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Solving the Linear Oscillatory Problem without Damping with Random Loading Condition Using the Decomposition Method

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Abstract

In this paper we study the solution of random linear oscillatory equation $\ddot{x}+w^2x=F(t;\omega)$ without damping and with random leading condition using the method. Finally, the time evolution of the mean, variance and standard deviation has been plotted for a range of values of the natural frequency *w*.

Keywords

Linear Stochastic Differential Equations, Adomian Decompositions, Linear Oscillatory, Mathematica

1. Introduction

The Adomian decomposition technique was firstly introduced by Adomian in 1975. This technique can be used to solve differential, integral, algebraic and many other equations (linear or nonlinear) [1]-[12]. The method is based on a suggestion by Adomian G. that the solution can be decomposed into components. In the coming sections we will see that the Adomian decomposition method is also very convenient computationally and offers some significant advantages [13]-[20]. The Adomian decomposition method is not a perturbation procedure, so no assumption concerning the size of randomness is necessary, where each term from the decomposed solution depends only on the preceding terms. A little work in the convergence of the procedure had been done [21] [22] [23] [24] [25].

2. Problem Formulation

In this paper, we focus on solving the following Solving the linear oscillatory problem

$$\ddot{x} + w^2 x = F(t;\omega) \tag{1}$$

$$F(t;\omega) = e(t)\left[1 + \varepsilon n(t;\omega)\right] \tag{2}$$

under stochastic excitation $F(t;\omega)$ with the deterministic initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

where

w: frequency of oscillation,

ε : deterministic nonlinearity scale,

$\omega \in (\Omega, \sigma, P)$: a triple probability space with Ω as the sample space, where σ is a σ-algebra on event in Ω and P is a probability measure, and $n(t;\omega)$ is a white noise with the following properties:

$$En(t;\omega) = 0 \tag{3}$$

$$En(t_1;\omega) \cdot n(t_2;\omega) = \operatorname{cov}\left[n(t_1), n(t_2)\right] = \delta(t_1 - t_2) \tag{4}$$

By obtaining the P.d.f. of $x(t)$, the average and variance of the solution process in terms of t: time, the general solution is

$$x(t) = x_0 \cos wt + \frac{\dot{x}_0}{w} \sin wt + \frac{1}{\omega} \int_0^t \sin w(t-s) F(s;q) \mathrm{d}s \tag{5}$$

The ensemble average is given by

$$\begin{aligned} Ex(t) = \mu_{x(t)} &= x_0 \cos wt + \frac{\dot{x}_0}{w} \sin wt + \frac{1}{w} \int_0^t \sin w(t-s) EF(s;q) \mathrm{d}s \\ &= x_0 \cos wt + \frac{\dot{x}_0}{w} \sin wt + \frac{1}{\omega} \int_0^t \sin w(t-s) e(s) \mathrm{d}s \end{aligned} \tag{6}$$

The covariance takes the form

$$\begin{aligned} \operatorname{cov}\left(x(t_1), x(t_2)\right) &= E\left(x(t_1) - \mu_{x(t_1)}\right) \cdot \left(x(t_2) - \mu_{x(t_2)}\right) \\ &= \frac{\varepsilon^2}{w^2} \int_0^{t_1} \sin w(t_1 - s) \sin w(t_2 - s) e^2(s) \mathrm{d}s \end{aligned} \tag{7}$$

The variance is

$$\sigma_x^2(t) = \frac{\varepsilon^2}{w^2} \int_0^t \sin^2 w(t-s) e^2(s) \mathrm{d}s \tag{8}$$

Due to linearity and the deterministic properties of $x_0, \dot{x}_0$ and the frequency w we obtain a Gaussian solution process:

$$f_{x(t)} = \frac{1}{\sigma_{x(t)} \sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x(t) - \mu_{x(t)}}{\sigma_{x(t)}}\right)^2} \tag{9}$$

where $\mu_{x(t)} = x_0 \cos wt + \frac{\dot{x}_0}{\omega} \sin wt + \frac{1}{\omega} \int_0^t \sin w(t-s) e(s) \mathrm{d}s$.

$$\sigma_{x(t)}^2 = \frac{\varepsilon^2}{w^2}\int_0^t \sin^2 w(t-s)e^2(s)\,\mathrm{d}s$$

Equation (9) represents a closed form solution of problem (1) with random loading condition.

