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Existence of Multiple Positive Solutions for Third-Order Three-Point Boundary Value Problem

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Abstract

In this paper, we study the existence of positive solutions for a class of third-order three-point boundary value problem. By employing the fixed point theorem on cone, some new criteria to ensure the three-point boundary value problem has at least three positive solutions are obtained. An example illustrating our main result is given. Moreover, some previous results will be improved significantly in our paper.

Keywords

Third-Order Three-Point Boundary Value Problem, Fixed Point Theorem, Three Positive Solutions

1. Introduction

As we all know, the earliest boundary value problem studied is Dirichlet problem. We need to find the solution of Laplace equation. Boundary value problems are most common in physics, such as wave equation. With the development of boundary value problems, many scholars began to pay attention to the study of higher-order boundary value problems. The third-order three-point problems have a wide range of applications in the fields of mathematics and physics [1] [2] [3] [4] [5]. Many works on the third-order boundary value problems have been established. In [6] [7] [8] [9] [10], the authors have studied the third-order three-point boundary value problem and proved that the model has at least one positive solution. Recently, there have been many papers dealing with the positive solutions of boundary value problems for nonlinear differential equations with various boundary conditions. For example, Anderson [11] obtained some existence results for positive solutions for the following system:

$$\begin{cases} x'''(t) = f(x(t)), 0 < t < 1, \\ x(0) = x'(t_2) = x''(1) = 0, \end{cases} \tag{1.1}$$

where $f : R \to R$ is continuous, f is nonnegative for $x \ge 0$ and $\frac{1}{2} \le t_2 < 1$.

Moreover, Yao [12] considered the following system:

$$\begin{cases} u'''(t) + \lambda f(t, u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $f(t,u)$ is a Caratheodory function. The author proved that (1.2) has at least one positive solution by Krasnoselskii fixed point theorems.

With the development of third-order boundary value problems, Guo *et al.* [2] considered the existence of a positive solution to the third-order three-point boundary value problem as follows

$$\begin{cases} u'''(t) + a(t) f(u(t)) = 0,\ t \in (0,1), \\ u(0) = u'(0) = 0.\ u'(1) = \alpha u'(\eta), \end{cases} \tag{1.3}$$

where $f : [0,+\infty) \to [0,+\infty)$ is continuous; $a : [0,1] \to [0,+\infty)$ is continuous and not identically zero on $\left[\frac{\eta}{\alpha}, \eta\right]$, $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$. By using the Guo-Krasnoselskii fixed point theorem, they proved that the system (1.3) has at least one positive solution.

To our best knowledge, few papers can be found in the literature for three positive solutions of third-order three-point boundary value problems. Motivated greatly by the above-mentioned excellent works, in this paper, we will consider the following model

$$\begin{cases} x'''(t) + h(t) f(t, x(t), x'(t), x''(t)) = 0, 0 < t < 1, \\ x(0) = x'(0) = 0, x'(1) = \xi x'(\eta), \end{cases} \tag{1.4}$$

where $E = \{x \in C^3([0,1], R) : x(0) = x'(0) = 0, x'(1) = \xi x'(\eta)\}$, $0 < \eta < 1$, $1 < \xi < \frac{1}{\eta}$, $f : [0,1] \times [0,+\infty) \times R \times R \to [0,+\infty)$ is a continuous function; $h : [0,1] \to [0,+\infty)$ is continuous and not identically zero on $\left[\frac{\eta}{\alpha}, \eta\right]$.

Obviously, this model is new because the nonlinear f depends not only on the unknown function but also the derivative of unknown function. In particular, the system (1.2) is special case of system (1.4). By the properties of the Green's function, existence results of at least three positive solution for the third-order three-point boundary value problem are established by a new method which is different from the method in [13]. The paper is organized as follows. In Section 2, we present some notation and lemmas. In Section 3, we give the main results. In Section 4, an example is given to illustrate the main results of this paper.

2. Preliminaries

Definition 2.1. Let E be a real Banach space. $K \subset E$ is a nonempty closed convex set. If it satisfies the following two conditions:

1) $x \in K, \lambda > 0$ implies $\lambda x \in K$;

2) $x \in K, -x \in P$ implies $x = 0$.

Then, K is called a cone of E.

Definition 2.2. Suppose K is a cone. The map $\alpha : K \to [0, +\infty)$ is continuous and satisfies the following inequality

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y) \text{ for any } x, y \in K \text{ and } 0 \le t \le 1.$$

Then the map α is a nonnegative continuous convex function on K.

