Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2018, 6, 2475-2497
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Uniform Attractors for a Non-Autonomous Thermoviscoelastic Equation with Strong Damping

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How to cite this paper: Ma, Z.Y. (2018) Uniform Attractors for a Non-Autonomous Thermoviscoelastic Equation with Strong Damping. *Journal of Applied Mathematics and Physics*, **6**, 2475-2497.
https://doi.org/10.4236/jamp.2018.612209

Received: October 26, 2018
Accepted: December 8, 2018
Published: December 11, 2018

Abstract

This paper considers the existence of uniform attractors for a non-autonomous thermoviscoelastic equation with strong damping in a bounded domain $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ by establishing the uniformly asymptotic compactness of the semi-process generated by the global solutions.

Keywords

Thermoviscoelastic Equation, Uniform Attractors, Strong Damping

1. Introduction

In this paper we investigate the existence of uniform attractors for a nonlinear non-autonomous thermoviscoelastic equation with strong damping

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} g(s)\Delta u(t-s)\mathrm{d}s - \Delta u_t + \nabla\theta = \sigma(x,t),\ x \in \Omega, t > \tau, \quad (1.1)$$

$$\theta_t - \Delta\theta + \mathrm{div} u_t = f(x,t),\ x \in \Omega, t > \tau, \quad (1.2)$$

$$\theta(x,t) = u(x,t) = 0, \quad \text{on } \partial\Omega \times [\tau, +\infty), \quad (1.3)$$

$$u(x,\tau) = u_0^\tau(x), u_t(x,\tau) = u_1^\tau(x), u(x,t) = u_\tau(x,t), \theta(x,\tau) = \theta_0^t(x), x \in \Omega, \quad (1.4)$$

where $\Omega \subseteq \mathbb{R}^n (n = 1,2)$ is a bounded domain with smooth boundary $\partial\Omega$, u and θ are displacement and temperature difference, respectively. $u_\tau(x,t)$ (the past history of u) is a given datum which has to be known for all $t \leq \tau$, the function g represents the kernel of a memory, $\sigma = \sigma(x,t), f = f(x,t)$ are non-autonomous terms, called symbols, and ρ is a real number such that

$$1 < \rho \leq \frac{2}{n-2} \quad \text{if } n \geq 3; \quad \rho > 1 \quad \text{if } n = 1,2. \quad (1.5)$$

Now let us recall the related results on nonlinear one-dimensional thermoviscoelasticity. Dafermos [1], Dafermos and Hsiao [2], proved the global existence of a classical solution to the thermoviscoelastic equations for a class of solid-like materials with the stress-free boundary conditions at one end of the rod. Hsiao and Jian [3], Hsiao and Luo [4] obtained the large-time behavior of smooth solutions only for a special class of solid-like materials. Ducomet [5] proved the asymptotic behavior for a non-monotone fluid in one-dimension: the positive temperature case. Watson [6] investigated the unique global solvability of classical solutions to a one-dimensional nonlinear thermoviscoelastic system with the boundary conditions of pinned endpoints held at the constant temperature and where the pressure is not monotone with respect to u and may be of polynomial growth. Racke and Zheng [7] proved the global existence and asymptotic behavior of weak solutions to a model in shape memory alloys with a stress-free boundary conditions at least at one end of the rod. Qin [8] [9] obtained the global existence, and asymptotic behavior of smooth solutions under more general constitutive assumptions, and more recently. Qin [10] has further improved these results and established the global existence, exponential stability and the existence of maximal attractors in $H^i\left(i=1,2,4\right)$. As for the existence of global (maximal) attractors, we refer to [11] [12] [13]. More recently, Qin and Lü [12] obtained the existence of (uniformly compact) global attractors for the models of viscoelasticity; Qin, Liu and Song [13] established the existence of global attractors for a nonlinear thermoviscoelastic system in shape memory alloys.

