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Self-Adaptive Algorithms for the Split Common Fixed Point Problem of the Demimetric Mappings

Xinhong Chen[1], Yanlai Song[1], Jianying He[1], Liping Gong[2]

[1]College of Science, Zhongyuan University of Technology, Zhengzhou, China
[2]School of Foreign Languages, Zhongyuan University of Technology, Zhengzhou, China
Email: songyanlai2009@163.com, chenxh6@163.com

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Abstract

The split common fixed point problem is an inverse problem that consists in finding an element in a fixed point set such that its image under a bounded linear operator belongs to another fixed-point set. In this paper, we present new iterative algorithms for solving the split common fixed point problem of demimetric mappings in Hilbert spaces. Moreover, our algorithm does not need any prior information of the operator norm. Weak and strong convergence theorems are given under some mild assumptions. The results in this paper are the extension and improvement of the recent results in the literature.

Keywords

Hilbert Space, Demimetric Mapping, Split Common Fixed Point Problem, Self-Adaptive Algorithm

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be two nonlinear mappings. We denote the fixed point sets of S and T by $F(S)$ and $F(T)$, respectively. Let $A: H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Then, we consider the following split common fixed point problem:

$$\text{Finding } x \in H_1 \text{ such that } x \in F(S) \text{ and } Ax \in F(T). \tag{1.1}$$

The split common fixed point problem (1.1) is a generalization of the split feasibility problem arising from signal processing and image restoration; see

[1]-[7] for instance. It was first introduced and studied by Censor and Segal [8]. Note that solving (1) can be translated to solve the fixed point equation

$$x^* = S\left(x^* - \tau A^*\left(I-T\right)Ax^*\right), \quad \tau > 0.$$

Censor and Segal also proposed the following algorithm for directed mappings.

Algorithm 1.1 *Initialization: let* $x^* \in H_1 := \mathbb{R}_n$ *be arbitrary. Iterative step: let*

$$x_{n+1} = S\left(x_n - \tau A^*\left(I-T\right)Ax_n\right), \quad n \geq 0,$$

where $S:\mathbb{R}_n \to \mathbb{R}_n$ and $T:R_m \to \mathbb{R}_m$ are two directed mappings and $\tau \in \left(0, \frac{2}{\lambda}\right)$ with λ being the spectral radius of the operator A^*A.

Since then, there has been growing interest in the split common fixed point problem; please, see [9]-[15].

Recently, Wang [16] introduced the following new iterative algorithms for the split common fixed point problem of directed mappings.

Algorithm 1.2 *Choose an arbitrary initial guess* x_0. *Assume* x_n *has been constructed. If*

$$\left\|x_n - Sx_n + A^*\left(I-T\right)Ax_n\right\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$x_{n+1} = x_n - \tau_n\left[x_n - Sx_n + A^*\left(I-T\right)Ax_n\right], \quad \forall n \geq 0,$$

where τ_n is chosen self-adaptively as

$$\tau_n = \frac{\left\|x_n - Sx_n\right\|^2 + \left\|\left(I-T\right)Ax_n\right\|^2}{\left\|x_n - Sx_n + A^*\left(I-T\right)Ax_n\right\|^2}.$$

Algorithm 1.3 *Let* $u \in H$ *and start an initial guess* $x_0 \in H$. *Assume* x_n *has been constructed. If*

$$\left\|x_n - Sx_n + A^*\left(I-T\right)Ax_n\right\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$x_{n+1} = \alpha_n u + \left(1-\alpha_n\right)\left[x_n - Sx_n + A^*\left(I-T\right)Ax_n\right], \quad \forall n \geq 0,$$

where the stepsize sequence τ_n is chosen self-adaptively as

$$\tau_n = \frac{\left\|x_n - Sx_n\right\|^2 + \left\|\left(I-T\right)Ax_n\right\|^2}{\left\|x_n - Sx_n + A^*\left(I-T\right)Ax_n\right\|^2}.$$

Wang obtained the weak and strong convergence of Algorithms 1.2 and 1.3, respectively. Inspired by the above work in the literature, Yao, *et al.* [17] extend Wang's results in [16] from the directed mappings to the demicontractive mappings. Further, they construct the following two self-adaptive algorithms for solving the split common fixed point problem (1.1).

