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The Additive Operator Preserving Birkhoff Orthogonal

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Abstract

The Birkhoff orthogonal plays an important role in the geometric study of Banach spaces. It has been confirmed that a Birkhoff orthogonality preserving linear operator between two normed linear spaces must necessarily be a scalar multiple of a linear isometry. In this paper, the author gives a new result that a Birkhoff orthogonality preserving additive operator between two-dimensional normed linear spaces is necessarily a scalar multiple of a linear isometry.

Keywords

Birkhoff Orthogonal, Additive Operator, Smooth Point

1. Introduction

There are several notions of orthogonality in normed linear space, such as isosceles orthogonality, Birkhoff orthogonality, Euclidean orthogonality and so on. It is obvious that all the above types of orthogonality are equivalent to the usual (Euclidean) orthogonality arising from the inner product of spaces. In fact, they are equivalent to each other if and only if the considered space is an inner product space; cf. [[1], [2], (3.1), (3.4)].

Recall that an element x in real normed linear space X is said to be a Birkhoff orthogonal to y in X, written as, $x \perp_B y$, if:

$$\| x \| \leq \| x + ky \|, \forall k \in \mathbb{R}.$$

The Birkhoff orthogonal plays an important role in the geometric study of Banach spaces. It holds great significance for various geometric properties of the norm, such as strict convexity and smoothness. One can find some details in [3–6].

We say that an operator T between two normed linear spaces X and Y preserves Birkhoff orthogonal(OP), if:

$$x \perp_B y \Longrightarrow T(x) \perp_B T(y).$$

Alexander proved that operators from a real Banach space into itself that preserve Birkhoff orthogonality are isometrically multiplied by a constant [7]. Recently, Chmieliński found that an orthogonality preserving linear map between two inner product spaces is necessarily a scalar multiple of a linear isometry [8]. This result was extended by [9] to normed linear spaces in the sense of Birkhoff orthogonality with a linear operator.

In this paper, we will show that a Birkhoff orthogonality preserving additive map between two-dimensional normed linear spaces is necessarily a scalar multiple of a linear isometry.

While finishing the editing of the paper, I was informed that Pawel Wójcik has also obtained the result in [10]. But the proof is essentially constructive, and some of the techniques may be useful when studying the property of the operator which preserves orthononality.

2. Preliminaries

At the beginning of this section, let us recall the notion of a linear operator in linear space. An operator is linear if and only if it is additive and homogeneous. We can find an example operator which is additive but not linear, even in the one-dimensional linear space $\mathbb{R}$. For example, some solutions of Cauchy's equation on $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is additive if it satisfies Cauchy's functional equation,

$$f(x+y) = f(x) + f(y), \tag{2.1}$$

for all $x, y \in \mathbb{R}$. By the axiom of choice, there are infinitely many nonlinear functions that satisfy the equation. This was proved in 1905 by Georg Hamel using Hamel bases. Such functions are sometimes called Hamel functions. There are some conditions under which the solution to Cauchy's equation is linear. The following results can be found in [11].

Theorem 2.1. *An additive function $f : \mathbb{R} \to \mathbb{R}$ is linear if one of the following conditions holds:*
$\{i\}$ *f is continuous.*
$\{ii\}$ *f is monotonic in any interval.*
$\{iii\}$ *f is bounded in any interval.*
$\{iv\}$ *f is Lebesgue measurable.*

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function and $f([0, +\infty]) \subset [0, +\infty]$, then f is a linear.*

Proof. Since f is additive, there exists $\alpha > 0$ such that $f(a) = \alpha a$, for any $a \in [0, +\infty] \cap \mathbb{Q}$. Without loss of generality, we assume that $\alpha = 1$, then $f(a) = a$, for any $a \in [0, +\infty] \cap \mathbb{Q}$. We claim that $f(a) = a$ for all $a \in [0, +\infty]$. If not, there exists $b \in [0, +\infty] \setminus \mathbb{Q}$ such that $f(b) \neq b$; without loss of generality, we assume that $f(b) < b$. While there exists $c \in [0, +\infty] \cap \mathbb{Q}$ such that $b - c > 0$ and $f(c) - f(b) > 0$, since $\mathbb{Q}$ is dense in $\mathbb{R}$. So $f(c - b) > 0$, which contradicts to $c - b < 0$. Then, $f(a) = a$ for all $a \in [0, +\infty]$. Since f is additive and $f(a) = a$ for all $a \in [0, +\infty]$, it follows that f is linear. □

At the end of this section, we give an example to show that there exists a Birkhoff orthogonal preserving operator that is homogeneous, but is nonlinear.