Our problem is derived from the form

$$f\left(u_t\right)u_{tt}-\Delta u-\Delta u_{tt}=0, \tag{1.6}$$

which has several modeling features. The aim of this paper is to extend the decay results in [14] for a viscoelastic system to those for the thermoviscoelastic system (1.1-1.2) and then to establish the existence of the uniform attractor for this thermoviscoelastic systems. In the case $f\left(u_t\right)$ is a constant, Equation (1.6) has been used to model extensional vibrations of thin rods (see Love [15], Chapter 20). In the case $f\left(u_t\right)$ is not a constant, Equation (1.6) can model materials whose density depends on the velocity u_t. For instance, a thin rod which possesses a rigid surface and with an interior which can deforms slightly. We refer the reader to Fabrizio and Morro [16] for several other related models.

Let us recall some results concerning viscoelastic wave equations. In [17], the author concerned with the quasilinear viscoelastic equation

$$u_{tt}-\Delta u+\int_0^t g\left(t-\tau\right)\Delta u=\left|u\right|^{\gamma+1}, \tag{1.7}$$

he proved that the energy decays similarly with that of *g*. In [18], Wu considered the nonlinear viscoleastic wave equation

$$\left|u_t\right|^{\rho}u_{tt}-\Delta u-\Delta u_{tt}+g*\Delta u+\left|u\right|^{p}u=0 \tag{1.8}$$

with the same boundary and initial conditions as (1.7), the author proved that, for a class of kernels g which is singular at zero, the exponential decay rate of the solution energy. Later, Han and Wang [19] considered a similar system like:

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u + |u_t|^{m} u_t = 0, \tag{1.9}$$

with Dirichlet boundary condition, where $\rho > 0, m > 0$ are constants, they proved the energy decay for the viscoelastic equation with nonlinear damping. Then Park and Park [20] established the general decay for the viscoelastic problem with nonlinear weak damping

$$|u_t|^{\rho} u_{tt} - \Delta u_{tt} - \Delta u + g * \Delta u + h(u_t) = 0, \tag{10}$$

with the Dirichlet boundary condition, where $\rho > 0$ is a constant. In [14], Cavalcanti *et al.* studied the following equation with Dirichlet boundary conditions

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u - \gamma \Delta u_t = 0 \tag{1.11}$$

where $g * \Delta u = \int_0^t g(t-s)\Delta u(s)\mathrm{d}s$. They established a global existence result for $\gamma \geq 0$ and an exponential decay of energy for $\gamma > 0$, and studied the interaction within the $|u_t|^{\rho} u_{tt}$ and the memory term $g * \Delta u$. Messaoudi and Tatar [21] established, for small initial data, the global existence and uniform stability of solutions to the equation

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u = b|u|^{p-2} u, \tag{1.12}$$

with Dirichlet boundary condition, where $\gamma \geq 0, \rho, b > 0, p > 2$ are constants. In the case $b = 0$ in (1.12), Messaoudi and Tatar [22] proved the exponential decay of global solutions to (1.12) without smallness of initial data, considering only the dissipation effect given by the memory. Considering nonlinear dissipation. Recently, Araújo *et al.* [23] studied the following equation

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} \mu(s)\Delta u(t-s)\mathrm{d}s + f(u) = h(x),$$

and proved the global existence, uniqueness and exponential stability, and the global attractor was also established, but they did not establish the uniform attractors for non-autonomous equation. Then, Qin *et al.* [24] established the existence of uniform attractors for a non-autonomous viscoelastic equation with a past history

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} g(s)\Delta u(t-s)\mathrm{d}s + u_t = \sigma(x,t),\ x \in \Omega,\ t > \tau,$$

Moreover, we would like to mention some results in [25] [26] [27] [28] [29].

For problem (1.1)-(1.4) with $\sigma(x,t) = 0$, when $\int_0^{+\infty} g(s)\Delta u(t-s)\mathrm{d}s$ was replaced by $g * \Delta u$, Han and Wang [30] established the global existence of weak solutions and the uniform decay estimates for the energy by using the Faedo-Galerkin method and the perturbed energy method, respectively. To the best of our knowledge, there is no result on the existence of uniform attractors for non-autonomous thermoviscoelastic problem (1.1)-(1.4). Therefore in this paper, we shall establish the existence of uniform attractors for problem (1.1)-(1.4) by establishing uniformly asymptotic compactness of the semi-process generated by their global solutions. Noting that the symbol $\sigma(x,t), f(x,t)$, which are dependent in t, so our estimates are more complicated than [23] [24] and we must use new methods to deal with the symbol $\sigma(x,t), f(x,t)$ as the change of time. Therefore we improved the results in [23] [24]. For more results concerning attractors, we can refer to [31]-[37].