Algorithm 1.4. *Initialization: let* $x_0 \in H_1$ *be arbitrary. For* $n \geq 0$, *assume the current iterate* x_n *has been constructed. If*

$$\left\| x_n - Sx_n + A^*\left(I-T\right)Ax_n \right\| = 0,$$

then stop; otherwise, calculate the next iterate x_{n+1} *by the following formula*

$$\begin{cases} y_n = x_n - Sx_n + A^*\left(I-T\right)Ax_n, \\ x_{n+1} = x_n - \gamma\tau_n y_n, \quad \forall n \ge 0, \end{cases}$$

where $\gamma \in \left(0, \min\left\{1-\beta, 1-\mu\right\}\right)$ *is a positive constant and* τ_n *is chosen self-adaptively as*

$$\tau_n = \frac{\left\|x_n - Sx_n\right\|^2 + \left\|\left(I-T\right)Ax_n\right\|^2}{\left\|y_n\right\|^2}.$$

Algorithm 1.5. *Initialization: Let* $u \in H_1$ *be a fixed point and let* $x_0 \in H_1$ *be arbitrary. Iterative step: for* $n \ge 0$, *assume the current iterate* x_n *has been constructed. If*

$$\left\| x_n - Sx_n + A^*\left(I-T\right)Ax_n \right\| = 0,$$

then stop; otherwise, calculate the next iterate x_{n+1} *by the following formula*

$$\begin{cases} y_n = x_n - Sx_n + A^*\left(I-T\right)Ax_n, \\ x_{n+1} = \alpha_n u + \left(1-\alpha_n\right)\left(x_n - \gamma\tau_n y_n\right), \quad \forall n \ge 0, \end{cases}$$

where $\gamma \in \left(0, \min\left\{1-\beta, 1-\mu\right\}\right)$ is *a positive constant and* τ_n *is chosen self-adaptively as*

$$\tau_n = \frac{\left\|x_n - Sx_n\right\|^2 + \left\|\left(I-T\right)Ax_n\right\|^2}{\left\|y_n\right\|^2}.$$

They also obtained the weak and strong convergence of Algorithms 1.4 and 1.5, respectively. Motivated and inspired by the work in the literature, the main purpose of this paper is to extend the results of Wang [16] and Yao, *et al.* [17] from the directed mappings or demicontractive mappings to the demicontractive mappings. We present two self-adaptive algorithms for solving the split common fixed point problem (1.1). Weak and strong convergence theorems are given under some mild assumptions. Our results improve essentially the corresponding results in [16] [17]. Further, some other results are also improved; see [9]-[22].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H.

Definition 2.1. *A mapping* $T: C \to C$ *is said to be*:

1) *directed if*

$$\left\|Tx - x^*\right\|^2 \le \left\|x - x^*\right\|^2 - \left\|Tx - x\right\|^2, \quad \forall x \in C, x^* \in F\left(T\right);$$

2) β-demicontractive if there exists a constant $\beta \in \left[0,1\right)$ such that

$$\left\|Tx - x^*\right\|^2 \le \left\|x - x^*\right\|^2 + \beta\left\|Tx - x\right\|^2, \quad \forall x \in C, x^* \in F\left(T\right);$$

3) k-demimetric if there exists a constant $k \in \left(-\infty, 1\right)$ such that

$$\left\langle x-x^*, x-Tx\right\rangle \geq \frac{1-k}{2}\|x-Tx\|^2, \quad \forall x\in C, x^*\in F(T). \tag{2.1}$$