3. The Adomian Decomposition Method

Case-study:

Let us consider

$$F(t;\omega) = \mathrm{e}^{-t} + \varepsilon n(t;\omega) \quad (10)$$

In the **Adomian decomposition** method, differential operators are decomposed. Thus Equation (1) is rewritten in the following form:

$$(L+R)x = F(t;q) \quad (11)$$

where:

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \text{ and } R = \omega^2$$

Hence,

$$Lx = F(t;q) - Rx \quad (12)$$

Solving for x we obtain

$$x = L^{-1}F(t;q) - L^{-1}Rx + \phi(t) \quad (13)$$

where $\phi(t)$ is the solution of $Lx = 0$

$$Lx = 0 \Rightarrow \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0 \Rightarrow x = at + c \quad (14)$$

Subject to the initial conditions:

$$\phi(t) = x_0 + \dot{x}_0 t \quad (15)$$

Thus, the solution of equation takes the form:

$$x = x_0 + \dot{x}_0 t + \int_0^t\int_0^t F(t;q)\,\mathrm{d}t\mathrm{d}t - w^2\int_0^t\int_0^t x(t)\,\mathrm{d}t\mathrm{d}t \quad (16)$$

We now assume that the solution can be written in the following form:

$$x(t) = x^{(0)}(t) + x^{(1)}(t) + \cdots = \sum_{i=0}^{\infty} x^{(i)}(t) \quad (17)$$

Substituting (17) in (16) we obtain:

$$\sum_{i=0}^{\infty} x^{(i)} = x_0 + \dot{x}_0 t + \int_0^t\int_0^t F(t;q)\,\mathrm{d}t\mathrm{d}t - \omega^2 \sum_{i=0}^{\infty}\int_0^t\int_0^t x^{(i)}(t)\,\mathrm{d}t\mathrm{d}t \quad (18)$$

By matching the boundaries, we obtain:

$$x^{(0)}(t) = x_0 + \dot{x}_0 t + \int_0^t\int_0^t F(t;q)\,\mathrm{d}t\mathrm{d}t \quad (19)$$

$$x^{(1)}(t) = -w^2 \int_0^t \int_0^t x^{(0)} \mathrm{d}t\mathrm{d}t \tag{20}$$

$$x^{(2)}(t) = -w^2 \int_0^t \int_0^t x^{(1)}(t) \mathrm{d}t\mathrm{d}t \tag{21}$$

And the nth term will be:

$$x^{(n)}(t) = -w^2 \int_0^t \int_0^t x^{(n-1)}(t) \mathrm{d}t\mathrm{d}t, \quad n \geq 1 \tag{22}$$

By applying this procedure to equation, we obtain:

$$x^{(1)}(t) = -w^2 x_o \frac{t^2}{2!} - w^2 \dot{x}_0 \frac{t^3}{3!} - w^2 L^{-1} L^{-1} F(t;q) \tag{23}$$

$$x^{(2)}(t) = w^4 x_0 \frac{t^4}{4!} + w^4 \dot{x}_0 \frac{t^5}{5!} + w^4 L^{-1} L^{-1} L^{-1} F(t;q) \tag{24}$$

$$x^{(3)}(t) = -w^6 x_0 \frac{t^6}{6!} - w^6 \dot{x}_0 \frac{t^7}{7!} - w^6 \left(L^{-1}\right)^4 F(t;q) \tag{25}$$

$$x^{(4)}(t) = w^8 x_0 \frac{t^8}{8!} + w^8 \dot{x}_0 \frac{t^9}{9!} + w^8 \left(L^{-1}\right)^5 F(t;q) \tag{26}$$