Example 2.3. *Let* $e_1 = (1, 0) \in \mathbb{R}^2$*, and let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined as*

$$T(x) = (|< \frac{x}{\|x\|}, e_1 > | + 1)x, \forall x \in X. \tag{2.2}$$

where $< \cdot, \cdot >$ *is the usual inner product of* $\mathbb{R}^2$*. Obviously,* $T(kx) = kT(x)$ *for all* $k \in \mathbb{R}$*, and* T *preserves the Birkhoff orthogonal, but it is not additive.*

3. Main Result

Throughout this paper, the unit sphere of normed space X will be denoted as $S(X)$, while the smooth points in $S(X)$ are denoted as $sm(X)$. For a set $A \subset X$, the notation $x \perp_B A$ means $x \perp_B y$ for any $y \in A$. Let $D(x) = \{x^* \in S(X^*) | x^*(x) = \|x\|\}$, then x is a smooth point if and only if $D(x)$ contains only one point. Next, $in(A)$ is a relative interior point of A. For $x, y \in X$, let $[x, y] = \{\lambda x + (1-\lambda)y | 0 \leq \lambda \leq 1\}$, $span\{x\} = \{z | z \in \alpha x, \alpha \in \mathbb{R}\}$.

Definition 3.1. *Let* X *be a real normed space. For any* $x, y \in S(X)$*,* $x \neq -y$*, we define the arc of* x *and* y *to be the set*

$$A(x, y) = \{z \mid z = \frac{\lambda x + (1-\lambda)y}{\|\lambda x + (1-\lambda)y\|}, \quad \lambda \in [0, 1]\}.$$

The arc $A(x, y)$ *is non-trivial if* $x \neq y$*.*

Lemma 3.2. *Let* X *be a two-dimensional normed space, then* $x \in S(X)$ *is not a smooth point if and only if there exists a non-trivial arc* $A(y_1, y_2)$ *such that* $x \perp_B A(y_1, y_2)$*.*

Proof. Since $x \in S(X)$ is not a smooth point, there exist $x_1^*, x_2^* \in D(x)$, $y_1 \in Ker(x_1^*) \cap S(X)$ and $y_2 \in Ker(x_2^*) \cap S(X)$. Without loss of generality, we assume that $t = \frac{x_2^*(y_1)}{x_1^*(y_2)} < 0$. For any $0 < \lambda < 1$, let $\eta = \frac{\lambda t}{\lambda + \lambda t - 1}$, then $0 < \eta < 1$, $\eta x_1^* + (1-\eta)x_2^* \in D(x)$ and

$$(\eta x_1^* + (1-\eta)x_2^*)(\frac{\lambda y_1 + (1-\lambda)y_2}{\|\lambda y_1 + (1-\lambda)y_2\|}) = 0$$

So $x \perp_B A(y_1, y_2)$. □

It is obvious that for any $x \in S(X)$, the set $\{y | x \perp_B y, y \in S(X)\}$ is closed. Then, x must be a Birkhoff orthogonal to an unique maximal arc, denoted as $A(x)$.

The following two lemmas are obvious from the definition of the smooth point, so we have omitted the proof.