Clearly, (2.1) is equivalent to the following:

$$\left\|Tx-x^*\right\|^2 \leq \left\|x-x^*\right\|^2 + k\|Tx-x\|^2, \quad \forall x\in C, x^*\in F(T).$$

It is obvious that the demimetric mappings include the directed mappings and the demicontractive mappings as special cases. Furthermore, this class mapping also contains the classes of strict pseudo-contractions, firmly-quasinon expansive mappings, 2-generalized hybrid mappings and quasi-non-expansive mappings. The class of demimetric mappings is fundamental because many common types of mappings arising in optimization belong to this class, see for example [23] [24] and references therein.

Definition 2.2 *A sequence* $\{x_n\}$ *is called Fejér-monotone with respect to a given nonempty set* Ω, *if for every* $x\in\Omega$,

$$\|x_{n+1}-x\| \leq \|x_n-x\|, \quad \forall n\geq 0.$$

Next we adopt the following notations:

a) $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively;

b) $\omega_w(x_n) := \left\{x : \exists x_{n_j} \rightharpoonup x\right\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Recall that a mapping $f: C\to C$ is said to be contractive if there exists a constant $v\in(0,1)$ such that

$$\|fx-fy\| \leq v\|x-y\|, \ \forall x,y\in C.$$

We use Π_C to denote the collection of mappings f verifying the above inequality. That is

$$\Pi_C = \{f : C\to H : f \text{ is a contraction with constant } v\}.$$

Let D be a nonempty subset of C. A sequence $\{f_n\}$ of mappings of C into H is said to be stable on D (see [25]) if $\{f_n(x) : n\geq 0\}$ is a singleton for every $x\in D$. It is clear that if $\{f_n\}$ is stable on D, then $f_n(x) = f_0(x)$ for all $n\geq 0$ and $x\in D$.

Recall that the (nearest point or metric) projection from H onto C, denoted P_C, assigns to each $u\in H$, the unique point $P_C(u)\in C$ with the property

$$\|u-P_C(u)\| = \inf\{\|u-v\| : v\in C\}.$$

The metric projection $P_C(u)$ of H onto C is characterized by

$$\left\langle u-P_C(u), y-P_C(u)\right\rangle \leq 0, \quad \forall y\in C, u\in H.$$

Lemma 2.1 ([26]) *Let* Ω *be a nonempty closed convex subset in H. If the sequence* $\{x_n\}$ *is Fejér monotone with respect to* Ω, *then we have the following conclusions*:

1) $x_n \rightharpoonup x^*\in\Omega$ iff $\omega_w(x_n)\subset\Omega$;

2) the sequence $\{P_\Omega(x_n)\}$ converges strongly;

3) if $x_n \rightharpoonup x^*\in\Omega$, then $x^* = \lim_{n\to\infty} P_\Omega(x_n)$.

Lemma 2.2 ([27]) *Let* $\{\alpha_n\}$ *be a sequence of nonnegative numbers satisfying the property*:

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n c_n, \quad n \ge 0,$$

where $\{\gamma_n\},\{c_n\}$ satisfy the restrictions:

1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
2) $\limsup_{n\to\infty} c_n \le 0$ or $\sum_{n=1}^{\infty} c_n\gamma_n < \infty$.

Then, $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.3 ([23] [24]) *Let E be a smooth, strictly convex and reflexive Banach space and let k be a real number with* $k \in (-\infty,1)$. *Let U be an k-demimetric mapping of E into itself. Then* $F(U)$ *is closed and convex.*

3. Main Results

Now we study the split common fixed points problem (1) under the following hypothesis:

- H_1 and H_2 are two real Hilbert spaces;
- $S: H_1 \to H_1$ and $T: H_2 \to H_2$ are two demimetric mappings with constants $\beta \in (-\infty,1)$ and $\mu \in (-\infty,1)$, respectively;
- $A: H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*;
- $\{f_n\} \subset \Pi_C$ is stable on Ω, where Ω denotes the solution set of problem (1.1).