Suppose K is a cone. The map $\varphi : K \to [0, +\infty)$ is continuous and satisfies the following inequality

$$\varphi(tx + (1-t)y) \ge t\varphi(x) + (1-t)\varphi(y) \text{ for any } x, y \in K \text{ and } 0 \le t \le 1.$$

Then the map φ is a nonnegative continuous concave function on K.

Lemma 2.1 [2] Assume $\xi\eta \ne 1$, then the system

$$\begin{cases} x'''(t) + y(t) = 0, 0 < t < 1, \\ x(0) = x'(0) = 0, x'(1) = \xi x'(\eta), \end{cases} \tag{1.5}$$

has a unique solution $x(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s$ for $y \in C[0,1]$,

where

$$G(t,s) = \frac{1}{2(1-\xi\eta)} \begin{cases} (2ts - s^2)(1-\xi\eta) + t^2 s(\xi - 1), s \le \min\{\eta, t\}, \\ t^2(1-\xi\eta) + t^2 s(\xi - 1), t \le s \le \eta, \\ (2ts - s^2)(1-\xi\eta) + t^2(\xi\eta - s), \eta \le s \le t, \\ t^2(1-s), \max\{\eta, t\} \le s. \end{cases} \tag{1.6}$$

If we denote $g(s) = \frac{1+\xi}{1-\xi\eta} s(1-s), s \in [0,1]$, then we have the following lemma.

Lemma 2.2 [2] Let $0 < \eta < 1, 1 < \xi < \frac{1}{\eta}$, then

1) $0 \le G(t,s) \le g(s)$, for any $(t,s) \in [0,1] \times [0,1]$;

2i) $G(t,s) \ge \beta g(s)$, for any $(t,s) \in \left[\frac{\eta}{\xi}, \eta\right] \times [0,1]$,

where $0 < \beta = \frac{\eta^2}{2\xi^2(1+\xi)} \min\{\xi - 1, 1\} < 1$.

For positive real numbers a, b, c, d, we define the following convex sets:

$$P(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$

$$P(\gamma, \varphi, b, d) = \{x \in K \mid b \le \varphi(x), \gamma(x) \le d\},$$

$$P(\gamma,\alpha,\varphi,b,c,d)=\{x\in K \mid b\le\varphi(x),\alpha(x)\le c,\gamma(x)\le d\},$$

$$R(\gamma,\psi,a,d)=\{x\in K \mid a\le\psi(x),\gamma(x)\le d\}.$$

Lemma 2.3 [14] **(Arzela-Ascoli theorem)** Let $\mu\subset C[0,1]$ be a operator, then μ is sequentially compact in $C[0,1]$ if and only if μ is uniformly bounded and equicontinuous.

Lemma 2.4 [5] **(Krasnoselskii fixed point theorem)** Let E be a real *Banach* space. $K\subset E$ is a cone. Suppose γ,α are nonnegative continuous convex functions on *K*. φ is a nonnegative continuous concave function on *K*. ψ is a nonnegative continuous function on *K*, which satisfied $\psi(\lambda x)\le\lambda\psi(x),\lambda\in[0,1]$ and for positive numbers of q,d, we have

$$\varphi(x)\le\psi(x),\ \|x\|\le q\gamma(x),\ \forall x\in\overline{P(\gamma,d)}. \tag{1.7}$$

Let $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ be a completely continuous operator. There exist positive numbers of a,b,c and $a<b$ satisfing the following conditions:

1) $\{x\in P(\gamma,\alpha,\varphi,b,c,d)\mid\varphi(x)>b\}\ne\varnothing$, and $\alpha(Tx)>b$, for all $x\in P(\gamma,\alpha,\varphi,b,c,d)$;

2) $\varphi(Tx)>b$, for $x\in P(\gamma,\varphi,b,d)$, and $\alpha(Tx)>c$;

3) $0\notin R(\gamma,\psi,a,d)$, and $\psi(Tx)<a$, for $x\in R(\gamma,\psi,a,d)$, $\psi(x)=a$;

then T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\gamma,d)}$ such that

$$\gamma(x_i)\le d, i=1,2,3;\ b<\varphi(x_1);$$

$$a<\psi(x_2)\text{, as }\ \varphi(x_2)<b;\ \psi(x_3)<a.$$

3. The Existence of Three Positive Solutions

We define the norm

$$\|x\|=\max\left\{\max_{t\in[0,1]}|x(t)|,\max_{t\in[0,1]}|x'(t)|,\max_{t\in[0,1]}|x''(t)|\right\}.$$