The nth term is:

$$x^{(n)}(t) = w^{2n} x_0 \frac{t^{2n}}{2n!} + w^{2n} \dot{x}_0 \frac{t^{2n+1}}{(2n+1)!} + w^{2n} \left(L^{-1}\right)^{n+1} F(t;q) \tag{27}$$

Thus,

$$\begin{aligned} x(t) &= x^{(0)} + x^{(1)} + x^{(2)} + \cdots \\ &= x_0 \left[1 - \frac{(wt)^2}{2!} + \frac{(wt)^4}{4!} + \cdots\right] + \frac{\dot{x}_0}{\omega}\left[(wt) - \frac{(wt)^3}{3!} + \frac{(wt)^5}{5!} + \cdots\right] \\ &\quad + \frac{1}{w}\left[wL^{-1} - w^3\left(L^{-1}\right)^2 + w^5\left(L^{-1}\right)^3 - w^7\left(L^{-1}\right)^4 + w^9\left(L^{-1}\right)^5 + \cdots\right] F(t;q) \\ &= x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{1}{\omega}\left[\omega L^{-1} - \omega^2\left(L^{-1}\right)^2\right] F(t;q) \end{aligned} \tag{28}$$

where,

$$\int_0^t \cdots \int_0^t F(u) \mathrm{d}u^n = \int_0^t \frac{(t-u)^{n-1}}{(n-1)!} F(u) \mathrm{d}u \tag{29}$$

$$L^{-1} F(t;q) = \int_0^t \int_0^t F(t;q) \mathrm{d}t^2 = \int_0^t (t-u) F(u;q) \mathrm{d}u \tag{30}$$

$$L^{-1} L^{-1} F(t;q) = \int_0^t \int_0^t \int_0^t \int_0^t F(t;q) \mathrm{d}t^4 = \int_0^t \frac{(t-u)^3}{3!} F(u;q) \mathrm{d}u \tag{31}$$

$$L^{-1} L^{-1} L^{-1} F(t;q) = \int_0^t \int_0^t \int_0^t \int_0^t \int_0^t \int_0^t F(t;q) \mathrm{d}t^6 = \int_0^t \frac{(t-u)^5}{5!} F(u;q) \mathrm{d}u \tag{32}$$

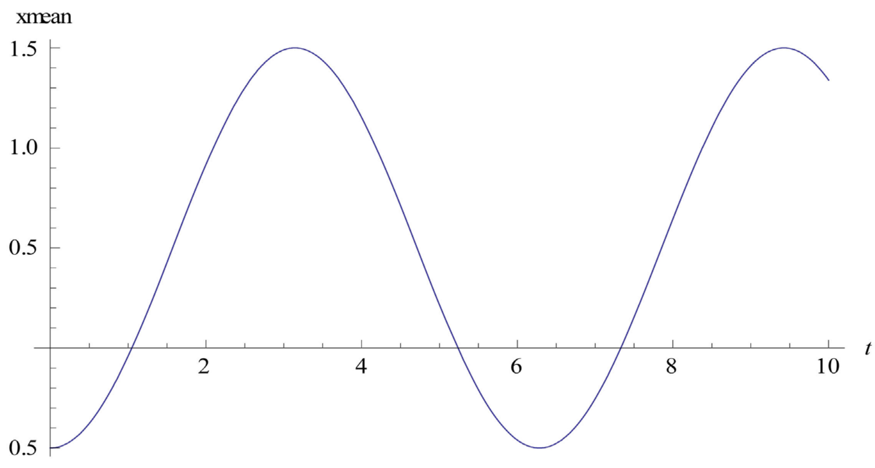

Figure 1. The mean of $x(t)$ at $\omega = 1$.

