, Stability Analysis, Nonstandard Methods, Interpolation Methods

## 1. Introduction

### 1.1. Formulation of the Interpolating Function

Finite difference schemes have been in the forefront of the methods of using discrete models to approximate the solution of ordinary differential equations. Among the techniques used in building finite difference, scheme is the use of interpolation which requires the design of a basis function that is adequately differentiable in the domain of the numerical integration. Such basis function is then used to create a discrete version of the differential equation involved. This method has been used in the works of [1]-[6], etc. Notable among latest works on interpolating with trigonometric function includes [7], who constructed a new piecewise rational quadratic trigonometric spline with four local positive shape parameters in each subinterval is to visualize some given planar data. The order of approximation of the developed interpolating function was found to $0\left(h^3\right)$.

[8] developed a new method for smooth rational cubic trigonometric interpolation based on values of function which is being interpolated. This rational cubic trigonometric spline is used to constrain the shape of the interpolant in such a way that the uniqueness of the interpolating function for the given data would be replaced by uniqueness of the interpolating curve for the given data.

[9] constructed a new quadratic trigonometric B-spline with control parameters to address the problems related to two dimensional digital image interpolation. The newly constructed spline is then used to design an image interpolation scheme. Most of these works concentrated on application of trigonometric splines to digital imaging. We are interested in a general use of trigonometric interpolation functions for creating discrete models for the solution of ordinary differential Equation.

### 1.2. Nonstandard Modeling Techniques

The need for the nonstandard method came up due to some shortcomings of the standard method, in which the qualitative properties of the exact solutions are not usually transferred to the numerical solution. These shortcomings may create a lot of problems, which may affect the stability properties of the standard approach [10].

The concept of numerical instability and its proof by [10] [11], has led to the establishment of five major modeling rules proposed for the construction of difference schemes that will exhibit numerical stability. He also used these techniques to derive numerical models that are exact schemes for some classes of ordinary differential equations.

A finite difference scheme is called nonstandard finite difference method, if at least one of the following conditions is met [12]:

a) In the discrete derivative, the traditional denominator is replaced by a non-negative function $\varphi$ such that, $\varphi(h) = h + 0(h^2)$ as $h \to 0$;

b) Non-linear terms that occur in the differential equation are approximated in a non-local way, *i.e.* by a suitable function of several points of the mesh. The concept of nonstandard finite difference schemes was proposed by [10] as a solution to the numerical instability that exist in the use of finite difference schemes.

Since the discovery of this method, researchers like [5] [12] among others have suggested ways of building numerically reliable schemes using the nonstandard modeling rules. In this work we will create a scheme using the both of these two techniques above.

## 2. Derivation of the Scheme

Let assume that a solution of a differential equation can be represented by a function

$$F(x) = A\cos x + B\mathrm{e}^{-\propto x} - Qx \tag{1}$$

where $-\propto$ and $Q$ are simulation parameters and $A$ and $B$ are arbitrary constants.

Let a first order ordinary differential equation possess a real valued solution and be differentiable in its domain several times, then from (1) we can write:

$$y = A\cos x + B\mathrm{e}^{-\propto x} - Qx$$

$$y' = -A\sin x - \propto B\mathrm{e}^{-\propto x} - Q \tag{2}$$

$$y'' = -A\cos x + \propto^2 B\mathrm{e}^{-\propto x} \tag{3}$$

$$y''' = +A\sin x - \propto^3 B\mathrm{e}^{-\propto x} \tag{4}$$

$$y^{iv} = A\cos x + \propto^4 B\mathrm{e}^{-\propto x} \tag{5}$$

From (3) and (5) we have

$$y'' + y^{iv} = B\left(\propto^4 + \propto^2\right)\mathrm{e}^{-\propto x}$$

$$B = \frac{\left(y'' + y^{iv}\right)}{\left(\propto^4 + \propto^2\right)\mathrm{e}^{-\propto x}} \tag{6}$$

From (2) and (4) we have

$$y' + y''' = -B\left(\propto^3 + \propto\right)\mathrm{e}^{-\propto x}$$

$$y''' - y' = 2A\sin x + B\left(\propto - \propto^3\right)\mathrm{e}^{-\propto x}$$

$$A = \frac{\left(y''' - y'\right) - B\left(\propto - \propto^3\right)\mathrm{e}^{-\propto x}}{2\sin x} \tag{7}$$

Also from (3) and (5) we have

$$y^{iv} - y'' = 2A\cos x + \left(\propto^4 - \propto^2\right)B\mathrm{e}^{-\propto x}$$

$$A = \frac{\left(y^{iv} - y''\right) - B\left(\propto^4 - \propto^2\right)\mathrm{e}^{-\propto x}}{2\cos x} \tag{8}$$

For the discrete representation

$$y = A\cos x + B\mathrm{e}^{-\propto x} - Qx$$

$$y(x) = A\cos x + B\mathrm{e}^{-\propto x} - Qx$$

$$y(x_n) = A\cos x_n + B\mathrm{e}^{-\propto x_n} - Qx_n \tag{9}$$

$$y(x_{n+1}) = A\cos x_{n+1} + B\mathrm{e}^{-\propto x_{n+1}} - Qx_{n+1} \tag{10}$$

$$y(x_{n+1}) \equiv y_{n+1} \quad \text{and} \quad y(x_n) \equiv y_n$$

$$y(x_{n+1}) - y(x_n) \equiv y_{n+1} - y_n$$

$$\mathrm{e}^{\propto x_{n+1}} - \mathrm{e}^{\propto x_n} = \mathrm{e}^{\propto x_n}\left(\mathrm{e}^{\propto h} - 1\right)$$

$$M = \left(\mathrm{e}^{\propto h} - 1\right)$$

$$y_{n+1} = y_n - 2A\sin\left(x_n + \frac{h}{2}\right)\sin\frac{h}{2} + BM\mathrm{e}^{-\propto x_n} - Qh \tag{11}$$