Motivated by [38] [39] [40], we shall add a new variable $\eta = \eta^t(x,s)$ to the system which corresponds to the relative displacement history. Let us define

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s),\ t \ge \tau, (x,s) \in \Omega \times \mathbb{R}^+. \tag{1.13}$$

A direct computation yields

$$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t),\ t \ge \tau, (x,s) \in \Omega \times \mathbb{R}^+, \tag{1.14}$$

and we can take as initial condition ($t = \tau$)

$$\eta^\tau(x,s) = u_0^\tau(x) - u_0^\tau(x, \tau - s), (x,s) \in \Omega \times \mathbb{R}^+. \tag{1.15}$$

Thus, the original memory term can be written as

$$\int_0^{+\infty} g(s) \Delta u(t-s) \mathrm{d}s = \int_0^{+\infty} g(s) \mathrm{d}s \cdot \Delta u - \int_0^{+\infty} g(s) \Delta \eta^t(s) \mathrm{d}s, \tag{1.16}$$

and we get a new system

$$|u_t|^\rho u_{tt} - \left(1 - \int_0^{+\infty} g(s) \mathrm{d}s\right) \Delta u - \Delta u_{tt} - \int_0^{+\infty} g(s) \Delta \eta^t(s) \mathrm{d}s - \Delta u_t + \nabla \theta = \sigma(x,t), \tag{1.17}$$

$$\theta_t - \Delta \theta + \mathrm{div} u_t = f(x,t) \tag{1.18}$$

$$\eta_t^t + \eta_s^t = u_t, \tag{1.19}$$

with the boundary conditions

$$u = 0 \ \text{ on } \partial\Omega \times \mathbb{R}^+, \quad \eta^t = 0 \ \text{ on } \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.20}$$

and initial conditions

$$u(x,\tau) = u_0^\tau(x), u_t(x,\tau) = u_1^\tau(x), \eta^t(x,0) = 0, \eta^\tau(x,s) = u_0^\tau(x) - u(x, \tau - s). \tag{1.21}$$

The rest of our paper is organized as follows. In Section 2, we give some preparations for our consideration and our main result. The statements and the proofs of our main results will be given in Section 3 and Section 4, respectively.

For convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. C_1 denotes a general positive constant, which may be different in different estimates.

2. Preliminaries and Main Result

We assume the memory kernel $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function such that

$$g(s) < +\infty, l = 1 - \int_0^{+\infty} g(s) \mathrm{d}s > 0 \tag{2.1}$$

and suppose that there exists a positive constant ξ_2 verifying

$$g'(t) \le -\xi_2 g(t), \quad \forall\, t \ge 0, \tag{2.2}$$

In order to consider the relative displacement η as a new variable, one introduces the weighted L^2-space

$$\mathcal{M}=L_g^2\left(\mathbb{R}^+;H_0^1(\Omega)\right)=\left\{u:\mathbb{R}^+\to H_0^1(\Omega)\mid \int_0^{+\infty}g(s)\|\nabla u(s)\|^2\,\mathrm{d}s<+\infty\right\},$$

which is a Hilbert space equipped with inner product and norm

$$(u,v)_{\mathcal{M}}=\int_0^{+\infty}g(s)\left(\int_\Omega \nabla u(s)\nabla v(s)\,\mathrm{d}x\right)\mathrm{d}s \text{ and } \|u\|_{\mathcal{M}}^2=\int_0^{+\infty}g(s)\|\nabla u(s)\|^2\,\mathrm{d}s,$$

respectively.