Lemma 3.3. *Let* X *be a two-dimensional normed linear space,* $x, y \in sm(X)$*. If there exists a non-zero element* $z \in X$*, such that* $x \perp_B z$ *and* $y \perp_B z$*, then* $[x, y] \subset S(X)$ *or* $[-x, y] \subset S(X)$*.*

Lemma 3.4. *Let* X *be a normed linear space. Then, let* x *be a smooth point in* X*. Then there exists an unique number* a *such that* $x \perp_B ax + y$*.*

Lemma 3.5. *Let* X *be a two-dimensional normed linear space. Then, let* $x \perp_B y$ *and* x *be a convex point in* $S(X)$*. If* $x' \in X$ *and* $x' \perp_B y$*, then there exists a number* $\mu \in \mathbb{R}$ *such that* $x' = \mu x$*.*

Proof. If $x' \in S(X)$ and $x' \perp_B y$, then we have $[x, x'] \subset S(X)$ or $[-x, x'] \subset S(X)$ from Lemma 3.3. Since x is a convex point in $S(X)$, then $x = x'$ or $x = -x'$. Similarly, if $x' \in X$ and $x' \perp_B y$, we can easily find there must exist a number $\mu \in \mathbb{R}$ such that $x' = \mu x$. □

Lemma 3.6. *Let X be a two-dimensional normed linear space, $x, y \notin sm(X)$. If $in(A(x)) \cap A(y) \neq \emptyset$, then x and y must be linearly dependent.*

Proof. Since $x, y \notin sm(X)$ and $in(A(x)) \cap in(A(y)) \neq \emptyset$, there exists $z_1, z_2 \in A(x) \cap A(y)$. Because $dim(X) = 2$, by the monotonic subdifferential of a convex function there exists $x_1^*, x_2^* \in D(x) \cap D(y)$ such that $x_1^*(z_1) = x_2^*(z_2) = 0$. Without loss of generality, let us assume that $\| y \| = \| x \|$. Hence, we obtain that

$$x_1^*((1-\lambda)x + \lambda y) = 1, x_2^*((1-\mu)x + \mu y) = 1, \forall \lambda, \mu \in [0, 1].$$

Let $x_0 = \frac{1}{2}x + \frac{1}{2}y$, then we have $x_0 \in sm(X)$, which contradicts to $x_1^*, x_2^* \in D(x_0)$ and $x_1^* \neq x_2^*$. □

Lemma 3.7. *Let X and Y both be two-dimensional normed linear spaces. Then, we can suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then for any $x \in X$, there exists a function $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ such that $T(\alpha x) = f_x(\alpha)T(x)$, for any $\alpha \in \mathbb{R}$. Moreover, f_x is additive, and $f_x(a) = a$ for any $a \in \mathbb{Q}$.*

Proof. We only need to prove that $T(\alpha x) \in span\{T(x)\}$, for any $\alpha \in \mathbb{R}$. If not, there exists $\alpha_0 \in \mathbb{R}$ such that $T(\alpha_0 x) \notin span\{T(x)\}$. Let $y \in X$ such that $x \perp_B y$.

Since $T(y) \in span\{T(x), T(\alpha_0 x)\}$, there exists $p, q \in \mathbb{Q}$ such that

$$\| pT(x) + qT(\alpha_0 x) - T(y) \| < \frac{\| T(y) \|}{2} \tag{3.1}$$

Then

$$\begin{aligned} \| T(px + q\alpha_0 x) \| &= \| pT(x) + qT(\alpha_0 x) \| \\ &\geq \| T(y) \| - \| pT(x) + qT(\alpha_0 x) - T(y) \| \\ &> \frac{\| T(y) \|}{2} \\ &> \| T(px + q\alpha_0 x) - T(y) \| \end{aligned}$$

which contradicts to $(px + q\alpha_0 x) \perp_B y$ and T preserves the Birkhoff orthogonal.

Moreover, since T is additive, for any $a_1, a_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(a_1 x) + T(a_2 x) &= f_x(a_1)T(x) + f_x(a_2)T(x) \\ &= T((a_1 + a_2)x) = f_x(a_1 + a_2)T(x) \end{aligned} \tag{3.2}$$

Thus f_x is additive. It is obvious that $f_x(1) = 1$, since f_x is additive, it follows that $f_x(a) = a$, for any $a \in \mathbb{Q}$. □

Lemma 3.8. *Let X and Y be both two-dimensional normed linear spaces, and then suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then for any $x \in X$, x is a smooth point in X if and only if $T(x)$ is a smooth point in Y.*

Proof. Without loss of generality, we may assume that $x \in S(X)$ and $\|T(x)\| = \alpha$.