From (7), (8) and (11)

$$y_{n+1} = y_n - 2\left\{\frac{\left(y^{iv} - y''\right) - B\left(\propto^4 - \propto^2\right)\mathrm{e}^{-\propto x_n}}{2\cos x}\right\}\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right) + \left\{\frac{\left(y'' + y^{iv}\right)}{\left(\propto^4 + \propto^2\right)\mathrm{e}^{-\propto x_n}}\right\}\left(\mathrm{e}^{\propto h} - 1\right)\mathrm{e}^{-\propto x_n} - Qh \tag{12}$$

Or

$$y_{n+1} = y_n - 2\left\{\frac{\left(y''' - y'\right) - B\left(\propto - \propto^3\right)\mathrm{e}^{-\propto x}}{2\sin x_n}\right\}\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right) + \left\{\frac{\left(y'' + y^{iv}\right)}{\left(\propto^4 + \propto^2\right)\mathrm{e}^{-\propto x_n}}\right\}\left(\mathrm{e}^{\propto h} - 1\right)\mathrm{e}^{-\propto x_n} - Qh \tag{13}$$

Let $f_n = y'$, $f_n^1 - y''$, $f_n^2 = y'''$, $f_n^3 = y^{iv}$

Then

$$y_{n+1} = y_n - 2\left\{\frac{\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\cos x_n}\left(f_n^3 - f_n^1\right) - \left\{\frac{\propto^4 - \propto^2}{\propto^4 + \propto^2}\right\}\left(f_n^3 + f_n^1\right)\right\} + \left\{\frac{\left(\mathrm{e}^{\propto h} - 1\right)\left(f_n^3 + f_n^1\right)}{\left(\propto^4 + \propto^2\right)}\right\} - Qh \tag{14}$$

Or

$$y_{n+1} = y_n - 2\left\{\frac{\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\sin x_n}\left(f_n^2 - f_n\right) - \left\{\frac{\propto - \propto^3}{\propto^4 + \propto^2}\right\}\left(f_n^3 + f_n^1\right)\right\} + \left\{\frac{\left(\mathrm{e}^{\propto h} - 1\right)\left(f_n^3 + f_n^1\right)}{\left(\propto^4 + \propto^2\right)}\right\} - Qh \tag{15}$$

$$E = \frac{\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\cos x_n},\quad F = \frac{\propto^4 - \propto^2}{\propto^4 + \propto^2},$$

$$G = \frac{\mathrm{e}^{\propto h} - 1}{\propto^4 + \propto^2},\quad H = \frac{\propto - \propto^3}{\propto^4 + \propto^2} \quad \text{and} \quad I = \frac{\sin\left(x_n + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\sin x_n} \tag{16}$$

$$y_{n+1} = y_n - 2\left\{E\left(f_n^3 - f_n^1\right) - F\left(f_n^3 + f_n^1\right)\right\} + \left\{G\left(f_n^3 + f_n^1\right)\right\} - Qh,\quad \cos x_n \neq 0$$

$$y_{n+1} = y_n - 2\left\{I\left(f_n^2 - f_n\right) - H\left(f_n^3 + f_n^1\right)\right\} + \left\{G\left(f_n^3 + f_n^1\right)\right\} - Qh,\quad \sin x_n \neq 0$$

$$y_{n+1} = y_n + \left(2F - 2E + G\right)f_n^3 + \left(2E + 2F + G\right)f_n^1 - Qh,\quad \cos x_n \neq 0 \tag{17}$$

$$y_{n+1} = y_n + \left(2H + G\right)f_n^3 - \left(2I\right)f_n^2 + \left(2H + G\right)f_n^1 + \left(2I\right)f_n - Qh,\quad \sin x_n \neq 0 \tag{18}$$

Let

$$R=\left(2F-2E+G\right),\ S=\left(2E+2F+G\right),$$
$$U=\left(2H+G\right),\ T=\left(-2I\right),\ M=\left(2I\right) \tag{19}$$

Substitute (19) into (17) and (18), we have the integrator in the form (20) and (21) respectively

$$y_{n+1}=y_n+Rf_n^3+Sf_n^1-Qh \tag{20}$$

$$y_{n+1}=y_n+Uf_n^3+Tf_n^2+Uf_n^1+Mf_n-Qh \tag{21}$$

## 3. Properties of the Integration Method

### 3.1. Definition [13]

Any algorithm for solving a differential equation in which the approximation $y_{n+1}$ to the solution at $x_{n+1}$ can be calculated iff $x_n, y_n$ and $h$ are known is called a one step method. It is a common practice to write the functional dependence $y_{n+1}$ on the quantities $x_n, y_n$ and $h$ in the form

$$y_{n+1}=y_n+\phi\left(x_n,y_n,h\right)$$

where $\phi\left(x_n,y_n,h\right)$ is the incremental function.

### 3.2. Definition [1]

A numerical scheme with an incremental $\phi\left(x_n,y_n,h\right)$ is said to be consistent with the initial value problem $y'=f\left(x,y\right), y\left(x_0\right)=y_0$ if the incremental function is identically zero at $t_0$ when $h=0$.

