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Regularity of Global Attractors for the Kirchhoff Wave Equation

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Abstract

In this paper, we mainly use operator decomposition technique to prove the global attractors which in $H_0^1 \times L^2$ for the Kirchhoff wave equation with strong damping and critical nonlinearities, are also bounded in $H^2 \times H_0^1$.

Keywords

The Kirchhoff Wave Equation, Critical Exponent, The Regularity of Global Attractor

1. Introduction

In this paper, we discuss the regularity of global attractors for the following Kirchhoff wave equation

$$u_{tt} - \left(1 + \epsilon \|\nabla u\|^2\right) \Delta u - \Delta u_t + f(u_t) + h(u) = g(x) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with the smooth boundary $\partial\Omega$, $\epsilon \in [0,1]$, $f(m)$ and $h(m)$ are nonlinear functions and $g(x)$ is an external force term which is independent of time.

G. Kirchhoff [1] introduced the Equation (1.1) in $\mathbb{R}^1$ without dissipation $-\Delta u_t$ and nonlinear perturbations $f(u_t)$ and $h(u)$, and described the oscillation of an elastic stretched string. Furthermore, if the string is made up of the viscoelastic material of rate-type, the equation with the strong damping $-\Delta u_t$ appeared [2]. Since $\epsilon = 0$, the Equation (1.1) became the following strongly damped semi-linear wave equation

$$u_{tt} - \Delta u - \Delta u_t + f(u_t) + h(u) = g(x), \tag{1.3}$$

which described the thermal evolution and $h(u)$ denoted a source term depending nonlinearly on displacement, $f(u_t)$ denoted a nonlinearly temperature-dependent internal source term [3]. With different conditions about the growth exponents q and p of the nonlinearities $f(u_t)$ and $h(u)$, some scholars [4] [5] analyzed the longtime behaviour of solutions of (1.3)-(1.2) by the global and exponential attractors in a bounded region of $\mathbb{R}^3$. When the nonlinearities are of fully supercritical growth, which lead to that the weak solutions of the equation lose their uniqueness. Z. J. Yang and Z. M. Liu [6] established the existence of global attractor for the subclass of limit solutions of (1.3)-(1.2) by using J. Ball's attractor theory on the generalized semiflow. Recently, I. Chueshov [7] founded that the Kirchhoff wave equation with strong nonlinear damping was still well-posed and the related evolution semigroup had a finite-dimensional global attractor in $H = H_0^1 \cap L^{p+1} \times L^2$ in the sense of "partially strong topology". Without "partially strong topology", P. Y. Ding, Z. J. Yang [8] proved the existence of a finite-dimensional global attractor in the natural energy space. And H. L. Ma and C. K. Zhong [9] proved that global attractors for the Kirchhoff equations with strong nonlinear damping attracted $H = H_0^1 \times L^2$ -bounded set with respect to the $H_0^1 \times H_0^1$ norm.

Since $\epsilon > 0$, the following quasi-linear wave equation of Kirchhoff type

$$u_{tt} - \left(1 + \left\|\nabla u(t)\right\|^2\right)\Delta u + u_t + g(u) = f(x) \tag{1.4}$$

was studied by M. Nakao, and the author proved the existence and absorbing properties of attractors in a local sense [10]. Replacing u_t with $-\Delta u_t$, Y. H. Wang and C. K. Zhong [11] proved the upper semicontinuity of pullback attractors in non-autonomous case. Then Z. J. Yang and Y. Q. Wang [12] studied the longtime behavior of the Kirchhoff type equation with a strong dissipation and proved that the continuous semigroup $S(t)$ possessed global attractors in the phase spaces with low regularity. As for the Kirchhoff wave equation with strong damping and critical nonlinearities, Z. J. Yang and F. Da [13] also studied the stability for the Kirchhoff wave equation with strong damping and critical nonlinearities and proved the existence of global attractors and exponential attractors. Comparing with many researches about the longtime dynamic behavior of solutions for the Kirchhoff wave equation with different types of dissipations [14]-[23], there are few researches about problem of (1.1)-(1.2). And the attractor is a key point for studying these properties, we introduce readers to see the classical book [24].