Let

$$\mathscr{H}=H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{M}. \tag{2.3}$$

Define the generalized energy of problem (1.17)-(1.21)

$$F(t)=\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{l}{2}\|\nabla u(t)\|^2+\frac{1}{2}\|\nabla u_t(t)\|^2+\frac{1}{2}\|\theta\|^2+\frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2. \tag{2.4}$$

To present our main result, we need the following global existence and uniqueness results.

Theorem 2.1. Let $\left(u_0^\tau,u_1^\tau,\theta_0^\tau,\eta^\tau\right)\in\mathscr{H}\left(\forall\tau\in\mathbb{R}^+\right)$, $\mathbb{R}_\tau=[\tau,+\infty)$, and any fixed $\sigma,f\in E_1$. Assume (2.1) and (2.2) hold. Then problem (1.17)-(1.21) admits a unique global solution $\left(u,u_t,\theta,\eta^t\right)\in C\left([0,T],\mathscr{H}\right)$ such that

$$u\in L^\infty\left(\mathbb{R}_\tau,H_0^1(\Omega)\right),u_t\in L^\infty\left(\mathbb{R}_\tau,H_0^1(\Omega)\right),u_{tt}\in L^2\left(\mathbb{R}_\tau,H_0^1(\Omega)\right), \tag{2.5}$$

$$\theta\in L^\infty\left(\mathbb{R}_\tau,H_0^1(\Omega)\right),\eta^t\in L^\infty\left(\mathbb{R}_\tau,\mathcal{M}\right). \tag{2.6}$$

We now define the symbol space for (1.17)-(1.21).

Let

$$G=(\sigma,f,0)\in E_1\equiv L^2\left(\mathbb{R}^+,\left(L^2(\Omega)\right)^3\right). \tag{2.7}$$

Observe the following important fact: The properly defined (uniform) attractor A of problem (1.17)-(1.21) with the symbol G_0 must be simultaneously the attractor of each problem (1.17)-(1.21) with the symbol $G(t)\in H_+(G_0)$, which is called the hull of G_0 and defined as

$$\Sigma=H_+(G_0)=\left[G_0(t+h)\mid h\in\mathbb{R}^+\right]_{E_1} \tag{2.8}$$

where $[\cdot]_{E_1}$ denotes the closure in Banach space E_1.

We note that

$$G_0\in E_1\subseteq\hat{E}_1=L_{loc}^2\left(\mathbb{R}^+,\left(L^2(\Omega)\right)^3\right).$$

where G_0 is a translation compact function in $\hat{E}_1$ in the weak topology, which means that G_0 is compact in $\hat{E}_1$. We consider the Banach space $L_{loc}^p(\mathbb{R}^+,E_1)$ of functions $\mu(s),s\in\mathbb{R}^+$ with values in a Banach space E_1 that are locally p-power integrable in the Bochner sense. In particular, for any time interval $[t_1,t_2]\subseteq\mathbb{R}^+$,

$$\int_{t_1}^{t_2}\|\mu(s)\|_{E_1}^p\,\mathrm{d}s<+\infty.$$

Let $\mu(s)\in L_{loc}^p\left(\mathbb{R}^+,E_1\right)$, consider the quantity

$$\eta_\mu(h)=\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\mu(s)\|_{E_1}^p\,\mathrm{d}s.$$

Lemma 2.1. Let Σ defined as before and $G_0\in E_1$, then

1) G_0 is a translation compact in $\hat{E}_1$ and any $G\in\Sigma=H_+(G_0)$ is also a translation compact in $\hat{E}_1$, moreover, $H_+(G)\subseteq H_+(G_0)$;

2) The set $H_+(G_0)$ is bounded in $L^2(\mathbb{R}^+,L^2(\Omega))$ such that

$$\eta_G(h)\le\eta_{G_0}(h)<+\infty,\quad \text{for all } G\in\Sigma.$$

Proof. See, e.g., Chepyzhov and Vishik [41].

Lemma 2.2. For every $\tau\in\mathbb{R}$, every non-negative locally summable function ϕ_0 on $R_\tau\equiv[\tau,+\infty)$ and every $\nu>0$, we have

$$\sup_{t\ge\tau}\int_\tau^t\phi_0(s)\mathrm{e}^{-\nu(t-s)}\mathrm{d}s\le\frac{1}{1-\mathrm{e}^{-\nu}}\sup_{t\ge\tau}\int_t^{t+1}\phi_0(s)\mathrm{d}s$$

for a.a. $t\ge\tau$.