Define the cone by

$$K=\left\{x\in E\mid x(t)\ge0,t\in[0,1],\ \min_{t\in\left[\frac{\eta}{\xi},\eta\right]}x(t)\ge\beta\max_{t\in[0,1]}x(t)\right\}.$$

Suppose

$$\gamma(x)=\max_{t\in[0,1]}|x''(t)|,\psi(x)=\alpha(x)=\max_{t\in[0,1]}|x(t)|,\varphi(x)=\min_{t\in\left[\frac{\eta}{\xi},\eta\right]}|x(t)|.$$

Lemma 3.1. Let $T:K\to E$ be the operator defined by

$$Tx(t)=\int_0^1G(t,s)h(s)f(s,x(s),x'(s),x''(s))\mathrm{d}s,x\in K.$$

Then $T:K\to K$ is completely continuous.

Proof From the fact that f is nonnegative continuous function and Lemma 2.2, we know that $Tx(t)\ge0,t\in[0,1]$. Let $x\in K$, from Lemma 2.2, we have

$$\int_0^1G(t,s)h(s)f(s,x(s),x'(s),x''(s))\mathrm{d}s$$

$$\le\int_0^1g(s)h(s)f(s,x(s),x'(s),x''(s))\mathrm{d}s,$$

so

$$\max_{t\in[0,1]}|Tx(t)| = \max_{t\in[0,1]}\left|\int_0^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))\,\mathrm{d}s\right| \le \int_0^1 g(s)h(s)f(s,x(s),x'(s),x''(s))\,\mathrm{d}s.$$

and

$$\begin{aligned}\min_{t\in\left[\frac{\eta}{\xi},\eta\right]} Tx(t) &= \min_{t\in\left[\frac{\eta}{\xi},\eta\right]}\int_0^1 G(t,s)h(s)f(s,x(s),x'(s),x''(s))\,\mathrm{d}s\\ &\ge \beta\int_0^1 g(s)h(s)f(s,x(s),x'(s),x''(s))\,\mathrm{d}s\\ &\ge \beta\max_{t\in[0,1]}(Tx)(t).\end{aligned}$$

thus $T:K\to K$. According to the Arzela-Ascoli theorem, we prove that T is a completely continuous operator.

For convenience, we note that

$$A=\max\left\{\int_0^1 h(s)\left|\frac{\partial^2 G(t,s)}{\partial t^2}\right|_{t=0}\mathrm{d}s, \int_0^1 h(s)\left|\frac{\partial^2 G(t,s)}{\partial t^2}\right|_{t=1}\mathrm{d}s\right\},$$

$$B=\min\left\{\int_{\frac{\eta}{\xi}}^{\eta} G\left(\frac{\eta}{\xi},s\right)h(s)\,\mathrm{d}s, \int_{\frac{\eta}{\xi}}^{\eta} G(\eta,s)h(s)\,\mathrm{d}s\right\},$$

$$C=\int_0^1 g(s)h(s)\,\mathrm{d}s.$$

Theorem 3.1. Suppose there exist $0<a<b<\dfrac{b}{\beta}\le d$ such that

(H_1) $f(t,u,v,w)\le\dfrac{d}{A},(t,u,v,w)\in[0,1]\times[0,d]\times[-d,d]\times[-d,d]$,

(H_2) $f(t,u,v,w)>\dfrac{b}{B},(t,u,v,w)\in\left[\dfrac{\eta}{\xi},\eta\right]\times\left[b,\dfrac{b}{\beta}\right]\times[-d,d]\times[-d,d]$,

(H_3) $f(t,u,v,w)<\dfrac{a}{C},(t,u,v,w)\in[0,1]\times[0,a]\times[-d,d]\times[-d,d]$,

then the system (1.4) has at least three positive points x_1, x_2 and x_3 satisfying

$$\max_{t\in[0,1]}|x_i''(t)|\le d,\ i=1,2,3;$$

$$\max_{t\in[0,1]}|x_i'(t)|\le d,\ i=1,2,3;$$

$$b<\min_{t\in\left[\frac{\eta}{\xi},\eta\right]}|x_1(t)|;\ \max_{t\in[0,1]}|x_3(t)|<a;$$

$$a<\max_{t\in[0,1]}|x_2(t)|\le\frac{b}{\beta},\ \text{for}\ \min_{t\in\left[\frac{\eta}{\xi},\eta\right]}|x_2(t)|<b.$$

