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A Note on the Perturbation of MF Algebras and Quasidiagonal C*-Algebras

Wenjuan Zhan, Liguang Wang

School of Mathematical Sciences, Qufu Normal University, Qufu, China
Email: zhanwenjuan0624@163.com, wangliguang0510@163.com

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Abstract

Perturbation problem of operator algebras was first introduced by Kadison and Kastler. In this short note, we consider the uniform perturbation of two classes of operator algebras, *i.e.*, MF algebras and quasidiagonal C*-algebras. We show that the sets of MF algebras and quasidiagonal C*-algebras of a given C*-algebra are closed under the perturbation of uniform norm.

Keywords

MF Algebra, Quasidiagonal C*-Algebra

1. Introduction and Preliminaries

Kadison and Kastler in [1] initiated the study of uniform perturbations of operator algebras. They considered a fixed C*-algebra $\mathscr{U}$ and equipped the set of all C*-subalgebras of $\mathscr{U}$ with a metric arising from Hausdorff distance between the unit balls of these subalgebras. We first recall the following definition of the metric d defined on the set of all C*-subalgebras of a C*-algebra $\mathscr{U}$ (see [1]).

Definition 1.1. Let $\mathscr{A}$ and $\mathscr{B}$ be C*-subalgebras of a C*-algebra $\mathscr{U}$. The Kadison-Kastler metric $d(\mathscr{A},\mathscr{B})$ between $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$d(\mathscr{A},\mathscr{B}) = \max\left\{\sup_{a\in(\mathscr{A})_1}\inf_{b\in(\mathscr{B})_1}\|a-b\|,\ \sup_{b\in(\mathscr{B})_1}\inf_{a\in(\mathscr{A})_1}\|a-b\|\right\}$$

where $(\mathscr{A})_1$ and $(\mathscr{B})_1$ denote the unit ball of $\mathscr{A}$ and $\mathscr{B}$ respectively.

Kadison and Kastler conjectured in [1] that sufficiently close von Neumann algebras (or C*-algebras) are necessarily unitarily conjugate. The first positive answer to Kadison-Kastler's conjecture was given by Christensen [2] when either $\mathscr{A}$ or $\mathscr{B}$ is a von Neumann algebra of type I. Many results related to this conjecture have been obtained during the past 40 years ([3] [4] [5] [6]). One-sided version of Kadison-Kastler's conjecture was introduced and studied by Christensen in [4] as well. Christensen showed in [4] that a nuclear C^*-algebra that is nearly contained in an injective von Neumann algebra is unitarily conjugate to this von Neumann algebra. Christensen, Sinclair, Smith and White showed in [5] that the property of having a positive answer to Kadison's similarity problem transfers to close C^*-algebras. Very recently, Kadison-Kastler's conjecture has been proved for the class of separable nuclear C^*-algebras in the remarkable paper [6].

The problem we are going to consider is as follows: Suppose $\mathscr{A}, \mathscr{B}$ are C^*-subalgebras of a C^*-algebra $\mathscr{U}$. If $d(\mathscr{A}, \mathscr{B}) < \gamma$, is $\mathscr{A}$ and $\mathscr{B}$ share similar properties?

In this short note, we show that the sets of matricial field algebras (MF algebras) and quasidiagonal C^*-algebras of a given C^*-algebra are closed under the perturbation of uniform norm.

2. Main Results

In this section, we consider some topological properties of the set of all MF algebras and quasidiagonal C^*-subalgebras under the perturbation of uniform norm. For basics of C^*-algebras, we refer to [7] and [8]. We first recall the definition of MF algebras ([9]).