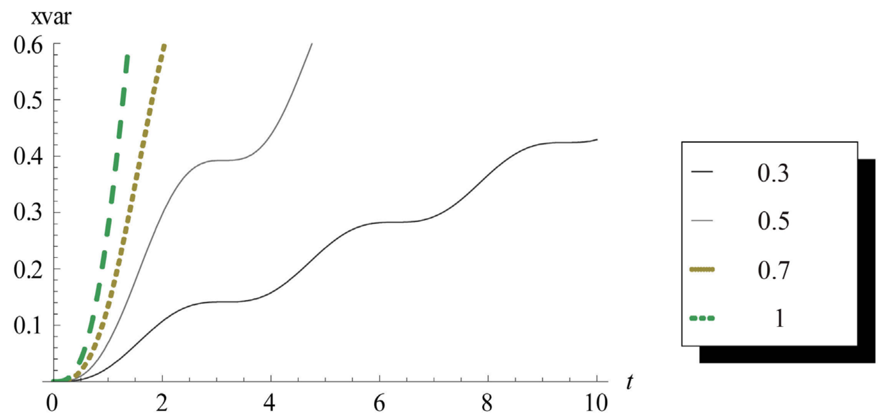

Figure 2. The variance of $x(t)$ at $\omega = 1$.

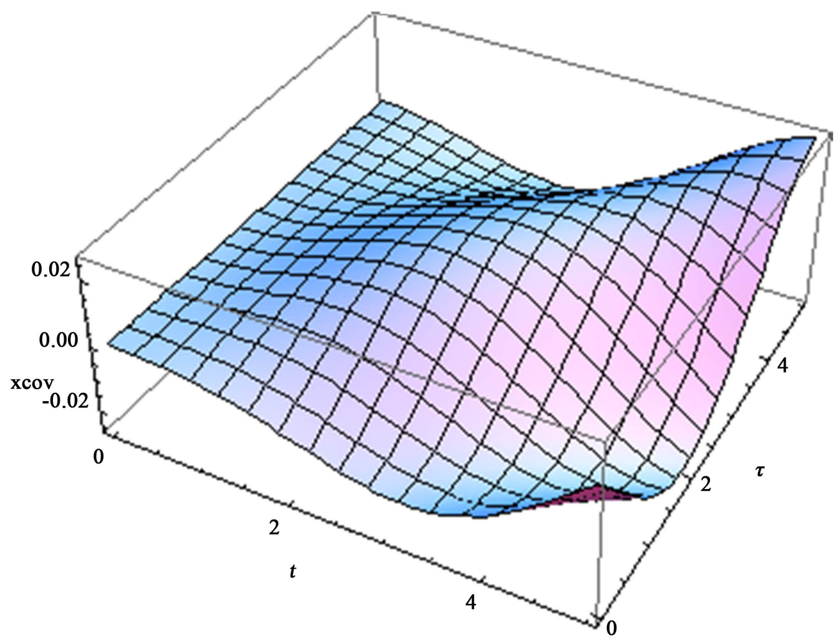

Figure 3. The covariance of $x(t)$ at $\varepsilon = 0.1, \omega = 1$.