Lemma 3.1 z^* *solves problem* (1) *iff* $\left\|z^* - Sz^* + A^*(I-T)Az^*\right\| = 0$.

Proof. If z^* solves problem (1), then $z^* = Sz^*$ and $(I-T)Az^* = 0$. Therefore, we get $\left\|z^* - Sz^* + A^*(I-T)Az^*\right\| = 0$. To see the converse, suppose that $\left\|z^* - Sz^* + A^*(I-T)Az^*\right\| = 0$. Then, we have for any $z \in \Omega$ that

$$\begin{aligned} 0 &= \left\|z^* - Sz^* + A^*(I-T)Az^*\right\|\left\|z^* - z\right\| \\ &\ge \left\langle z^* - Sz^* + A^*(I-T)Az^*, z^* - z\right\rangle \\ &\ge \left\langle z^* - Sz^*, z^* - z\right\rangle + \left\langle A^*(I-T)Az^*, z^* - z\right\rangle \\ &\ge \left\langle z^* - Sz^*, z^* - z\right\rangle + \left\langle (I-T)Az^*, Az^* - Az\right\rangle. \end{aligned} \tag{3.1}$$

Since S and T are demimetric, we have that

$$\left\langle z^* - Sz^*, z^* - z\right\rangle \ge \frac{1-\beta}{2}\left\|z^* - Sz^*\right\|^2 \tag{3.2}$$

and

$$\left\langle (I-T)Az^*, Az^* - Az\right\rangle \ge \frac{1-\mu}{2}\left\|Az^* - TAz^*\right\|^2. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain that

$$0 \ge \frac{1-\beta}{2}\left\|z^* - Sz^*\right\|^2 + \frac{1-\mu}{2}\left\|Az^* - TAz^*\right\|^2. \tag{3.4}$$

Since $\beta,\mu \in (-\infty,1)$, we infer that $z^* \in F(S)$ and $Az^* \in F(T)$ by (3.4). Therefore, z^* solves problem (1.1). This completes the proof.

Next we construct the following self-adaptive algorithm to solve problem

(1.1).

Algorithm 3.1. *Initialization: let* $x_0 \in H_1$ *be arbitrary. For* $n \geq 0$, *assume the current iterate* x_n *has been constructed. If*

$$\left\| x_n - Sx_n + A^*(I-T)Ax_n \right\| = 0,$$

then stop (*in this case* x_n *solves problem* (1.1) *by Lemma* 3.1); *otherwise, calculate the next iterate* x_{n+1} *by the following formula*

$$\begin{cases} y_n = x_n - Sx_n + A^*(I-T)Ax_n, \\ x_{n+1} = x_n - \gamma\tau_n y_n, \quad \forall n \geq 0, \end{cases} \tag{3.5}$$

where $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ *is a positive constant and* τ_n *is chosen self adaptively as*

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|y_n\|^2}.$$

We assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 is infinite. In other words, Algorithm 3.1 does not terminate in a finite number of iterations.

Theorem 3.2. *Assume that S and T are demiclosed at zero. If* $\Omega \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by* (3.5) *converges weakly to a solution* z^* ($= \lim_{n\to\infty} P_\Omega(x_n)$) *of problem* (1.1).

Proof. Since A is linear and continuous, noticing Lemma 2.3, we see Ω is closed and convex. Thus we have that P_Ω is well defined.