Proof. See, e.g., Chepyzhov, Pata and Vishik [42].

Similar to Theorem 2.1, we have the following existence and uniqueness result.

Theorem 2.2. Let $\Sigma=H_+(G_0)=\left[G_0(t+h)\,|\,h\in\mathbb{R}^+\right]_{E_1}$, where $G_0\in E_1$ is an arbitrary but fixed symbol function. Assume (2.1) and (2.2) hold. Then for any $G\in\Sigma$ and for any $(u_0^\tau,u_1^\tau,\theta_0^\tau,\eta^\tau)\in\mathscr{H}\ (\forall\tau\in\mathbb{R}^+)$, problem (1.17)-(1.21) admits a unique global solution $(u,u_t,\theta,\eta^t)\in\mathscr{H}$, which generates a unique semi-process $\{U_G(t,\tau)\},(t\ge\tau\in\mathbb{R}^+,G\in\Sigma)$ on $\mathscr{H}$ of a two-parameter family of operators such that for any $t\ge\tau,\tau\in\mathbb{R}^+,\mathbb{R}_\tau=[\tau,+\infty)$,

$$U_G(t,\tau)(u_0^\tau,u_1^\tau,\theta,\eta^\tau)=(u,u_t,\theta,\eta^t)\in\mathscr{H},\tag{2.9}$$

$$u\in L^\infty(\mathbb{R}_\tau,H_0^1(\Omega)),u_t\in L^\infty(\mathbb{R}_\tau,H_0^1(\Omega)),u_{tt}\in L^2(\mathbb{R}_\tau,H_0^1(\Omega)),$$
$$\theta\in L^\infty(\mathbb{R}_\tau,H_0^1(\Omega)),\eta^t\in L^\infty(\mathbb{R}_\tau,\mathcal{M}).\tag{2.10}$$

Our main result reads as follows.

Theorem 2.3. Assume that $G\in E_1$ and Σ is defined by (2.8), then the family of processes $\{U_{G,f}(t,\tau)\}(G\in\Sigma,t\ge\tau,\tau\in\mathbb{R}^+)$ corresponding to (1.17)-(1.21) has a uniformly (w.r.t. $G\in\Sigma$) compact attractor $\mathcal{A}_\Sigma$.

3. The Well-Posedness

The global existence of solutions is the same as in [23] [30] [40], so we omit the details here. Next we prove the uniqueness of solutions.

We consider two symbols σ_1,f_1 and σ_2,f_2 and the corresponding solutions (u,θ_1,η^t) and (v,θ_2,ξ^t) of problem (1.17)-(1.21) with initial data $(u_0^\tau,u_1^\tau,\theta_{10},\eta^\tau)$ and $(v_0^\tau,v_1^\tau,\theta_{20},\xi^\tau)$ respectively. Let $\omega(t)=u(t)-v(t)$, $p(t)=\theta_1(t)-\theta_2(t)$, $\zeta^t(x,s)=\eta^t(x,s)-\xi^t(x,s)$.

Then (ω,p,ζ^t) verifies

$$\begin{aligned}&|u_t|^\rho\omega_{tt}+v_{tt}\left(|u_t|^\rho-|v_t|^\rho\right)-l\Delta\omega-\Delta\omega_{tt}\\&-\int_0^{+\infty}g(s)\Delta\zeta(s)\mathrm{d}s-\Delta\omega_t+\nabla p=\sigma_1-\sigma_2,x\in\Omega,t>\tau,\end{aligned}\tag{3.1}$$

$$p_t - \Delta p + \operatorname{div}\omega = f_1 - f_2, \tag{3.2}$$

$$\zeta_t^t + \zeta_s^t = \omega_t, \tag{3.3}$$

with Dirichlet boundary conditions and initial conditions

$$\omega(x,\tau) = \omega_0^\tau,\ \omega_t(x,\tau) = \omega_1^\tau,\ p(x,\tau) = p_1^\tau,\ \zeta^\tau = \eta^\tau - \xi^\tau. \tag{3.4}$$