If x is not smooth point of X, from Lemma 3.2, there exists $z_1, z_2 \in X$ such that $x \perp_B z_1$, $x \perp_B z_2$ and z_1, z_2 are linearly independent. Since T preserves Birkhoff orthogonal, and from Lemma 3.6 we have $T(x) \perp_B T(z_1)$; hence $T(x) \perp_B T(z_2)$ and $T(z_1), T(z_2)$ are linearly independent. From Lemma 3.2, we obtain that $T(x)$ is not a smooth point in Y.

If x is a smooth point in X, and $T(x)$ is not smooth point in Y, there exists a non-trivial arc $A(y_1, y_2)$ such that $T(x) \perp_B A(y_1, y_2)$ via Lemma 3.2. We can choose $z \in X$ such that $x \perp_B z$ and $T(z) \in A(y_1, y_2)$. Then we can find $x_1 \notin span\{x\}$ and $z_1 \in X$ such that $x_1 \perp_B z_1$ and $T(z_1) \in in(A(y_1, y_2))$. We claim that x_1 must be a smooth point in X, indeed, if not, then $T(x_1)$ is not a s-mooth point in Y by the above proof. Assume $T(x_1) \perp_B A(y_1', y_2')$, since $in(A(y_1, y_2)) \cap A(y_1', y_2') \neq \emptyset$. From Lemma 3.7, we obtain that $T(x_1) \in span\{T(x)\}$ which contradicts to $x_1 \notin span\{x\}$. Similarly, we can find $x_2, x_3, z_2, z_3 \in X$, such that x, x_1, x_2, x_3 are multiply linear independent, $x_i \perp_B z_i$ and $T(z_i) \in in(A(y_1, y_2))$, for $i = 1, 2, 3$. Since $T(x) \perp_B T(z_1)$ and $T(x_1) \perp_B T(z_1)$, from Lemma 3.3, we obtain that $[\alpha \frac{T(x_1)}{\|T(x_1\|}, T(x)] \subset S_\alpha(Y)$. Similarly, we obtain that $[\alpha \frac{T(x_2)}{\|T(x_2\|}, T(x)] \subset S_\alpha(Y)$ and $[\alpha \frac{T(x_3)}{\|T(x_3\|}, T(x)] \subset S_\alpha(Y)$. Since $dim(Y) = 2$, it follows that there exists $i, j \in \{1, 2, 3\}$ such that

$$\alpha \frac{T(x_i)}{\|T(x_i\|} \in [\alpha \frac{T(x_j)}{\|T(x_j\|}, T(x)]. \tag{3.3}$$

So $T(x_i) \perp_B T(z_i)$ and $T(x_i) \perp_B T(z_j)$, and from Lemma 3.7, we obtain that $T(x_i) \in span\{T(x)\}$, which contradicts to $x_i \notin span\{x\}$. □

Lemma 3.9. *(cf. [9]) A Birkhoff orthogonality preserving linear map between two normed linear spaces is necessarily a scalar multiple of a linear isometry, where map T is said to be Birkhoff orthogonality preserving if the property that x is orthogonal to y, in the sense of Birkhoff-James, implies that $T(x)$ is orthogonal to $T(y)$, in the sense of Birkhoff-James.*

Lemma 3.10. *Let X and Y be both two-dimensional normed linear spaces, and X is smooth. Suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then we have $f_x = f_y$, for any $x, y \in X$, where f_x and f_y as Lemma 3.7.*

Proof. For any $x, y \in X$ and $\alpha \in \mathbb{R}$, let $z \in X$ such that $z \perp_B x + y$. So $z \perp_B \alpha(x + y)$. Since T is additive and preserves Birkhoff orthogonality, we obtain that

$$T(z) \perp_B T(x) + T(y). \tag{3.4}$$

And

$$T(z) \perp_B f_x(\alpha)T(x) + f_y(\alpha)T(y). \tag{3.5}$$

Since X is smooth, and from Lemma 3.8 we obtain that $T(z)$ is smooth and

$$f_x(\alpha)T(x) + f_y(\alpha)T(y) = T(\alpha(x + y)) \in span\{T(x) + T(y)\}, \tag{3.6}$$

there thus exists an $\beta \in \mathbb{R}$ such that

$$f_x(\alpha)T(x) + f_y(\alpha)T(y) = \beta((T(x) + T(y)). \tag{3.7}$$

However, $T(x) \notin spanT(y)$, so $f_x(\alpha) = f_y(\alpha)$. □

Theorem 3.11. *Let X and Y be both two-dimensional normed linear spaces, and X is smooth. Suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then T is linear.*