Based on these, the purpose of this paper is to prove the global attractor of problem (1.1)-(1.2), which attracts every $H_0^1(\Omega) \times L^2(\Omega)$ -bounded set that is compacted in $H^2(\Omega) \times H_0^1(\Omega)$ by the way in ([25], Theorem 3.1). And we also establish the asymptotic compactness of the global attractor by operator decomposition technique ([24], Theorem 1.1). So these jobs provide a way to research the longtime dynamic behaviour of such Kirchhoff wave equations, and also reflect the strong damped properties of Δu_t to some extent.

The paper is arranged as follows. In Section 2, we verify some preliminaries. In Section 3, we prove the existence of the global attractor. In Section 4, we prove the regularity of the global attractor.

2. Preliminaries

Let $A=-\Delta$ on $L^2(\Omega)$ with $D(A)=H^2\cap H_0^1$, and A strictly positive on H_0^1. We define the spaces $H^m=D\left(A^{\frac{m}{2}}\right),(m\in\mathbb{R})$ are Hilbert spaces with the following scalar products and the norms

$$\langle u,v\rangle_m=\left\langle A^{\frac{m}{2}}u,A^{\frac{m}{2}}v\right\rangle,\quad \|u\|_{H^m}=\left\|A^{\frac{m}{2}}u\right\|. \tag{2.1}$$

Let $\lambda_1(>0,\lambda<\lambda_1)$ be the first eigenvalue of A, then $B=A-\lambda I$ with $D(B)=D(A)$.

We define the phase space $X=H_0^1\times L^2$ with usual graph norm. Let $\varphi(\xi)=f(\xi)+\lambda\xi$, then problem (1.1)-(1.2) becomes

$$u_{tt}+\left(1+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\right)Au+Bu_t+\varphi(u_t)+h(u)=g, \tag{2.2}$$

$$u(0)=u_0,\quad u_t(0)=u_1. \tag{2.3}$$

For any $s>r$, we have the continuous embeddings $H^s\hookrightarrow H^r$,

$$H^s\hookrightarrow L^{\frac{6}{3-2s}}(\Omega),\ \forall s\in\left[0,\frac{3}{2}\right), \tag{2.4}$$

and the following inequalities hold true:

Interpolation inequality: if $r=\theta s+(1-\theta)q$, where $r,s,q\in\mathbb{R},s\geq q$ and $\theta\in[0,1]$, then there exists a constant $C>0$ such that

$$\|u\|_r\leq C\|u\|_s^{\theta}\|u\|_q^{1-\theta},\quad \forall u\in H^s. \tag{2.5}$$

The Generalized Poincare inequality:

$$\lambda_1\|u\|_\alpha^2\leq\|u\|_{\alpha+1}^2,\quad \forall u\in H^{\alpha+1}, \tag{2.6}$$

where $\lambda_1>0$ is the first eigenvalue of A.

The Young's inequality with ε: Let $a>0,b>0,\varepsilon>0,p>1,q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\leq\frac{\varepsilon a^p}{p}+\frac{\varepsilon^{-\frac{q}{p}}b^q}{q}, \tag{2.7}$$

especially, when $p=q=2$, then

$$ab\leq\varepsilon a^2+\frac{b^2}{4\varepsilon}. \tag{2.8}$$

The Gronwall inequality (differential form): let $\eta(\cdot)$ is nonnegative continuous differentiable function (or nonnegative absolutely continuous function), and satisfy

$$\eta'(t) \le \phi(t)\eta(t) + \varphi(t), \quad t \in [0,T], \tag{2.9}$$

here $\phi(t), \varphi(t)$ are nonnegative integrable functions, then

$$\eta(t) \le e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \varphi(s)ds\right], \quad \forall t \in [0,T]. \tag{2.10}$$

Throughout this paper, we will denote by C a positive constant which is various in different line or even in the same line and use the following abbreviations:

$$L^p = L^p(\Omega), \quad \|\cdot\| = \|\cdot\|_{L^2}, \quad \|u\|_m = \|u\|_{H^m}, \quad \|u\|_1 = \|u\|_{H_0^1}$$

with $p \ge 1$.

Assumption 2.1.