### 3.3. Theorem [13]

Let the incremental function of the scheme defined in the one step scheme above be continuous and jointly as a function of its arguments in the region defined by

$$x\in\left[a,b\right] \text{ and } y\in\left(-\infty,\infty\right),\ 0\le h\le h_0$$

where $h_0>0$ and let there exists a constant $L$ such that $\phi\left(x_n,y_n,h\right)-\phi\left(x_n,y_n^*,h\right)\le L\left|y_n-y_n^*\right|$ for all $\left(x_n,y_n,h\right)$ and $\left(x_n,y_n^*,h\right)$ in the region just defined then the relation $\left(x_n,y_n,0\right)=\left(x_n,y_n^*\right)$ is a necessary condition for the convergence of the new scheme.

### 3.4. Theorem [1]

Let $y_n=y\left(x_n\right)$ and $p_n=p\left(x_n\right)$ denote two different numerical solution of the differential equation with the initial condition specified a $y_0=y\left(x_0\right)=\xi$ and $p_0=p\left(x_0\right)=\xi^*$ respectively such that $\left|\xi-\xi^*\right|<\varepsilon,\ \varepsilon>0$.

If the two numerical estimates are generated by the integration scheme, we have

$$y_{n+1}=y_n+h\phi\left(x_n,y_n,h\right)$$

$$p_{n+1}=p_n+h\phi\left(x_n,p_n,h\right)$$

The condition that $\left|y_{n+1}-p_{n+1}\right|\le K\left|\xi-\xi^*\right|$ is the necessary and sufficient condition for the stability and convergence of the schemes.

### 3.5. Proof of Convergence of the Integration Method

The increment function $\varnothing(x_n, y_n; h)$ can be written in the form

$$\begin{aligned}&\varnothing(x_n, y_n; h)\\&= \left\{Mf(x_n, y_n) + Uf^{(1)}(x_n, y_n) + Tf^{(2)}(x_n, y_n) + Uf^{(3)}(x_n, y_n) - Qh\right\}\end{aligned} \tag{22}$$

Consider Equation (19), we can also write

$$\varnothing(x_n, y_n^*; h) = \left\{Mf(x_n, y_n^*) + Lf^{(1)}(x_n, y_n^*) + Tf^{(2)}(x_n, y_n^*) + Uf^{(3)}(x_n, y_n^*) - Qh\right\}$$

$$\begin{aligned}&\varnothing(x_n, y_n^*; h) - \varnothing(x_n, y_n; h)\\&= M\left[f(x_n, y_n^*) - f(x_n, y_n)\right] + U\left[f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n)\right]\\&+ T\left[f^{(2)}(x_n, y_n^*) - f^{(2)}(x_n, y_n)\right] + U\left[f^{(3)}(x_n, y_n^*) - f^{(3)}(x_n, y_n)\right] - Qh + Qh\end{aligned} \tag{23}$$

Let $\bar{y}$ be defined as a point in the interior of the interval whose points are $y$ and $y^*$, applying mean value theorem, we have

$$f(x_n, y_n^*) - f(x_n, y_n) = \frac{\partial f(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)$$

$$f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n) = \frac{\partial f^{(1)}(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)$$

$$f^{(2)}(x_n, y_n^*) - f^{(2)}(x_n, y_n) = \frac{\partial f^{(2)}(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)$$

$$f^{(3)}(x_n, y_n^*) - f^{(3)}(x_n, y_n) = \frac{\partial f^{(3)}(x_n, \bar{y})}{\partial y_n}(y_n^* - y_n)$$

We define

$$L = \sup_{(x_n, y_n)\in D}\frac{\partial f(x_n, y_n)}{\partial y_n}, \quad L_1 = \sup_{(x_n, y_n)\in D}\frac{\partial f^{(1)}(x_n, y_n)}{\partial y_n}$$

$$L_2 = \sup_{(x_n, y_n)\in D}\frac{\partial f^{(2)}(x_n, y_n)}{\partial y_n}, \quad L_3 = \sup_{(x_n, y_n)\in D}\frac{\partial f^{0}(x_n, y_n)}{\partial y_n}$$

Therefore

$$\begin{aligned}&\varnothing(x_n, y_n^*; h) - \varnothing(x_n, y_n; h)\\&= M\left[f(x_n, y_n^*) - f(x_n, y_n)\right] + U\left[f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n)\right]\\&+ T\left[f^{(2)}(x_n, y_n^*) - f^{(2)}(x_n, y_n)\right] + U\left[f^{(3)}(x_n, y_n^*) - f^{(3)}(x_n, y_n)\right]\\&= ML(y_n^* - y_n) + UL_1(y_n^* - y_n) + TL_2(y_n^* - y_n) + UL_3(y_n^* - y_n)\end{aligned} \tag{24}$$

Taking the absolute value of both sides

$$\begin{aligned}&\left|\varnothing(x_n, y_n^*; h) - \varnothing(x_n, y_n; h)\right|\\&\le \left|ML(y_n^* - y_n) + UL_1(y_n^* - y_n) + TL_2(y_n^* - y_n) + UL_3(y_n^* - y_n)\right|\\&\le \left|ML + UL_1 + TL_2 + UL_3\right|\left|y^* - y\right|\end{aligned} \tag{25}$$

If we let $K = \left|ML + UL_1 + TL_2 + UL_3\right|$

then our Equation (23) turns to

$$\left|\varnothing\left(x_n, y_n^*; h\right) - \varnothing\left(x_n, y_n; h\right)\right| \le K\left|y^* - y\right| \tag{26}$$

which is the condition for convergence.