Proof. For any $x \in X$ and $a \in \mathbb{R}$, there exists $z \in X$ such that $x \perp_B ax + z$. So we have $\lambda x \perp_B \lambda ax + \lambda z$ for all $\lambda \neq 0$. Since T preserves Birkhoff orthogonality, and from Lemma 3.10, we obtain that

$$f(\lambda)T(x) \perp_B f(\lambda a)T(x) + f(\lambda)T(z). \tag{3.8}$$

That is

$$T(x) \perp_B \frac{f(\lambda a)}{f(\lambda)}T(x) + T(z). \tag{3.9}$$

From Lemma 3.7, $T(x), T(z)$ are both smooth points. Since $T(x) \perp_B f(a)T(x) + T(z)$, we get $f(\lambda a) = f(\lambda)f(a)$, which means f is an identity map. Thus, T is linear. □

Theorem 3.12. *Let X and Y be both two-dimensional normed linear spaces, and X is not smooth. Suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then T is linear.*

Proof. Suppose x is not a smooth point in X. Then for any $z \in X$, there exists $[A, B] \subset \mathbb{R}(A \neq B)$ such that $x \perp_B ax + z$ for any $a \in [A, B]$. From Lemma 3.7, $T(x)$ is not a smooth point in Y. Thus, there exists a set $[A', B'] \subset \mathbb{R}(A' \neq B')$ such that $T(x) \perp_B bT(x) + T(z)$ for any $b \in [A', B']$, which means $f_x(a) \in [A', B']$ for any $a \in [A, B]$. From Theorem 2.1 and Theorem 2.2, we obtain that f_x is an identity map.

Next, we will discuss two cases where x is a smooth point in X. In case one, if there exists $z \notin sm(X)$, such that $x \perp_B ax + z$, where $a \in \mathbb{R}$ and $a \neq 0$, then $\lambda x \perp_B \lambda ax + \lambda z$ for all $\lambda \neq 0$ and

$$f_x(\lambda)T(x) \perp_B f_x(\lambda a)T(x) + f_z(\lambda)T(z). \tag{3.10}$$

That is

$$T(x) \perp_B \frac{f_x(\lambda a)}{f_x(\lambda)}T(x) + \frac{\lambda}{f_x(\lambda)}T(z). \tag{3.11}$$

We also have $T(x) \perp_B f_x(a)T(x) + T(z)$, so we obtain $f_x(\lambda a) = \lambda f_x(a)$. Note that $f_x(\alpha) = \alpha$ for all $\alpha \in \mathbb{Q}$. Thus, we obtain that f_x is an identity map.

In case two, if $x \perp_B z$ for any $z \notin sm(X)$. Since x is a smooth point in X, it follows that $y \in sm(X)$ for any $y \notin span\{z\}$. So we can choose $y_1, y_2 \in sm(X)$ such that $y_1 = x + kz$ and $y_1 \perp_B y_2$, where $k \neq 0$. Then we obtain that

$$T(y_1) \perp_B T(x) + kT(z). \tag{3.12}$$

Since $y_1 \perp_B \lambda y_2$ for any $\lambda \in \mathbb{R}$, then we obtain that

$$T(y_1) \perp_B f_x(\lambda)T(x) + k\lambda T(z). \tag{3.13}$$

Thus, we get $f_x(\lambda) = \lambda$, which means f_x is an identity map.

Since f_x is an identity map for all $x \in X$, it is evident that T is linear.

□

According to Lemma 3.9, Theorem 3.11 and Theorem 3.12, we obtain the main results.

Corollary 3.13. *Let X and Y be both two-dimensional normed linear spaces. Suppose that $T : X \longrightarrow Y$ is an additive operator with OP. Then T is a scalar multiple of a linear isometry.*

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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