1) $\varphi \in C^1(\mathbb{R}), \varphi(0) = 0$, and

$$0 \le \varphi'(s) \le C\left(1 + |s|^{q-1}\right), s \in \mathbb{R}, \tag{2.11}$$

where $1 \le q \le p^* \equiv \frac{N+2}{N-2} = 5$ if $N = 3$

2) $h \in C^1(\mathbb{R}), h(0) = 0$,

$$\liminf_{|s|\to\infty} h'(s) > -\lambda_1, \left|h'(s)\right| \le C\left(1 + |s|^{p-1}\right), s \in \mathbb{R}, \tag{2.12}$$

where $1 \le p \le p^* = 5$ if $N = 3$.

3)

$$g \in L^2, (u_0, u_1) \in X \text{ with } \|(u_0, u_1)\|_X \le R \tag{2.13}$$

Definition 2.2. Let $S(t)_{t\ge0}$ be a semigroup on a metric space (E,d). A subset A of E is called a global attractor for the semigroup, if A is compact and enjoys the following properties:

1) A is invariant, *i.e.* $S(t)A = A, \forall t \ge 0$;

2) A attracts all bounded set of E. That is, for any bounded subset B of E,

$$dist\left(S(t)B, A\right) \to 0, \text{ as } t \to 0.$$

Next we only formulate the following results, which is proved in [13]:

Lemma 2.3. Let (2.11)-(2.13) be valid. Then problem (2.2)-(2.3) admits a unique weak solution u, with $(u, u_t) \in L^\infty(\mathbb{R}^+; X) \cap C(\mathbb{R}^+; X)$, $u_t \in L^2(\mathbb{R}^+; H_0^1)$. Moreover, this solution possesses the following properties:

(Dissipativity)

$$\|(u,u_t)(t)\|_X^2 + \int_t^\infty \left(\|u_t(\tau)\|_{H_0^1}^2 + (\varphi(u_t), u_t)\right) d\tau \le C(R)e^{-kt} + C_0, \quad t \ge 0, \tag{2.14}$$

where k denotes a small positive constant, $C(R)$ and $C_0 = C\left(\|f\|_{H^{-1}}\right)$ are positive constants.

Lemma 2.4. Let (2.11)-(2.13) be valid and when $p = 5, h \in C^2(\mathbb{R})$. Then

$$\|u_t(t)\|_{H_0^1}^2 + \|u_{tt}(t)\|_{L^2}^2 \le R_0^2, \ t > 0. \tag{2.15}$$

Actually, by exploiting (2.11) and (2.14), we can get u, u_t are respectively bounded in H_0^1, L^2.

3. Existence of Global Attractors in $H_0^1 \times L^2$

For every fixed $x \in B_0$, we split the solution $S(t)x = (u(t), u_t(t))$ into the sum $\hat{\eta}(t) + \hat{\zeta}(t)$, where $\hat{\eta}(t) = (\hat{v}(t), \hat{v}_t(t))$ and $\hat{\zeta}(t) = (\hat{w}(t), \hat{w}_t(t))$ solve the Cauchy problems

$$\begin{cases} \hat{v}_{tt} + \left(1 + \epsilon \left\| A^{\frac{1}{2}} \hat{u} \right\|^2\right) A\hat{v} + B\hat{v}_t + \varphi_0(\hat{v}_t) + h_0(\hat{v}) = 0, \\ \hat{\eta}(0) = x, \end{cases} \tag{3.1}$$

$$\begin{cases} \hat{w}_{tt} + \left(1 + \epsilon \left\| A^{\frac{1}{2}} \hat{u} \right\|^2\right) A\hat{w} + B\hat{w}_t = \hat{\rho}, \\ \hat{\zeta}(0) = 0, \end{cases} \tag{3.2}$$

here

$$\hat{\rho} = g - [\varphi_0(u_t) + h_0(u)] + [\varphi_0(\hat{v}_t) + h_0(\hat{v})] + [\varphi_1(u_t) + h_1(u)].$$

Having set $\varphi(u_t) + h(u) = [\varphi_0(u_t) + h_0(u)] + [\varphi_1(u_t) + h_1(u)]$, and satisfying

$$\begin{aligned} &\varphi_0(u_t)u_t \geq 0, \quad \varphi_0'(u_t) \geq C, \quad |\varphi_0(u_t) - \varphi_0(v_t)| \leq C|u_t - v_t|\left(|u_t|^4 + |v_t|^4\right), \\ &|\varphi_1(u_t)| \leq C(1 + |u_t|). \end{aligned} \tag{3.3}$$

$$h_0(u)u \geq 0, \quad |h_0(u) - h_0(v)| \leq C|u - v|\left(|u|^4 + |v|^4\right), \quad |h_1(u)| \leq C(1 + |u|). \tag{3.4}$$

From now on, $c_0, \upsilon_0 > 0$ and J_0 will denote generic constants and a generic function, respectively, depending only on B_0.