### 3.6. Consistence of the Integration Method

Consider an initial value problem of the form

$$y' = f(x, y), \quad y(x_o) = y_o \tag{27}$$

Having an integrator of the form

$$y_{n+1} = y_n + \phi\left(x_n, y_n, h\right)$$

which can be obtained using (17) and (18) above and applying the rule in Section 1.2 a).

The renormalized nonstandard form of the schemes will be

$$y_{n+1} = y_n + \varphi\left\{Rf_n^3 + Sf_n^1 - Qh\right\}, \quad \cos x_n \neq 0 \tag{28}$$

$$y_{n+1} = y_n + \varphi\left\{Uf_n^3 + Tf_n^2 + Uf_n^1 + Mf_n - Qh\right\}, \quad \sin x_n \neq 0 \tag{29}$$

where $y_{n+1} = y_n + \varphi\left(x_n, y_n; h\right)$, $\varphi = \sin(\propto h)$ then

If $h = 0$, $E = 0$, $F = \dfrac{\propto^4 - \propto^2}{\propto^4 + \propto^2}$, $G = 0$ and $H = \dfrac{\propto - \propto^3}{\propto^4 + \propto^2}$, $I = 0$ and $\varphi = 0$ then (28) and (29) reduced to $y_{n+1} = y_n$

$$\Rightarrow \varphi\left(x_n, y_n; 0\right) = 0 \tag{30}$$

It is a known fact that a consistent method has order of at least one. Therefore, the new numerical integrator is consistent since Equations (28) and (29) can be reduced to (30) when $h = 0$.

### 3.7. Stability Analysis of the Integration Method

We shall establish the stability analysis of the integrator by considering the theorem established by [14].

Let $y_n = y\left(x_n\right)$ and $P_n = P\left(x_n\right)$ denote two different numerical solutions of initial value problem of ordinary differential Equation (25) with the initial conditions specified as $y\left(x_o\right) = \eta$ and $p\left(x_o\right) = \eta^*$ respectively, such that $\left|\eta - \eta^*\right| < \varepsilon$, $\varepsilon > 0$. If the two numerical estimates are generated by the integrator (19). From the increment function (26), we have

$$y_{n+1} = y_n + \varphi\varnothing\left(x_n, y_n; h\right) \tag{31}$$

$$P_{n+1} = P_n + \varphi\varnothing\left(x_n, p_n; h\right) \tag{32}$$

The condition that

$$\left|y_{n+1} - P_{n+1}\right| \le K\left|\eta - \eta^*\right| \tag{33}$$

is the necessary and sufficient condition that our new method (19) be stable and convergent.

**Proof**

From (29) we have

$$y_{n+1} = y_n + \varphi\left(2H+G\right) f_n^3 - \varphi\left(2I\right) f_n^2 + \varphi\left(2H+G\right) f_n^1 + \varphi\left(2I\right) f_n - \varphi Qh$$

$$y_{n+1} = y_n + \varphi\left\{Uf_n^3 + Tf_n^2 + Uf_n^1 + Mf_n - Qh\right\} \tag{34}$$

Then let

$$y_{n+1} = y_n + \varphi\left\{Mf\left(x_n, y_n\right) + Uf^{(1)}\left(x_n, y_n\right) + Tf^{(2)}\left(x_n, y_n\right) + Uf^{(3)}\left(x_n, y_n\right) - Qh\right\} \tag{35}$$

and

$$p_{n+1} = p_n + \varphi\left\{Mf\left(x_n, p_n\right) + Uf^{(1)}\left(x_n, p_n\right) + Tf^{(2)}\left(x_n, p_n\right) + Uf^{(3)}\left(x_n, p_n\right) - Qh\right\} \tag{36}$$

Therefore,

$$\begin{aligned} & y_{n+1} - p_{n+1} \\ & = y_n - p_n + \varphi\left\{M\left[f\left(x_n, y_n\right) - f\left(x_n, p_n\right)\right] + U\left[f^1\left(x_n, y_n\right) - f^1\left(x_n, p_n\right)\right]\right. \\ & \left. + T\left[f^2\left(x_n, y_n\right) - f^2\left(x_n, p_n\right)\right] + U\left[f^3\left(x_n, y_n\right) - f^3\left(x_n, p_n\right)\right]\right\} \end{aligned} \tag{37}$$

Applying the mean value theorem as before, we have

$$\begin{aligned} y_{n+1} - p_{n+1} = y_n - p_n + \varphi & \left\{M\left[\frac{\partial f\left(x_n, p_n\right)}{\partial y_n}\left(y_n - p_n\right)\right]\right\} \\ & + \varphi\left\{U\left[\frac{\partial f^{(1)}\left(x_n, p_n\right)}{\partial y_n}\left(y_n - p_n\right)\right] + T\left[\frac{\partial f^{(2)}\left(x_n, p_n\right)}{\partial y_n}\left(y_n - p_n\right)\right]\right. \\ & \left. + U\left[\frac{\partial f^{(3)}\left(x_n, p_n\right)}{\partial y_n}\left(y_n - p_n\right)\right]\right\} \end{aligned} \tag{38}$$

We define

$$L = \sup_{(x_n, y_n)\in D} \frac{\partial f\left(x_n, \bar{y}_n\right)}{\partial y_n}, \quad L_1 = \sup_{(x_n, y_n)\in D} \frac{\partial f^{(1)}\left(x_n, \bar{y}\right)}{\partial y_n}$$

$$L_2 = \sup_{(x_n, y_n)\in D} \frac{\partial f^{(2)}\left(x_n, \bar{y}\right)}{\partial y_n}, \quad L_3 = \sup_{(x_n, y_n)\in D} \frac{\partial f^{3}\left(x_n, \bar{y}\right)}{\partial y_n}$$