We next prove that the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. Letting $z \in \Omega$, we then obtain that

$$\begin{aligned} &\langle y_n, x_n - z\rangle \\ &= \langle x_n - Sx_n + A^*(I-T)Ax_n, x_n - z\rangle \\ &= \langle x_n - Sx_n, x_n - z\rangle + \langle A^*(I-T)Ax_n, x_n - z\rangle \\ &\geq \frac{1-\beta}{2}\|x_n - Sx_n\|^2 + \frac{1-\mu}{2}\|(I-T)Ax_n\|^2 \\ &\geq \frac{1}{2}\min\{1-\beta, 1-\mu\}\left(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2\right). \end{aligned} \tag{3.6}$$

In view of Equation (3.5) and Equation (3.6), we deduce

$$\begin{aligned} &\|x_{n+1} - z\|^2 = \|x_n - \gamma\tau_n y_n - z\|^2 \\ &= \|x_n - z\|^2 - 2\gamma\tau_n\langle y_n, x_n - z\rangle + \gamma^2\tau_n^2\|y_n\|^2 \\ &\leq \|x_n - z\|^2 + \gamma^2\frac{\left(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2\right)^2}{\|y_n\|^2} \\ &\quad - \gamma\min\{1-\beta, 1-\mu\}\frac{\left(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2\right)^2}{\|y_n\|^2} \\ &\leq \|x_n - z\|^2 - \gamma\left(\min\{1-\beta, 1-\mu\} - \gamma\right)\frac{\left(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2\right)^2}{\|y_n\|^2}. \end{aligned} \tag{3.7}$$

This implies that the sequence $\{x_n\}$ is Fejér monotone.

Next, we show that every weak cluster point of the sequence $\{x_n\}$ belongs to the solution set of problem (1.1).

From the Fejér-monotonicity of $\{x_n\}$, it follows that the sequence $\{x_n\}$ is bounded. Further, we deduce from (3.7) that

$$\begin{aligned}&\gamma\left(\min\{1-\beta,1-\mu\}-\gamma\right)\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|y_n\|^2}\\&\le\|x_n-z\|^2-\|x_{n+1}-z\|^2.\end{aligned}$$

An induction induces that

$$\gamma\left(\min\{1-\beta,1-\mu\}-\gamma\right)\sum_{n=0}^{\infty}\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|y_n\|^2}\le\|x_0-z\|^2<\infty,$$

which implies that

$$\lim_{n\to\infty}\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|y_n\|^2}=0.$$

Observe that

$$\begin{aligned}&\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|y_n\|^2}\\&=\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|x_n-Sx_n+A^*(I-T)Ax_n\|^2}\\&\ge\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{2\left(\|x_n-Sx_n\|^2+\|A\|^2\|(I-T)Ax_n\|^2\right)}\\&\ge\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{2\max\left\{1,\|A\|^2\right\}\left(\|x_n-Sx_n\|^2+\|(I-T)Ax_n\|^2\right)}\\&=\frac{\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2}{2\max\left\{1,\|A\|^2\right\}}.\end{aligned}\tag{3.8}$$

By the demiclosedness (at zero) of S and T, we deduce immediately $\omega_w(x_n)\subset\Omega$. To this end, the conditions of Lemma 2.1 are all satisfied. Consequently, $x_n\rightharpoonup z^*=\lim_{n\to\infty}P_\Omega(x_n)$. This completes the proof.

Next, we study an iteration with strong convergence for solving problem (1.1).

Algorithm 3.3 *Initialization: Let* $x_0\in H_1$ *be arbitrary. Iterative step: for* $n\ge0$, *assume the current iterate* x_n *has been constructed. If*

$$\|x_n-Sx_n+A^*(I-T)Ax_n\|=0,$$

then stop (*in this case* x_n *solves problem* (1.1) *by Lemma* 3.1); *otherwise*, calculate the next iterate x_{n+1} by the following formula

$$\begin{cases} y_n = x_n - Sx_n + A^*(I-T)Ax_n, \\ x_{n+1} = \alpha_n f_n x_n + (1-\alpha_n)(x_n - \gamma\tau_n y_n), \quad \forall n \geq 0, \end{cases} \tag{3.9}$$

where $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ *is a positive constant and* τ_n *is chosen self-adaptively as*

$$\tau_n = \frac{\|x_n - Sx_n\|^2 + \|(I-T)Ax_n\|^2}{\|y_n\|^2}.$$

Theorem 3.4 *Assume that*:

$(C1)$ $\Omega \neq \varnothing$;

$(C2)$ *S and T are demiclosed at zero*;

$(C3)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.9) *converges strongly to the solution* $z(= P_\Omega f_0 z)$ *of problem* (1.1).