Theorem 3.1. Let (2.11)-(2.13) be valid, then the solution semigroup $S(t)$ possesses a global attractor $\mathcal{B}$ in *X*.

Proof. Estimate (2.14) shows

$$\|(u, u_t)(t)\|_X^2 \leq C(R)\mathrm{e}^{-kt} + C_0, \ t \geq 0,$$

such that the ball $B_0 = \left\{(u, u_t) \in X \mid \|(u, u_t)\|_X \leq R_0\right\}$ is an absorbing set of the semigroup $S(t)$ in *X* for $R_0 > C_0\left(\|g\|_{H^{-1}}\right)$.

In order to prove the existence of the global attractors, now we need to prove the asymptotic compactness.

Multiplying the first equation of (3.1) by $\hat{v}_t + \gamma\hat{v}$ and integrating over Ω, we get

$$\left\langle \hat{v}_{tt} + \left(1 + \epsilon \left\| A^{\frac{1}{2}} \hat{u} \right\|^2\right) A\hat{v} + B\hat{v}_t + \varphi_0(\hat{v}_t) + h_0(\hat{v}), \hat{v}_t + \gamma\hat{v} \right\rangle = 0.$$

By using $\epsilon \geq 0, \varphi_0(u_t)u_t \geq 0, h_0(u)u \geq 0$ and the generalized Poincare inequality, then

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\hat{v}_t\|^2 + \|\hat{v}\|_1^2 + \gamma\left(\|\hat{v}\|_1^2 + \langle \hat{v}_t, \hat{v} \rangle - \lambda\|\hat{v}\|^2\right)\right] \\ &\leq (\gamma - \lambda_1 + \lambda)\|\hat{v}_t\|^2 - \gamma\|\hat{v}\|_1^2 - \gamma\int_\Omega \varphi_0(\hat{v}_t)\hat{v}\mathrm{d}x - \int_\Omega h_0(\hat{v})\hat{v}_t\mathrm{d}x, \end{aligned}$$

By $\lambda < \lambda_1$, we know

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\hat{v}_t\|^2+\|\hat{v}\|_1^2+\gamma\left(\|\hat{v}\|_1^2+\langle\hat{v}_t,\hat{v}\rangle-\lambda\|\hat{v}\|^2\right)\right]$$
$$\geq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\hat{v}_t\|^2+\|\hat{v}\|_1^2+\gamma\left(\langle\hat{v}_t,\hat{v}\rangle-\lambda_1\|\hat{v}\|^2\right)\right],$$

where $\gamma>0$ is small enough such that

$$E(t)=\frac{1}{2}\left[\|\hat{v}_t\|^2+\|\hat{v}\|_1^2+\gamma\left(\langle\hat{v}_t,\hat{v}\rangle-\lambda_1\|\hat{v}\|^2\right)\right]\sim\frac{1}{2}\left[\|\hat{v}_t\|^2+\|\hat{v}\|_1^2\right]. \tag{3.5}$$

Actually, noting that $\varphi_0'(\hat{v}_t)\geq C$, and by exploiting (2.8) and (2.12), we deduce that

$$\begin{aligned}-\int_\Omega\varphi_0(\hat{v}_t)\hat{v}\mathrm{d}x&=-\int_\Omega\frac{\varphi_0(\hat{v}_t)-\varphi(0)}{\hat{v}_t-0}\hat{v}_t\hat{v}\mathrm{d}x\\&=-\int_\Omega\varphi_0'(\hat{v}_t)\hat{v}_t\hat{v}\mathrm{d}x\\&\leq-C\langle\hat{v}_t,\hat{v}\rangle,\end{aligned} \tag{3.6}$$

and

$$-\int_\Omega h_0(\hat{v})\hat{v}_t\mathrm{d}x\leq\int_\Omega\lambda_1\hat{v}_t\hat{v}\mathrm{d}x\leq\frac{1}{2\gamma}\|\hat{v}\|^2+\frac{\gamma\lambda_1}{2}\|\hat{v}_t\|^2. \tag{3.7}$$