Proof For $x\in K$, we have

$$x(t)=x(0)+\int_0^t x'(s)\,\mathrm{d}s\le t\max_{t\in[0,1]}|x'(t)|\le\max_{t\in[0,1]}|x'(t)|,$$

so

$$\max_{t\in[0,1]}|x(t)|\le\max_{t\in[0,1]}|x'(t)|.$$

Since

$$x'(t) = x'(0) + \int_0^t x''(s)\,ds$$

which also implies that

$$|x'(t)| \le \max_{t\in[0,1]} |x''(t)|$$

therefore,

$$\|x(t)\| \le \gamma(x), x \in \overline{P(\gamma,d)}, \ \beta\alpha(x) \le \varphi(x) \le \alpha(x) = \psi(x).$$

So we show that (1.7) of the Lemma 2.4 holds.

If $x \in \overline{P(\gamma,d)}$, we have $\gamma(x) = \max_{t\in[0,1]} |x''(t)| \le d$.

And $T'''x(t) = -h(t) f(t, x(t), x'(t), x''(t)) \le 0$ for any $t \in [0,1]$. From assumption (H_1), we have $f(t, x(t), x'(t), x''(t)) \le \frac{d}{A}$, therefore,

$$\begin{aligned}
\gamma(Tx(t)) &= \max_{t\in[0,1]} |T''x(t)| \\
&= \max\{|T''x(1)|, |T''x(0)|\} \\
&\le \max\left\{\int_0^1 \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right|_{t=0} h(s) f(s, x(s), x'(s), x''(s))\,ds,\right. \\
&\qquad \left.\int_0^1 \left|\frac{\partial^2 G(t,s)}{\partial t^2}\right|_{t=1} h(s) f(s, x(s), x'(s), x''(s))\,ds\right\} \\
&\le \frac{d}{A} A = d,
\end{aligned}$$

hence

$$T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}.$$

Let $x(t) = \frac{b}{\beta}, t \in [0,1]$, it is easy to prove $x(t) = \frac{b}{\beta} \in P\left(\gamma, \alpha, \varphi, b, \frac{b}{\beta}, d\right)$, $\varphi(x) = \frac{b}{\beta} > b$, hence

$$\left\{x \in P\left(\gamma, \alpha, \varphi, b, \frac{b}{\beta}, d\right) \mid \varphi(x) > b\right\} \ne \varnothing.$$

If $x \in P\left(\gamma, \alpha, \varphi, b, \frac{b}{\beta}, d\right)$, then

$$b \le x(t) \le \frac{b}{\beta}, |x'(t)| \le d, \ |x''(t)| \le d, t \in \left[\frac{\eta}{\xi}, \eta\right].$$

From assumption (H_2), we have $f(t, x(t), x'(t), x''(t)) > \frac{b}{B}, t \in \left[\frac{\eta}{\xi}, \eta\right]$.

It can be divided into two situations:

(i) $\varphi(Tx) = Tx\left(\frac{\eta}{\xi}\right)$,

$$\begin{aligned}\varphi(Tx) &= Tx\left(\frac{\eta}{\xi}\right)\\ &= \int_0^1 G\left(\frac{\eta}{\xi}, s\right) h(s) f\left(s, x(s), x'(s), x''(s)\right) \mathrm{d}s\\ &> \frac{b}{B}\int_{\frac{\eta}{\xi}}^{\eta} G\left(\frac{\eta}{\xi}, s\right) h(s) \mathrm{d}s \geq b,\end{aligned}$$

(ii) $\varphi(Tx) = Tx(\eta)$,

$$\begin{aligned}\varphi(Tx) &= Tx(\eta)\\ &= \int_0^1 G(\eta, s) h(s) f\left(s, x(s), x'(s), x''(s)\right) \mathrm{d}s\\ &> \frac{b}{B}\int_{\frac{\eta}{\xi}}^{\eta} G(\eta, s) h(s) \mathrm{d}s \geq b,\end{aligned}$$

Therefore, we have $\varphi(Tx) > b$ for $x \in P\left(\gamma, \alpha, \varphi, b, \frac{b}{\beta}, d\right)$, that is to say, condition (i) of Lemma 2.4 is satisfied.

Since $x \in P(\gamma, \varphi, b, d), \alpha(Tx) > \frac{b}{\beta}$, we have

$$\varphi(Tx) \geq \beta\alpha(Tx) > \beta\frac{b}{\beta} = b.$$

Thus condition (ii) of Lemma 2.4 is satisfied.