Proof. Putting $z = P_\Omega f_0 z$, we obtain from (3.7) that

$$\begin{aligned} &\|x_n - \gamma\tau_n y_n - z\|^2 \\ &\leq \|x_n - z\|^2 - \gamma\left(\min\{1-\beta, 1-\mu\} - \gamma\right)\frac{\left(\|x_n - Sx_n\|^2 + \|Ax_n - TAx_n\|^2\right)^2}{\|y_n\|^2} \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.10}$$

Next, we show that the sequence $\{x_n\}$ is bounded. Indeed, we obtain from (3.9) and (3.10) that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n f_n x_n + (1-\alpha_n)(x_n - \gamma\tau_n y_n) - z\| \\ &\leq \alpha_n \|f_n x_n - z\| + (1-\alpha_n)\|x_n - \gamma\tau_n y_n - z\| \\ &\leq \alpha_n \left(\|f_n x_n - f_n z\| + \|f_n z - z\|\right) + (1-\alpha_n)\|x_n - z\| \\ &\leq \alpha_n \left(v\|x_n - z\| + \|f_0 z - z\|\right) + (1-\alpha_n)\|x_n - z\| \\ &\leq \alpha_n \|f_0 z - z\| + \left(1 - \alpha_n(1-v)\right)\|x_n - z\|. \end{aligned}$$

By induction, we get

$$\|x_{n+1} - z\| \leq \max\left\{\frac{\|f_0 z - z\|}{1-v}, \|x_0 - z\|\right\},$$

which gives that the sequence $\{x_n\}$ is bounded.

By virtue of (3.9), we deduce

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \langle \alpha_n f_n x_n + (1-\alpha_n)(x_n - \gamma\tau_n y_n) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)\langle (x_n - \gamma\tau_n y_n) - z, x_{n+1} - z\rangle + \alpha_n \langle f_n x_n - f_n z, x_{n+1} - z\rangle \\ &\quad + \alpha_n \langle f_0 z - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|x_n - \gamma\tau_n y_n - z\|\|x_{n+1} - z\| + \alpha_n \|f_n x_n - f_n z\|\|x_{n+1} - z\| \\ &\quad + \alpha_n \langle f_0 z - z, x_{n+1} - z\rangle \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|x_n-\gamma\tau_n y_n - z\|\|x_{n+1}-z\| + \alpha_n v\|x_n - z\|\|x_{n+1}-z\| \\
&\quad + \alpha_n \langle f_0 z - z, x_{n+1}-z\rangle \\
&= (1-\alpha_n)\left(\frac{1}{2}\|x_n-\gamma\tau_n y_n - z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2\right) \\
&\quad + \alpha_n\left(\frac{1}{2}v\|x_n-z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2\right) + \alpha_n\langle f_0 z - z, x_{n+1}-z\rangle,
\end{aligned}
$$

which implies

$$
\|x_{n+1}-z\|^2 \le (1-\alpha_n)\|x_n-\gamma\tau_n y_n - z\|^2 + \alpha_n v\|x_n-z\|^2 + 2\alpha_n\langle f_0 z - z, x_{n+1}-z\rangle.
$$