From (3.5)-(3.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq\left(\gamma+\lambda-\lambda_1+\frac{\gamma\lambda_1}{2}\right)\|\hat{v}_t\|^2-\gamma\|\hat{v}\|_1^2+\frac{1}{2\gamma}\|\hat{v}\|^2-C\langle v_t,v\rangle\leq-CE(t),$$

where $\gamma>0$ is small enough such that ($\gamma+\lambda-\lambda_1+\frac{\gamma\lambda_1}{2}$) is negative. Furthermore, by the Gronwall inequality, we can get

$$\|\hat{\eta}(t)\|_{H_0^1\times L^2}\leq c_0\mathrm{e}^{-\upsilon_0 t}\|x\|_{H_0^1\times L^2}. \tag{3.8}$$

Next multiplying the first equation of (3.2) by $A^{\frac{1}{4}}\hat{\omega}_t+\gamma A^{\frac{1}{4}}\hat{\omega}$ and integrating over Ω, we get

$$\begin{aligned}&\left|\left\langle\hat{w}_{tt}+\left(1+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\right)A\hat{w}+B\hat{w}_t,A^{\frac{1}{4}}\hat{w}_t+\gamma A^{\frac{1}{4}}\hat{w}\right\rangle\right|\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|\hat{w}_t\|_{\frac{1}{4}}^2+\frac{1}{2}\|\hat{w}\|_{\frac{5}{4}}^2+\gamma\left(\frac{1}{2}\|\hat{w}\|_{\frac{5}{4}}^2-\frac{\lambda}{2}\|\hat{w}\|_{\frac{1}{4}}^2+\left\langle\hat{w}_t,A^{\frac{1}{4}}\hat{w}\right\rangle\right)\right]\\&\quad+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\left\langle A\hat{w},A^{\frac{1}{4}}\hat{w}_t\right\rangle+\|\hat{w}_t\|_{\frac{5}{4}}^2-\lambda\|\hat{w}_t\|_{\frac{1}{4}}^2-\gamma\|\hat{w}_t\|_{\frac{1}{4}}^2+\gamma\|\hat{w}\|_{\frac{5}{4}}^2\\&\quad+\epsilon\gamma\left\|A^{\frac{1}{2}}u\right\|^2\left\langle A\hat{w},A^{\frac{1}{4}}\hat{w}\right\rangle\\&\geq\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\hat{w}_t\|_{\frac{1}{4}}^2+\frac{1}{2}\|\hat{w}\|_{\frac{5}{4}}^2\right)+\|\hat{w}_t\|_{\frac{5}{4}}^2-\lambda\|\hat{w}_t\|_{\frac{1}{4}}^2-\gamma\|\hat{w}_t\|_{\frac{1}{4}}^2+\gamma\|\hat{w}\|_{\frac{5}{4}}^2,\end{aligned} \tag{3.9}$$

where $\gamma>0$ is small enough. Then we define the energy functional

$$E_1(t)=\frac{1}{2}\left(\|\hat{w}_t\|_{\frac{1}{4}}^2+\|\hat{w}\|_{\frac{5}{4}}^2\right), \tag{3.10}$$

At the same time, by the interpolation inequality, we have

$$\begin{aligned}\|\hat{\rho}\|_{L^{\frac{4}{3}}(\Omega)}&=\left\|g+\left(\varphi_0(\hat{v}_t)-\varphi_0(u_t)\right)+\left(h_0(\hat{v})-h_0(u)\right)+\left(\varphi_1(u_t)+h_1(u)\right)\right\|\\&\le C\|g\|+C\|\hat{w}\|_{\frac{5}{4}}\left(\|\hat{v}\|_1^4+\|u\|_1^4\right)+C\|\hat{w}_t\|_{\frac{5}{4}}\left(\|\hat{v}_t\|_1^4+\|u_t\|_1^4\right)\\&\quad+C\left(1+\|u_t\|_1\right)+C\left(1+\|u\|_1\right)\\&\le c_0\mathrm{e}^{-\upsilon_0 t}\|x\|_{H^1\times L^2}^2\|\hat{w}\|_{\frac{5}{4}}+c\mathrm{e}^{-kt}\|\hat{w}_t\|_{\frac{5}{4}}+C\left(\|g\|+\|u\|_1+\|u_t\|_1+1\right)\\&\le c_0\mathrm{e}^{-\upsilon_0 t}\|\hat{w}\|_{\frac{5}{4}}+c\mathrm{e}^{-kt}\|\hat{w}_t\|_{\frac{5}{4}}+C\left(1+\mathrm{e}^{-kt}\right),\end{aligned}$$