Suppose $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ is a sequence of complex matrix algebras. We can introduce the full C*-direct product $\prod_{m=1}^{\infty} \mathcal{M}_{k_m}(\mathbb{C})$ of $\{\mathcal{M}_{k_n}(\mathbb{C})\}_{n=1}^{\infty}$ as follows:

$$\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \left\{ (Y_n)_{n=1}^{\infty} \mid \forall n \geq 1, Y_n \in \mathcal{M}_{k_n}(C) \text{ and } \sup_{n \geq 1} \|Y_n\| < \infty \right\}. \tag{1}$$

Furthermore, we can introduce a norm closed two sided ideal in $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ as follows,

$$\sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) = \left\{ (Y_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) : \lim_{n \to \infty} \|Y_n\| = 0 \right\}. \tag{2}$$

Let π be the quotient map from $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ to $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) \Big/ \sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$. It is known that $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) \Big/ \sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ is a unital C*-algebra. If we denote $\pi\left((Y_n)_{n=1}^{\infty}\right)$ by $\left[(Y_n)_n\right]$, then

$$\left\| \left[(Y_n)_n\right] \right\| = \limsup_{n \to 1} \|Y_n\|. \tag{3}$$

Now we are ready to recall an equivalent definition of MF algebras which is given by Blackadar and Kirchberg ([9]).

Definition 2.1. (Theorem 3.2.2, [9]) Let $\mathscr{U}$ be a separable C^*-algebra. If $\mathscr{U}$ can be embedded as a C^*-subalgebra of $\prod_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C}) \Big/ \sum_{n=1}^{\infty} \mathcal{M}_{k_n}(\mathbb{C})$ for a sequence $\{k_n\}_{n=1}$ of integers, then $\mathscr{U}$ is called an MF algebra.

Lemma 2.2. ([10] Lemma 2.12) Suppose that $\mathscr{U}$ is a separable C^*-algebra.

Assume for every finite family of elements $x_1, x_2, \cdots, x_n$ in $\mathscr{U}$ and every $\varepsilon > 0$, there is an MF algebra $\mathscr{U}_1$ such that $\{x_1, x_2, \cdots, x_n\} \subset_\varepsilon \mathscr{U}_1$, (in the sense of Definition 2.3 in [10]). Then $\mathscr{U}$ is also an MF algebra.

Proposition 2.3. Let $\mathscr{U}$ be a C*-algebra and $\mathfrak{F}$ be the subset of all separable MF algebras contained in $\mathscr{U}$. Then $\mathfrak{F}$ is closed under the metric *d*.

Proof. Let $\mathscr{A} \in \overline{\mathfrak{F}}$. Then there exist $\mathscr{A}_n \in F$ such that $d(\mathscr{A}_n, \mathscr{A}) \to 0$. For any $x_1, x_2, \cdots, x_m \in \mathscr{A}$, $\forall \varepsilon > 0$, there is an n_0 such that $d(\mathscr{A}_{n_0}, \mathscr{A}) < \frac{\varepsilon}{2\sum_{i=1}^m \|x_i\| + 1}$. Then there exist $y_1, y_2, \cdots, y_m \in \mathscr{A}_{n_0}$ such that

$$\|x_i - y_i\| < \frac{\varepsilon}{2\sum_{i=1}^m \|x_i\| + 1}\|x_i\| < \varepsilon \tag{4}$$

for all *i*. It follows from Lemma 2.2 that $\mathscr{A}$ is also a MF algebra.■

We will recall some results about quasidiagonal C*-algebras for the reader's convenience. We refer the reader to [11] for a comprehensive treatment of this important class of C*-algebras.

Definition 2.4. A subset $\Omega \subset \mathcal{B}(\mathscr{H})$ is called a quasidiagonal set of operators if for each finite set $\omega \subset \Omega$, finite set $\chi \subset \mathscr{H}$ and $\varepsilon > 0$, there exists a finite rank projection $P \in \mathcal{B}(\mathscr{H})$ such that $\|TP - PT\| \le \varepsilon$ and $\|P(x) - x\| \le \varepsilon$ for all $T \in \omega$ and $x \in \chi$.

Definition 2.5. A C*-algebra $\mathscr{U}$ is called quasidiagonal (QD) if there exists a faithful representation $\pi : \mathscr{U} \to B(\mathscr{H})$ such that $\pi(\mathscr{U})$ is a quasidiagonal set of operators.

The following result is Lemma 7.1.3 in [11] which is useful to determine whether a C*-algebra is quasidiagonal or not.