This together with (3.10) implies that

$$
\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&\le (1-\alpha_n(1-v))\|x_n-z\|^2 + 2\alpha_n\langle f_0 z - z, x_{n+1}-z\rangle \\
&\quad -(1-\alpha_n)\gamma(\min\{1-\beta,1-\mu\}-\gamma)\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|y_n\|^2} \qquad (3.11) \\
&\le (1-\alpha_n(1-v))\|x_n-z\|^2 + \alpha_n\Bigg(2\langle f_0 z - z, x_{n+1}-z\rangle \\
&\quad -\frac{(1-\alpha_n)\gamma(\min\{1-\beta,1-\mu\}-\gamma)}{\alpha_n}\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|x_n-Sx_n+A^*(I-T)Ax_n\|^2}\Bigg).
\end{aligned}
$$

Set $\delta_n = \|x_n - z\|^2$ and

$$
\begin{aligned}
\sigma_n &= 2\langle f_0 z - z, x_{n+1}-z\rangle - \frac{(1-\alpha_n)\gamma(\min\{1-\beta,1-\mu\}-\gamma)}{\alpha_n} \\
&\quad \times\frac{\left(\|x_n-Sx_n\|^2+\|Ax_n-TAx_n\|^2\right)^2}{\|x_n-Sx_n+A^*(I-T)Ax_n\|^2}
\end{aligned}
\qquad (3.12)
$$

for all $n \ge 0$. Returning to (3.11) to obtain

$$
\delta_{n+1} \le (1-\alpha_n(1-v))\delta_n + \alpha_n\sigma_n, \quad \forall n \ge 0. \qquad (3.13)
$$

From (3.12), we find

$$
\sigma_n \le 2\langle f_0 z - z, x_{n+1}-z\rangle \le 2\|f_0 z - z\|\|x_{n+1}-z\|.
$$

It follows that $\limsup_{n\to\infty}\sigma_n < +\infty$.

Next we show that $\limsup_{n\to\infty}\sigma_n \ge -1$.

If $\limsup_{n\to\infty}\sigma_n < -1$, then there exists n_0 such that $\sigma_n \le -1$ for all $n \ge n_0$. It then follows from (3.13) that

$$
\delta_{n+1} \le (1-\alpha_n(1-v))\delta_n - \alpha_n \le \delta_n - \alpha_n.
$$

for all $n \ge n_0$. By induction, we have

$$
\delta_{n+1} \le \delta_{n_0} - \sum_{i=n_0}^{n}\alpha_i. \qquad (3.14)
$$

By taking $\limsup$ as $n \to \infty$ in (3.14), we have

$$\limsup_{n\to\infty} \delta_n \le \delta_{n_0} - \lim_{n\to\infty} \sum_{i=n_0}^{n} \alpha_i = -\infty,$$

which induces a contradiction. So, $-1 \le \limsup_{n\to\infty} \sigma_n < +\infty$. Thus, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned} \limsup_{n\to\infty} \sigma_n &= \lim_{k\to\infty} \sigma_{n_k} \\ &= \lim_{k\to\infty} 2\left\langle f_0 z - z, x_{n_k+1} - z \right\rangle - \frac{\left(1-\alpha_{n_k}\right)\gamma\left(\min\{1-\beta, 1-\mu\} - \gamma\right)}{\alpha_{n_k}} \\ &\quad \times \frac{\left(\left\|x_{n_k} - Sx_{n_k}\right\|^2 + \left\|Ax_{n_k} - TAx_{n_k}\right\|^2\right)^2}{\left\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\right\|^2}. \end{aligned} \tag{3.15}$$

Since $\left\langle f_0 z, x_{n_k+1} - z \right\rangle$ is a bounded real sequence, without loss of generality, we may assume $\lim_{k\to\infty} \left\langle f_0 z, x_{n_k+1} - z \right\rangle$ exists. Consequently, from (3.15), the following limit also exists

$$\lim_{k\to\infty} \frac{\left(1-\alpha_{n_k}\right)\gamma\left(\min\{1-\beta, 1-\mu\} - \gamma\right)}{\alpha_{n_k}} \frac{\left(\left\|x_{n_k} - Sx_{n_k}\right\|^2 + \left\|Ax_{n_k} - TAx_{n_k}\right\|^2\right)^2}{\left\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\right\|^2}.$$