and by the embedding $H^{\frac{3}{4}}\hookrightarrow L^4$, then

$$\begin{aligned}&\left|\left\langle\hat{\rho},A^{\frac{1}{4}}\hat{w}_t+\gamma A^{\frac{1}{4}}\hat{w}\right\rangle\right|\\&\le\|\hat{\rho}\|_{L^{\frac{4}{3}}}\left(\left\|A^{\frac{1}{4}}\hat{w}_t\right\|_{L^4}+\gamma\left\|A^{\frac{1}{4}}\hat{w}\right\|_{L^4}\right)\\&\le\|\hat{\rho}\|\left(\left\|A^{\frac{1}{4}}\hat{w}_t\right\|_{\frac{3}{4}}+\gamma\left\|A^{\frac{1}{4}}\hat{w}\right\|_{\frac{3}{4}}\right)\\&\le c_0\mathrm{e}^{-\upsilon_0 t}\|\hat{w}\|_{\frac{5}{4}}^2+c\mathrm{e}^{-kt}\|\hat{w}_t\|_{\frac{5}{4}}^2+\delta\|\hat{w}_t\|_{\frac{5}{4}}^2+\gamma\delta\|\hat{w}\|_{\frac{5}{4}}^2+C\left(1+\mathrm{e}^{-kt}\right).\end{aligned} \tag{3.11}$$

By exploiting (2.8) and the generalized Poincare inequality, from (3.9)-(3.11), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)&\le\left(c_0\mathrm{e}^{-kt}+\delta-1\right)\|\hat{w}_t\|_{\frac{5}{4}}^2+(\lambda+\gamma)\|\hat{w}_t\|_{\frac{1}{4}}^2\\&\quad+\left(c_0\mathrm{e}^{-\upsilon_0 t}+\gamma\delta-\gamma\right)\|\hat{w}\|_{\frac{5}{4}}^2+C\left(1+\mathrm{e}^{-kt}\right)\\&\le-C\lambda_1\|\hat{w}_t\|_{\frac{1}{4}}^2+(\lambda+\gamma)\|\hat{w}_t\|_{\frac{1}{4}}^2+\left(c_0\mathrm{e}^{-\upsilon_0 t}-C\right)\|\hat{w}\|_{\frac{5}{4}}^2+C\left(1+\mathrm{e}^{-kt}\right)\\&\le-\left(C-c_0\mathrm{e}^{-\upsilon_0 t}\right)E_1(t)+C\left(1+\mathrm{e}^{-kt}\right),\end{aligned}$$

where $\delta>0$ is small enough and by $\lambda<\lambda_1$, we get $(-C\lambda_1+\lambda+\gamma),(\gamma\delta-\gamma)$ are negative. Then from the Gronwall inequality and noting that $\hat{\zeta}(0)=\left(\hat{\omega}(0),\hat{\omega}_t(0)\right)=0$, we get

$$E_1(t)\le c_0e^{-\nu t}E_1(0)+C\left(1+\mathrm{e}^{-kt}\right)\le C\left(1+\mathrm{e}^{-kt}\right)$$

which provides the following estimate

$$\left\|\hat{\zeta}(t)\right\|_{H^{\frac{5}{4}}\times H^{\frac{1}{4}}}\le C\left(1+\mathrm{e}^{-kt}\right), \tag{3.12}$$

From (3.8) and (3.12), we obtain that the evolution semigroup $S(t)$ is asymptotically compact in *X*, so the solution semigroup $S(t)$ possesses a global attractor $\mathcal{B}$ in $H_0^1\times L^2$, which

$$\mathcal{B}=\bigcap_{t_0\geq 0}\overline{\bigcup_{t\geq t_0}S(t)B_0},$$

where $t_0>0$ is chosen such that $S(t)B_0\subset B_0$ for $t\geq t_0$.