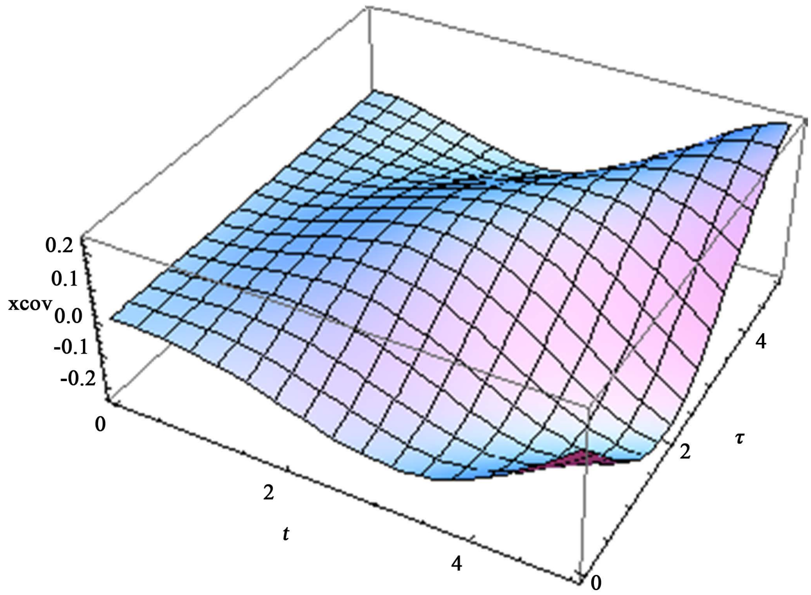

Figure 4. The covariance of $x(t)$ at $\varepsilon = 0.3, \omega = 1$.

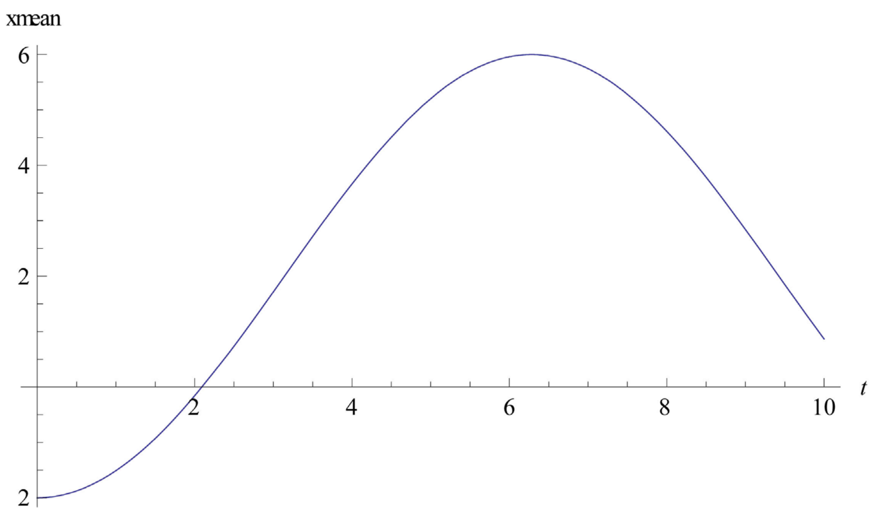

Figure 5. The mean of $x(t)$ at $\omega = 0.5$.

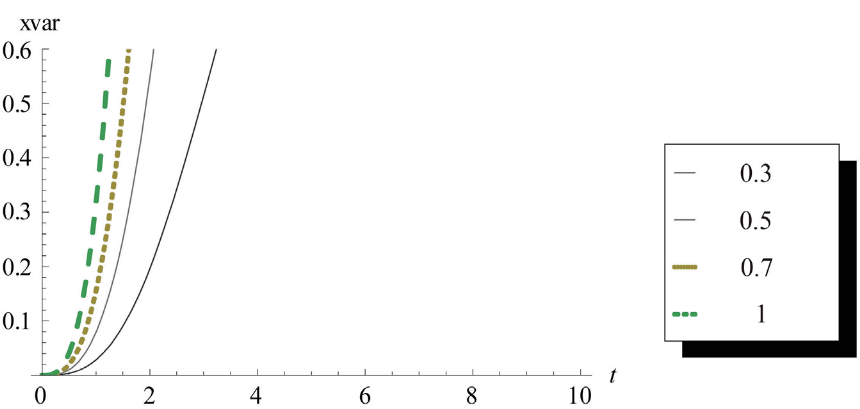

Figure 6. The variance of $x(t)$ at $\omega = 0.5$.