Therefore

$$\begin{aligned} y_{n+1} - p_{n+1} = & \varphi ML\left(y_n - p_n\right) + \varphi UL_1\left(y_n - p_n\right) \\ & + \varphi TL_2\left(y_n - p_n\right) + \varphi UL_3\left(y_n - p_n\right) \end{aligned} \tag{39}$$

Taking the absolute value of both sides

$$\begin{aligned} & \left|\varnothing\left(x_n, y_n^*; h\right) - \varnothing\left(x_n, y_n; h\right)\right| \\ & \le \left|\varphi ML\left(y_n - p_n\right) + \varphi UL_1\left(y_n - p_n\right) + \varphi TL_2\left(y_n - p_n\right) + \varphi UL_3\left(y_n - p_n\right)\right| \\ & \le \left|\varphi ML + \varphi UL_1 + \varphi TL_2 + \varphi UL_3\right|\left|y_n - p_n\right| \end{aligned} \tag{40}$$

If we let $K = \left|\varphi ML + \varphi UL_1 + \varphi TL_2 + \varphi UL_3\right|$

then our Equation (34) turns to

$$\left|\varnothing\left(x_n, y_n; h\right) - \varnothing\left(x_n, p_n; h\right)\right| \le K\left|y_n - p_n\right| \tag{41}$$

which is the condition for convergence.

and $y(x_o)=\eta$, $P(x_o)=\eta^*$, given $\varepsilon>0$, then

$$|y_{n+1}-p_{n+1}|\le N|y_n-p_n| \tag{42}$$

and

$$|y_{n+1}-p_{n+1}|\le N|\eta-\eta^*|<\varepsilon, \text{ for every } \varepsilon>0 \tag{43}$$

Then we conclude that our method (29) is stable and convergent.

**Note:** Similar arguments can be used to proof the stability, convergence and consistency of the scheme $y_{n+1}=y_n+\varphi\{Rf_n^3+Sf_n^1-Qh\}$.

## 4. The Implementation of the Integration Method

### 4.1. Application of the Finite Difference Schemes to a Deferential Equation I

We derive a scheme for the first equation thus

$$y'=25+y^2,\ y(0)=0,\ y=5\tan(5x) \tag{44}$$

$$y''=2yy'=2y(25+y^2)=50y+2y^3 \tag{45}$$

$$y'''=100+50y^2+150y^2+6y^4=100+200y^2+6y^4 \tag{46}$$

$$y^{iv}=5000+800y^3+24y^5 \tag{47}$$

Since we are integrating from zero and $\cos x_n\neq 0$

$$y_{n+1}=y_n+\varphi\{Rf_n^3+Sf_n^1-Qh\}$$

where $f_n=y'$, $f_n^1-y''$, $f_n^2=y'''$, $f_n^3=y^{iv}$.

### 4.2. Application of the Finite Difference Schemes to Logistic Model

We now derive a scheme for the first model

$$y'=y-y^2,\ y=\frac{y_0}{y_0+(1-y_0)e^t},\ y(0)=0.5 \tag{48}$$

$$y''=y'-2yy'$$

$$y''=y-3y^2+2y^3 \tag{49}$$

$$y'''=y-6y^2+6y^3-y^2+6y^3-6y^4 \tag{50}$$

$$y^{iv}=y-15y^2+50y^3-60y^4+24y^5 \tag{51}$$

Since we are integrating from zero we will use

$$y_{n+1}=y_n+\varphi\{Rf_n^3+Sf_n^1-Qh\},$$

where $f_n=y'$, $f_n^1=y''$, $f_n^2=y'''$, $f_n^3=y^{iv}$.

### 4.3. The Hybrid Nonstandard Schemes

The new scheme will be obtained by substituting the derivatives and applying it to our scheme

$$y_{n+1}=y_n+\varphi\{Ry''+Sy''-Qh\} \tag{52}$$

The new standard scheme will be named (New SCH STD) with

$$\varphi = 1, \quad h = h \tag{53}$$

For each of the examples the hybrid schemes will be obtained by applying the renormalization techniques.

The hybrid scheme (NEW SCH h) is obtained by using $\varphi = \sin(\propto h)$, step size

$$h = h \tag{54}$$

The hybrid scheme (NEW SCH SIN) by using $\varphi = \sin(\propto h)$, step size

$$h = \sin(rh), \quad r \in R \tag{55}$$

The hybrid scheme (NEW SCH EXP) by using $\varphi = \sin(\propto h)$, step size

$$h = \frac{\left(e^{\lambda h} - 1\right)}{\lambda}, \quad \lambda \in R \tag{56}$$

The choice of this denominator function as step size is informed by the works of [5] [12].

### 4.4. Experimentation and Result

The following are the 3D graphs obtained from the schemes when applied to the two models (**Figures 1-14**). We have used same parameters, step size, denominator functions and simulation parameters $\propto$ and $Q$ to test the two differential equation models.