4. Regularity of Global Attractors

Now we are in a position to state and prove the main result:

Theorem 4.1. The attractor $\mathcal{B}$ of the semigroup $S(t)$ on X is bounded in $H^2\times H_0^1$.

Proof. Having set $x=y+z$. For $y\in B_0, z\in H^2\times H_0^1$, we split the solution into the sum

$$S(t)x=Y(t)y+Z(t)z,$$

where $\eta(t)=Y(t)y=(v(t),v_t(t))$ and $\zeta(t)=Z(t)z=(w(t),w_t(t))$ solve the following equations with initial data $\eta(0)=y,\zeta(0)=z$,

$$\begin{cases}v_{tt}+\left(1+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\right)Av+Bv_t=0,\\ \eta(0)=x,\end{cases}\tag{4.1}$$

and

$$\begin{cases}w_{tt}+\left(1+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\right)Aw+Bw_t=\rho,\\ \zeta(0)=0,\end{cases}\tag{4.2}$$

where $\rho(t)=g+\varphi(u_t)+h(u)$.

Multiplying the first equation of (4.1) by $v_t+\gamma v$ and integrating over Ω, by $\epsilon>0$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v_t\|^2+\|v\|_1^2+\frac{\gamma}{2}\left(\|v\|_1^2-\lambda\|v\|^2+\langle v_t,v\rangle\right)\right]+\gamma\|v\|_1^2+\|v_t\|_1^2-\lambda\|v_t\|^2\leq 0,$$

where $\gamma>0$ is small enough such that

$$E_2(t)=\frac{1}{2}\left[\|v_t\|^2+\|v\|_1^2+\frac{\gamma}{2}\left(\|v\|_1^2-\lambda\|v\|^2+\langle v_t,v\rangle\right)\right]\sim\frac{1}{2}\left(\|v_t\|^2+\|v\|_1^2\right).$$

By $\lambda<\lambda_1$ and the generalized Poincare inequality, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)\leq(\lambda-\lambda_1)\|v_t\|^2-\gamma\|v\|_1^2\leq-\frac{C}{2}\left(\|v_t\|^2+\|v\|_1^2\right),\tag{4.3}$$

then by the Gronwall inequality, we get

$$\|\eta(t)\|_{H_0^1\times L^2}\leq c_0\mathrm{e}^{-\upsilon_0 t}\|y\|_{H_0^1\times L^2}.\tag{4.4}$$

Next multiplying the first equation of (4.2) by $Aw_t+\gamma Aw$ and integrating over Ω, exploiting (2.8) and the Hölder's inequality, the right side becomes

$$
\begin{aligned}
&\left|\left\langle \rho, Aw_t+\gamma Aw\right\rangle\right| \\
&\le \left|\left\langle g, Aw_t+\gamma Aw\right\rangle\right| + \left|\left\langle \varphi(u_t), Aw_t+\gamma Aw\right\rangle\right| + \left|\left\langle h(u), Aw_t+\gamma Aw\right\rangle\right| \\
&\le \frac{1}{2\delta}\|g\|^2 + \delta\|w_t\|_2^2 + \int_\Omega C\left(|u_t|+|u_t|^5\right)|Aw_t|\,\mathrm{d}x + \int_\Omega C\left(|u|+|u|^5\right)|Aw_t|\,\mathrm{d}x \\
&\quad + \gamma\left[\delta\|w\|_2^2 + \int_\Omega C\left(|u_t|+|u_t|^5\right)|Aw|\,\mathrm{d}x + \int_\Omega C\left(|u|+|u|^5\right)|Aw|\,\mathrm{d}x\right] \\
&\le \frac{1}{2\delta}\|g\|^2 + \delta\|w_t\|_2^2 + \int_\Omega C\left(1+|u_t|^6\right)|Aw_t|\,\mathrm{d}x + \int_\Omega C\left(1+|u|^6\right)|Aw_t|\,\mathrm{d}x \\
&\quad + \gamma\left[\delta\|w\|_2^2 + \int_\Omega C\left(1+|u_t|^6\right)|Aw|\,\mathrm{d}x + \int_\Omega C\left(1+|u|^6\right)|Aw|\,\mathrm{d}x\right] \\
&\le \frac{1}{2\delta}\|g\|^2 + C\delta\gamma\|w\|_2^2 + C\delta\|w_t\|_2^2 + \frac{C}{2\delta}\left(\|u\|_{L^2}^{12} + \|u_t\|_{L^2}^{12} + |\Omega|\right) \\
&\le C\delta\gamma\|w\|_2^2 + C\delta\|w_t\|_2^2 + J_0(t),
\end{aligned} \tag{4.5}
$$