It turns out that

$$\lim_{k\to\infty} \frac{\left(\left\|x_{n_k} - Sx_{n_k}\right\|^2 + \left\|Ax_{n_k} - TAx_{n_k}\right\|^2\right)^2}{\left\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\right\|^2} = 0. \tag{3.16}$$

Taking into consideration that

$$\frac{\left\|x_{n_k} - Sx_{n_k}\right\|^2 + \left\|Ax_{n_k} - TAx_{n_k}\right\|^2}{2\max\left\{1, \|A\|^2\right\}} \le \frac{\left(\left\|x_{n_k} - Sx_{n_k}\right\|^2 + \left\|Ax_{n_k} - TAx_{n_k}\right\|^2\right)^2}{\left\|x_{n_k} - Sx_{n_k} + A^*(I-T)Ax_{n_k}\right\|^2},$$

we then deduce from (3.16) that

$$\lim_{k\to\infty} \left\|x_{n_k} - Sx_{n_k}\right\| = \lim_{k\to\infty} \left\|Ax_{n_k} - TAx_{n_k}\right\| = 0. \tag{3.17}$$

It follows that any weak cluster point of $\{x_{n_k}\}$ belongs to Ω. Observe that

$$\begin{aligned} &\left\|x_{n+1} - x_n\right\| \\ &\le \alpha_n \left\|x_n - f_n x_n\right\| + \left(1-\alpha_n\right)\gamma\tau_n \left\|y_n\right\| \\ &= \alpha_n \left\|x_n - f_n x_n\right\| + \left(1-\alpha_n\right)\gamma \frac{\left(\left\|x_n - Sx_n\right\|^2 + \left\|Ax_n - TAx_n\right\|^2\right)^2}{\left\|x_n - Sx_n + A^*(I-T)Ax_n\right\|^2}. \end{aligned}$$

By (C3) and (3.16), we derive

$$\lim_{k\to\infty} \left\|x_{n_k+1} - x_{n_k}\right\| = 0.$$

This means that any weak cluster point of $\{x_{n_k+1}\}$ also belongs to Ω.

Without loss of generality, we assume that $\{x_{n_k+1}\}$ converges weakly to $\bar{x} \in \Omega$. Hence, we obtain

$$\limsup_{n\to\infty} \sigma_n \le \lim_{k\to\infty} 2\langle f_0 z - z, x_{n_k+1} - z\rangle = 2\langle f_0 z - z, \bar{x} - z\rangle \le 0.$$

due to the fact that $z = P_\Omega f_0 z$. Rewriting (3.13) as

$$\delta_{n+1} \le \left(1-\alpha_n(1-v)\right)\delta_n + \alpha_n(1-v)\frac{\sigma_n}{1-v}, \quad \forall n \ge 0,$$

and noticing Lemma 2.2, we get $x_n \to z$ as $n \to \infty$.

Theorem 3.5 *Let* $S: H_1 \to H_1$ *and* $T: H_2 \to H_2$ *be two demicontractive mappings with constants* $\beta \in [0,1)$ *and* $\mu \in [0,1)$, *respectively. Then the sequence* $\{x_n\}$ *generated by* (1.1) *converges strongly to the solution* $z(= P_\Omega f_0 z)$ *of problem* (3.9) *under the assumption of Theorem* 3.4.

4. Conclusion

In this paper, we consider a class of the split common fixed point problems. By extending results in [16] [17] from the directed mappings or the demicontractive mappings to the demimetric mappings, and a fixed point $u \in H_1$ to a sequence mappings $\{f_n\} \subset \Pi$, we construct two self-adaptive algorithms for solving the split common fixed point problem. Further, we also establish the weak and strong convergence theorems under some certain appropriate assumptions. The results in this paper are the extension and improvement of the recent results in the literature.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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