#### 4.4.1. The Graph for the Method of Differential Equation Model I

**Graph of the Schemes with parameters:** $h$ = 0.001, $\propto$ = (4.25 - 14), $Q$ = 36, $\lambda$ = 0.26.

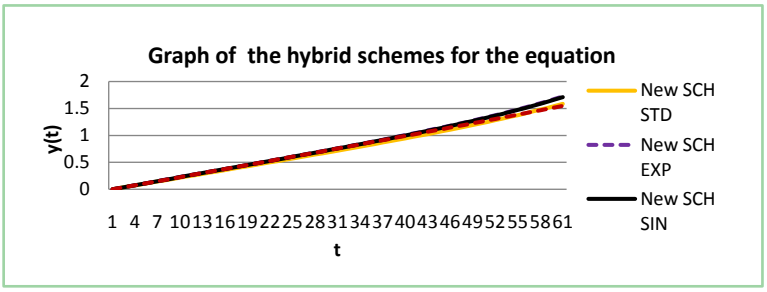


**Figure 1.** Solution curves for the schemes of Model I.

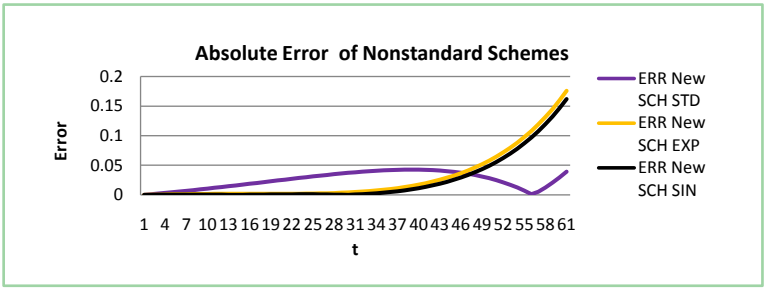


**Figure 2.** Graph of absolute error for the schemes in **Figure 1**.

**Graph of the Schemes with parameters:** $h = 0.0001$, $\propto = (4.25 - 14)$, $Q = 36$, $\lambda = 0.26$.

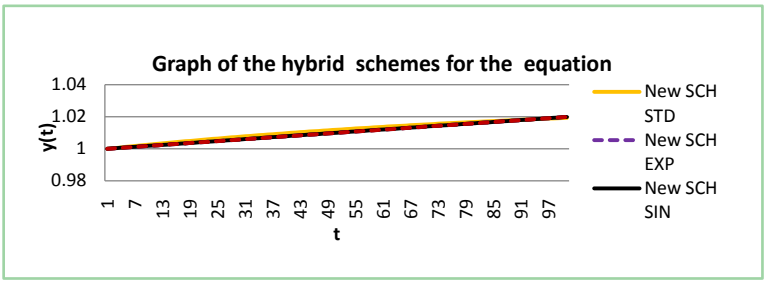


**Figure 3.** Solution curves for the schemes of model II.

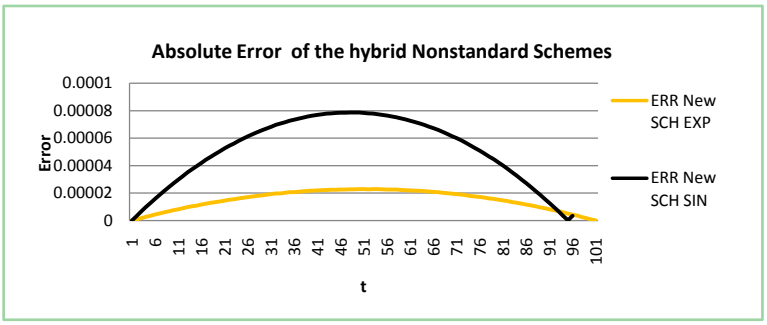


**Figure 4.** Graph of absolute Error for schemes in **Figure 3**.

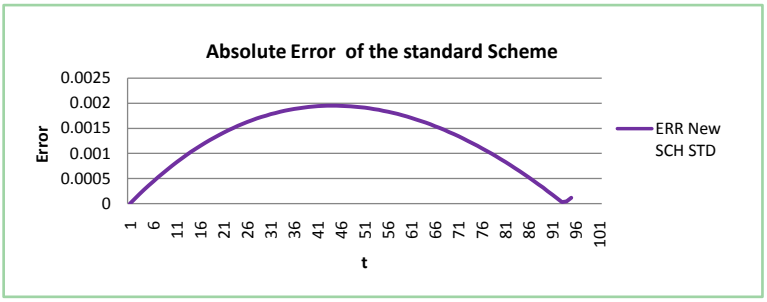


**Figure 5.** Graph of absolute Error for schemes in **Figure 3**.

### 4.4.2. The Graph for the Schemes of the Logistic Model II

**Graph of the Schemes with parameters:** $h = 0.001$, $\propto = (6.39 - 7.3)$, $Q = 36$, $\lambda = 0.0026$.

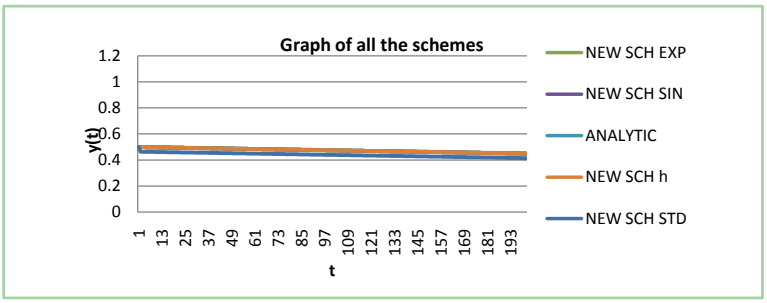


**Figure 6.** Solution curves for the schemes of Model II.

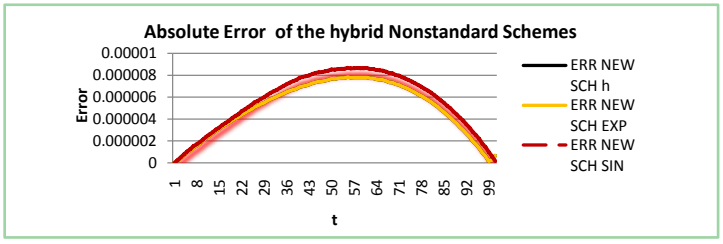


**Figure 7.** Graph of absolute error for the schemes in **Figure 6**.

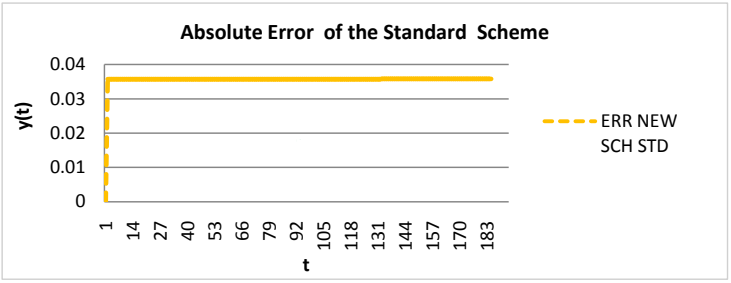


**Figure 8.** Graph of absolute error for the standard scheme in **Figure 6**.

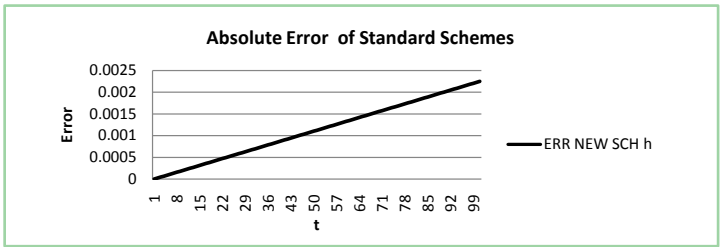


**Figure 9.** Error curve for the standard scheme in **Figure 6**.

**Graph of the Schemes of model II with parameters:** $h = 0.0001$, $\propto = (6.39 - 7.3)$, $Q = 36$, $\lambda = 0.0016$.