where $\delta>0$ is small enough, we know $J_0(t)$ is bounded by (2.13) and lemma 2.3. At the same time, the left side becomes

$$
\begin{aligned}
&\left|\left\langle w_{tt} + \left(1+\epsilon\left\|A^{\frac{1}{2}}u\right\|^2\right)Aw + Bw_t, Aw_t+\gamma Aw\right\rangle\right| \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|w_t\|_1^2 + \frac{1}{2}\|w\|_2^2 + \gamma\left(\frac{1}{2}\|w\|_2^2 - \frac{\lambda}{2}\|w\|_1^2 + \langle w_t, Aw\rangle\right)\right] \\
&\quad + \epsilon\left\|A^{\frac{1}{2}}u\right\|^2\langle Aw, Aw_t\rangle + \|w_t\|_2^2 - \lambda\|w_t\|_1^2 \\
&\quad + \epsilon\gamma\left\|A^{\frac{1}{2}}u\right\|^2\langle Aw, Aw_t\rangle + \gamma\|w\|_2^2 - \gamma\|w_t\|_1^2 \\
&\ge \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|w_t\|_1^2 + \frac{1}{2}\|w\|_2^2\right) + \|w_t\|_2^2 - \lambda\|w_t\|_1^2 + \gamma\|w\|_2^2 - \gamma\|w_t\|_1^2,
\end{aligned} \tag{4.6}
$$

then we define the energy functional

$$
E_3(t) = \frac{1}{2}\left[\|w_t\|_1^2 + \|w\|_2^2 + \gamma\left(\|w\|_2^2 - \lambda\|w\|_1^2 + \langle w_t, Aw\rangle\right)\right], \tag{4.7}
$$

where $\gamma>0$ is small enough such that $E_3(t)\sim\frac{1}{2}\left(\|w_t\|_1^2+\|w\|_2^2\right)$. By combining (4.5)-(4.7) and the embedding $H^2 \hookrightarrow H_0^1$, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_3(t) &\le (C\delta-1)\|w_t\|_2^2 + (C\gamma\delta-\gamma)\|w\|_2^2 + (\lambda+\gamma)\|w_t\|_1^2 + J_0(t) \\
&\le (-C\lambda_1+\lambda+\gamma)\|w_t\|_1^2 - C\|w\|_2^2 + J_0(t) \\
&\le -\frac{C}{2}\left(\|w_t\|_1^2 + \|w\|_2^2\right) + J_0(t),
\end{aligned}
$$

where $\delta>0$ is small enough and by $\lambda<\lambda_1$, we get $(-C\lambda_1+\lambda+\gamma),(C\gamma\delta-\gamma)$ are negative. From the Gronwall inequality, we get

$$
E_3(t) \le c_0\mathrm{e}^{-\nu t}E_3(0) + J_0(t),
$$

which provides the estimate

$$\left\|\zeta(t)\right\|_{H^2\times H_0^1} \le c_0 e^{-\nu t}\left\|z\right\|_{H^2\times H_0^1} + J_0(t). \tag{4.8}$$

From (4.4) and (4.8), for every bounded set $B \subset H_0^1 \times L^2$, we get

$$dist_{H_0^1\times L^2}\left(S(t)B, B_{H^2\times H_0^1}\left(J_0(t)\right)\right) \le Ce^{-\nu_0 t} \to 0, \text{ as } t\to\infty,$$

so

$$S(t)B \subseteq B_{H^2\times H_0^1}\left(J_0(t)\right), \ \forall t \le t_0.$$

Then we finish the proof.

5. Conclusion

In this paper, we first prove that the Kirchhoff wave equation with strong damping and critical nonlinearities possesses a global attractor in $H_0^1(\Omega)\times L^2(\Omega)$. Then we split the solution into two parts, one part decays exponentially and the other part satisfies asymptotic behaviour in spaces with higher regularity. By the operator decomposition technique, we get the global attractor which is compactly bounded in $H^2(\Omega)\times H_0^1(\Omega)$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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