Lemma 2.6. A C*-algebra $\mathscr{U}$ is quasidiagonal if and only if for each finite set $F \subset \mathscr{U}$ and $\varepsilon > 0$, there exists a completely positive map $\phi : \mathscr{U} \to M_n(\mathbb{C})$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\| < \epsilon \tag{5}$$

and

$$\|\phi(a)\| > \|a\| - \epsilon \tag{6}$$

for all $a, b \in F$.

Proposition 2.7. Let $\mathscr{U}$ be a separable C*-algebra. Let $\mathfrak{F} = QD(\mathscr{U})$ be the set of all quasidiagonal C*-subalgebras of $\mathscr{U}$. Then $\mathfrak{F}$ is closed under the metric *d*.

Proof. Let $\mathscr{A} \in \overline{\mathfrak{F}}$ and choose $\mathscr{A}_n \in \mathfrak{F}$ such that $d(\mathscr{A}_n, \mathscr{A}) \to 0$. Given finite subset $\{x_1, x_2, \cdots, x_k\}$ of the unit ball of $\mathscr{A}$ and $\epsilon > 0$. There is a $N \in \mathbb{N}$ such that $d(\mathscr{A}_N, \mathscr{A}) < \frac{\epsilon}{6}$. Choose $y_1, y_2, \cdots y_k$ in the unit ball of $\mathscr{A}_N$ such that $\|x_i - y_i\| < \frac{\epsilon}{6}$ for $i = 1, 2, \cdots, k$. Since $\mathscr{A}_N$ is QD, it follows from Lemma 2.6 that there is a c.c.p. map $\phi : \mathscr{A}_N \to M_t(\mathbb{C})$ such that

$$\left\|\phi\left(y_i y_j\right)-\phi\left(y_i\right)\phi\left(y_j\right)\right\| \le \frac{\varepsilon}{6} \tag{7}$$

and

$$\left\|\phi\left(y_j\right)\right\| \ge \left\|y_j\right\| - \frac{\epsilon}{6} \tag{8}$$

for all $i, j = 1, 2, \cdots, k$. Now use Arveson's extension theorem ([11]) to extend ϕ to a c.c.p. map $\tilde{\phi}$ from $\mathscr{U}$ to $M_t(\mathbb{C})$. Let $\psi : \mathscr{A} \to \mathscr{M}_t(\mathbb{C})$ be the restriction of $\tilde{\phi}$ to $\mathscr{A}$. Then for each $i, j = 1, 2, \cdots, k$, we have

$$\begin{aligned} &\left\|\psi\left(x_i x_j\right)-\psi\left(x_i\right)\psi\left(x_j\right)\right\| \\ &\le \left\|\psi\left(x_i x_j\right)-\psi\left(x_i y_j\right)\right\| + \left\|\psi\left(x_i y_j\right)-\psi\left(y_i y_j\right)\right\| \\ &\quad + \left\|\psi\left(y_i y_j\right)-\psi\left(y_i\right)\psi\left(y_j\right)\right\| + \left\|\psi\left(y_i\right)\psi\left(y_j\right)-\psi\left(x_i\right)\psi\left(y_j\right)\right\| \\ &\quad + \left\|\psi\left(x_i\right)\psi\left(y_j\right)-\psi\left(x_i\right)\psi\left(x_j\right)\right\| \\ &< \varepsilon \end{aligned} \tag{9}$$

and

$$\left\|\psi\left(x_i\right)\right\| = \left\|\psi\left(x_i - y_i\right)+\psi\left(y_i\right)\right\| \ge \left\|\psi\left(y_i\right)\right\| - \frac{\varepsilon}{6} > \left\|y_i\right\| - \frac{\varepsilon}{3} \ge \left\|x_i\right\| - \varepsilon. \tag{10}$$

Use Lemma 2.6 again we have that $\mathscr{A}$ is quasidiagonal.■

3. Conclusion

In this paper, we use some characterizations of MF algebras and quasidiagonal C*-algebras to show that these two sets of C*-subalgebras of a given C*-algebras are closed with respect to the topology induced by the Kadison-Kastler metric.

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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