Obviously, $\psi(0) = 0 < a$, so $0 \notin R(\gamma, \psi, a, d)$. We assume $x \in R(\gamma, \psi, a, d)$ and $\psi(x) = a$ hold.

From assumption (H_3). we have

$$\begin{aligned}\psi(Tx) &= \max_{t\in[0,1]}\left|Tx(t)\right|\\ &= \max_{t\in[0,1]}\int_0^1 G(t, s) h(s) f\left(s, x(s), x'(s), x''(s)\right) \mathrm{d}s\\ &\leq \int_0^1 g(s) h(s) f\left(s, x(s), x'(s), x''(s)\right) \mathrm{d}s\\ &< \frac{a}{C} C = a.\end{aligned}$$

Thus condition (iii) of Lemma 2.4 is also satisfied. From the above facts, the proof of Theorem 3.1 is completed.

4. Example

Example 4.1 Consider the following boundary value problem

$$\begin{cases} x'''(t) + 4tf\left(t, x(t), x'(t), x''(t)\right) = 0, 0 < t < 1,\\ x(0) = x'(0) = 0, x'(1) = \frac{6}{5}x'\left(\frac{1}{4}\right), \end{cases}$$

where,

$$f(t,u,v,w)=\begin{cases}\dfrac{u}{2}+\dfrac{e^{-v^2\cos^2(t)}}{1000}+\dfrac{e^{-w^2}}{1000}, \\ (t,u,v,w)\in[0,1]\times\left[0,\dfrac{1}{10}\right]\times[-100,100]\times[-100,100]; \\ 9995u-\dfrac{19989}{20}+\dfrac{e^{-v^2\cos^2(t)}}{1000}+\dfrac{e^{-w^2}}{1000}, \\ (t,u,v,w)\in[0,1]\times\left[\dfrac{1}{10},\dfrac{11}{100}\right]\times[-100,100]\times[-100,100]; \\ -\dfrac{2}{5}u+\dfrac{25011}{250}+\dfrac{e^{-v^2\cos^2(t)}}{1000}+\dfrac{e^{-w^2}}{1000}, \\ (t,u,v,w)\in[0,1]\times\left[\dfrac{11}{100},\dfrac{1011}{100}\right]\times[-100,100]\times[-100,100]; \\ \dfrac{100}{3483789}u+\dfrac{334401834}{3483789}+\dfrac{e^{-v^2\cos^2(t)}}{1000}+\dfrac{e^{-w^2}}{1000}, \\ (t,u,v,w)\in[0,1]\times\left[\dfrac{1011}{100},\dfrac{34848}{625}\right]\times[-100,100]\times[-100,100]; \\ -\dfrac{65625}{27652}u+\dfrac{6590152}{27652}+\dfrac{e^{-v^2\cos^2(t)}}{1000}+\dfrac{e^{-w^2}}{1000}, \\ (t,u,v,w)\in[0,1]\times\left[\dfrac{34848}{625},100\right]\times[-100,100]\times[-100,100]. \end{cases}$$

where $\xi=\dfrac{6}{5},\eta=\dfrac{1}{4},a=\dfrac{1}{10},b=\dfrac{11}{100},d=100$.

By the precise calculation, we have

$$A=\frac{157}{168},B=\frac{1375}{995328},C=\frac{22}{21},\frac{b}{\beta}=\frac{34848}{625},$$

$$f(t,u,v,w)\le 106<\frac{d}{A}\approx 107,$$
$$(t,u,v,w)\in[0,1]\times[0,100]\times[-100,100]\times[-100,100],$$

$$f(t,u,v,w)>96>\frac{b}{B}\approx 79.63,$$
$$(t,u,v,w)\in\left[\frac{5}{24},\frac{1}{4}\right]\times\left[\frac{11}{100},\frac{34848}{625}\right]\times[-100,100]\times[-100,100],$$

$$f(t,u,v,w)<0.052<\frac{a}{C}\approx 0.09,$$
$$(t,u,v,w)\in[0,1]\times\left[0,\frac{1}{10}\right]\times[-100,100]\times[-100,100].$$

All the conditions of theorem 3.1 are satisfied, so there are at least three positive solutions for the system.

5. Conclusion

In this paper, applying the fixed point theorem on the cone, we investigate the existence of positive solutions for a class of third-order three-point boundary value problem, which is a more general system. We obtain that the boundary value problem has at least three positive solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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