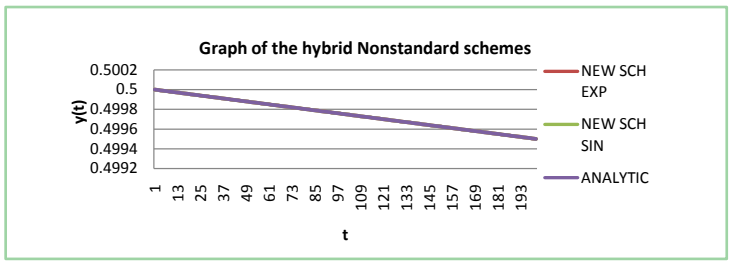


**Figure 10.** Solution curves for the schemes of model II.

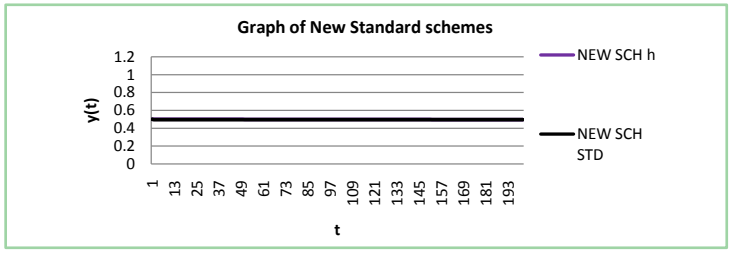


**Figure 11.** Solution curves for the standard schemes of model II.

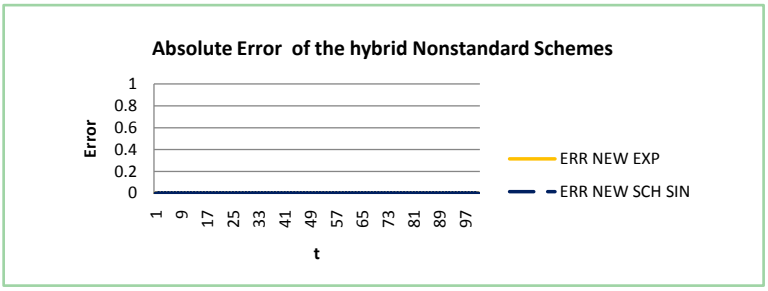


**Figure 12.** Graph of absolute Error for the hybrid Nonstandard schemes in **Figure 10**.

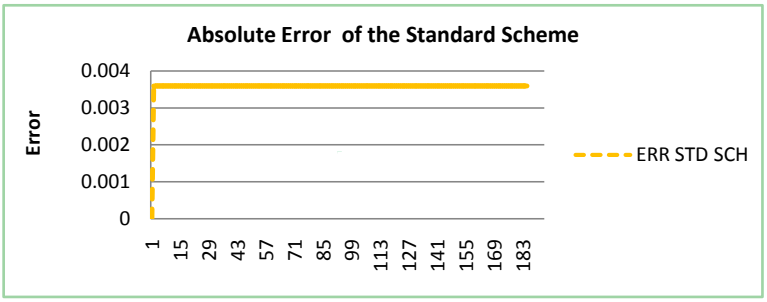


**Figure 13.** Graph of absolute Error for the Standard schemes in **Figure 11**.

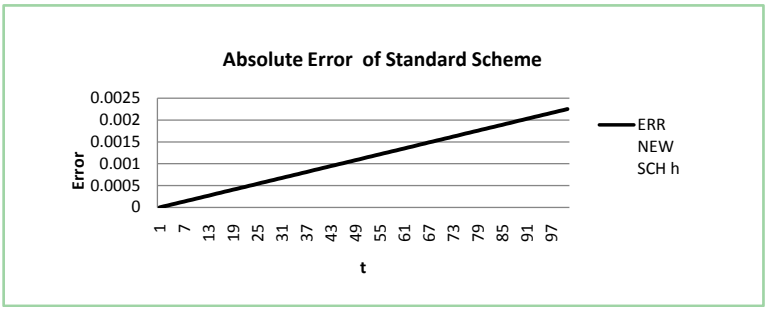


**Figure 14.** Graph of absolute Error for the Standard schemes in **Figure 11**.

## 5. Discussion and Conclusion

The derived simulation models have been tested with the control parameters $\propto$ and $Q$. We also applied the Nonstandard method by modifying denominator function $\varphi$, which also provides for parameters $\lambda$ and $r$ that can be chosen to obtain iteratively assigned step size as denominator. The discrete model worked for the tested differential equation. The solution curves of the hybrid schemes follow the analytical solutions of the respective equations monotonically are shown in **Figure 1**, **Figure 3**, **Figure 5**, **Figure 10** and **Figure 12**. The numerical properties of the schemes like linear stability, convergence and consistency have been proved analytically. During the course of simulating the equations, we varied these control parameters to obtain family of curves that are very close to the analytic solution. The scheme NEW SCH h and NEW SCH STD are the standard schemes because we maintain the $h = h$ as step size in both cases. The result showed that the two schemes do not possess smooth monotonicity properties (see **Fig-**

ure 6 and Figure 11) and diverge if $\varphi = 1$ (see Figure 14). These two schemes possess the highest absolute error of deviation from the analytic solution (see Figure 8, Figure 9, Figure 13 and Figure 14). These results confirm the good qualities of Nonstandard modeling techniques [15]. The choice of appropriate values for variables $\lambda$ and $r$ can be determined using the conditions set by [5] [12]. The graph of Absolute error in Figure 4, Figure 7 and Figure 12 has demonstrated these qualities. The Nonstandard schemes of Model II produced absolute errors that are very close to zero. It is also observed that these two hybrid schemes are exact schemes for the logistic equation when $h \le 0.0001$ because the result of the schemes is the same with the analytic solution (see Figure 12). This work can be extended by add higher degree polynomial to the basis function for a possible improved performance [14] [16]. We can conclude that the discrete model is suitable for the solution of first order ordinary differential equation as proposed.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References