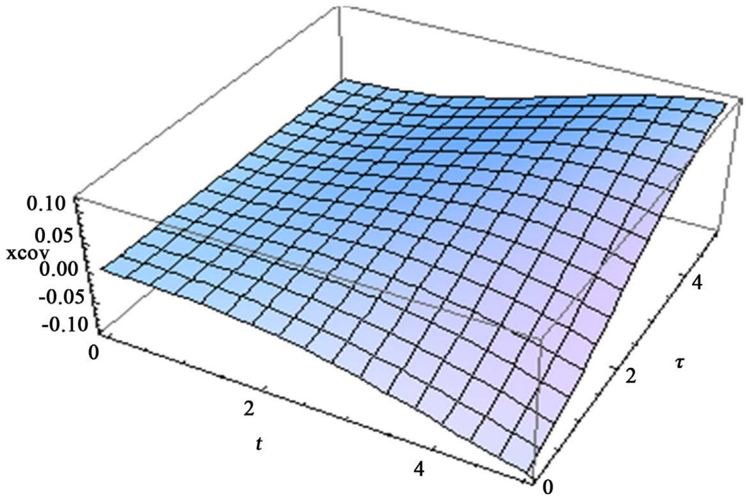

Figure 7. The covariance of $x(t)$ at $\varepsilon = 0.1, \omega = 0.5$.

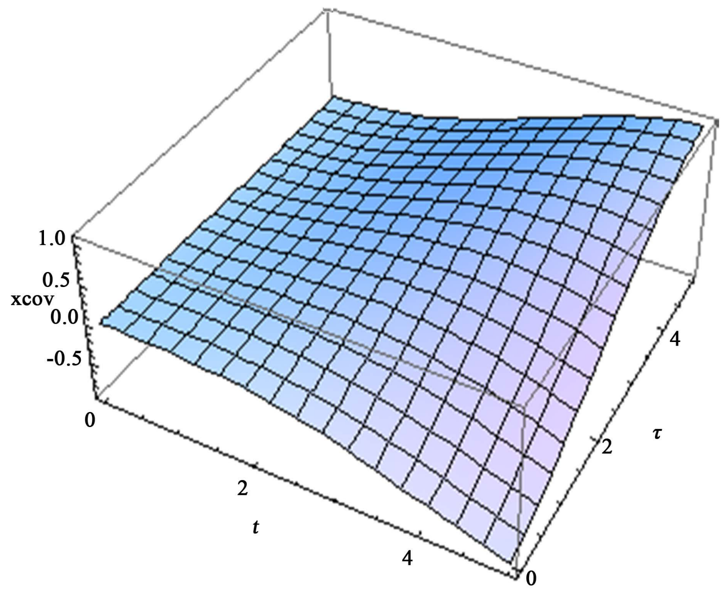

Figure 8. The covariance of $x(t)$ at $\varepsilon = 0.3, \omega = 0.5$.

$$\begin{aligned} x(t) &= x_0 \cos wt + \frac{\dot{x}_0}{\omega}\sin wt + \frac{1}{w}\left[w\int_0^t (t-u) F(u;q)\,\mathrm{d}u \right. \\ &\quad - w^3 \int_0^t \frac{(t-u)^3}{3!} F(u;q)\,\mathrm{d}u + w^5 \int_0^t \frac{(t-u)^5}{5!} F(u;q)\,\mathrm{d}u \\ &\quad \left. - w^7 \int_0^t \frac{(t-u)^7}{7!} F(u;q)\,\mathrm{d}u + \cdots \right] \\ &= x_0 \cos wt + \frac{\dot{x}_0}{\omega}\sin wt + \frac{1}{w}\int_0^t \left[w(t-u) - \frac{\left[w(t-u)\right]^3}{3!} + \cdots \right] F(u;q)\,\mathrm{d}u \\ &= x_0 \cos wt + \frac{\dot{x}_0}{w}\sin wt + \frac{1}{w}\int_0^t \sin w(t-u) F(u;q)\,\mathrm{d}u \end{aligned} \tag{33}$$

Example:
Let us consider

$$F(t;\omega)=e(t)\left[1+\varepsilon n(t;q)\right] \tag{34}$$

in the previous case-study. By using the decomposition method, the following results are obtained (**Figures 1-8**).

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