[1] Fatunla, S.O. (1988) Numerical Methods for Initial Values Problems on Ordinary Differential Equations. Academic Press, New York.

[2] Ibijola, E.A. (2000) Some Methods of Numerical Solutions of Ordinary Differential Equations. *Far East Journal of Applied Mathematics*, **4**, 371-390.

[3] Partidar, K.C. (2005) On the Use of Nonstandard Finite Difference Methods. *Journal of Difference Equations and Applications*, **11**, 735-758. https://doi.org/10.1080/10236190500127471

[4] Ibijola, E.A. and Obayomi, A.A. (2012) A New Numerical Scheme for the Solution of the Logistic Equation. *Advances in Differential Equations and Control Processes*, **10**, 1-18.

[5] Obayomi, A.A. and Oke, M.O. (2016) Development of New Nonstandard Denominator Function for Finite Difference Schemes. *Journal of Nigerian Association of Mathematical Physics*, **33**, 50-60.

[6] Obayomi, A.A. (2018) Development of a Discrete Model for the Tsunami Tidal Waves. *Journal of Mathematics and Computer Science*, **8**, 98-113.

[7] Liu, S.J., Chen, Z.L. and Zhu, Y.P. (2015) Rational Quadratic Trigonometric Interpolation Spline for Data Visualization. *Mathematical Problems in Engineering*, **2015**, Article ID: 983120. https://doi.org/10.1155/2015/983120

[8] Rana, S.S., Dube, M. and Tiwari, P. (2013) Rational Trigonometric Interpolation and Constrained Control of the Interpolant Curves. *International Journal of Computer Applications*, **67**, 40-44. https://doi.org/10.5120/11387-6671

[9] Hussain, M.Z., Abbas, S. and Irshad, M, (2017) Quadratic Trigonometric B-Spline for Image Interpolation Using GA. *PLoS ONE*, **12**, e0179721. https://doi.org/10.1371/journal.pone.0179721

[10] Mickens, R.E. (1994) Nonstandard Finite Difference Models of Differential Equations. *World Scientific*, **115**, 144-162. https://doi.org/10.1142/2081

[11] Mickens, R.E. (1981) Nonlinear Oscillations. Cambridge University Press, New York.

[12] Anguelov, R. and Lubuma, J.M.S. (2003) Nonstandard Finite Difference Method by Nonlocal Approximation. *Mathematics and Computers in Simulation*, **61**, 465-475. https://doi.org/10.1016/S0378-4754(02)00106-4

[13] Henrici, P. (1962) Discrete Variable Methods in ODE. John Willey & Sons, New York.

[14] Zill, D.G. and Cullen, R.M. (2005) Differential Equations with Boundary Value Problems. 6th Edition, Cole Thompson Learning Academic Resource Center, 90-101.

[15] Lambert, J.D. (1973) Introductory Mathematics for Scientists and Engineers. John Wiley & Sons, New York.

[16] Dreyer, T.P. (2005) Modeling with Ordinary Differential Equations. CRC Press, Boca Raton, FL.