The corresponding energy for (3.1)-(3.3) is defined

$$E_{\omega,p}(t) = \frac{1}{2}\int_\Omega |u_t|^\rho \omega_t^2 \mathrm{d}x + \frac{l}{2}\|\nabla\omega\|^2 + \frac{1}{2}\|\nabla\omega_t\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2. \tag{3.5}$$

It is easy to see that

$$\begin{aligned}(\zeta_s^t, \zeta^t)_{\mathcal{M}} &= \frac{1}{2}\int_\Omega\left(\int_0^{+\infty} g(s)\frac{\mathrm{d}}{\mathrm{d}s}|\nabla\zeta^t(s)|^2 \mathrm{d}s\right)\mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega\left(\int_0^{+\infty} g'(s)|\nabla\zeta^t(s)|^2 \mathrm{d}s\right)\mathrm{d}x.\end{aligned}$$

Noting that $x \to |x|^\rho$ is differentiable since $\rho > 1$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^\rho \omega_t^2 dx = \int_\Omega |u_t|^\rho \omega_{tt}\omega_t \mathrm{d}x + \frac{\rho}{2}\int_\Omega |u_t|^{\rho-2} u_t u_{tt}\omega_t^2 \mathrm{d}x,$$

and clearly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\omega,p}(t) &= -\|\nabla\omega_t\|^2 + \frac{1}{2}\int_0^{+\infty} g'(s)\|\nabla\zeta^t(s)\|^2 \mathrm{d}s \\ &\quad + \int_\Omega(\sigma_1 - \sigma_2)\omega_t \mathrm{d}x + \int_\Omega (f_1 - f_2)\theta \mathrm{d}x \\ &\quad + \frac{\rho}{2}\int_\Omega |u_t|^{\rho-1} u_{tt}\omega_t^2 \mathrm{d}x - \int_\Omega v_{tt}\omega_t\left(|u_t|^\rho - |v_t|^\rho\right)\mathrm{d}x.\end{aligned} \tag{3.6}$$

To simplify notations, let us say that the norm of the initial data is bounded by some $R > 0$. Then given $T > \tau$ we use C_{RT} to denote several positive constants which depend on R and T.

By Young's inequality and the interpolation inequalities, we derive

$$\left|\int_\Omega (\sigma_1 - \sigma_2)\omega_t \mathrm{d}x\right| \le \|\sigma_1 - \sigma_2\|\|\omega_t\| \le \|\sigma_1 - \sigma_2\|^2 + C_{RT}E_\omega(t), \tag{3.7}$$

$$\left|\int_\Omega (f_1 - f_2)\theta \mathrm{d}x\right| \le \|f_1 - f_2\|^2 + C_{RT}E_{\omega,p}(t), \tag{3.8}$$

$$\frac{\rho}{2}\left|\int_\Omega |u_t|^{\rho-1} u_{tt}\omega_t^2 \mathrm{d}x\right| \le \frac{\rho}{2}\|u_t\|_{2(\rho+1)}^{\rho-1}\|u_{tt}\|\|\omega_t\|_{2(\rho+1)}^2 \le C_{RT}\|\nabla u_{tt}\|\|\nabla\omega_t\|^2, \tag{3.9}$$

$$\begin{aligned}\left|-\int_\Omega v_{tt}\left(|u_t|^\rho - |v_t|^\rho\right)\omega_t \mathrm{d}x\right| &\le C_1\int_\Omega |v_{tt}|\left(|u_t|^{\rho-1} + |v_t|^{\rho-1}\right)\omega_t^2 \mathrm{d}x \\ &\le C_1\|v_{tt}\|\left(\|u_t\|_{2(\rho+1)}^{\rho-1} + \|v_t\|_{2(\rho+1)}^{\rho-1}\right)\|\omega_t\|_{2(\rho+1)}^2 \\ &\le C_1\|\nabla v_{tt}\|\|\nabla\omega_t\|^2,\end{aligned}$$

which, together with (3.6)-(3.9), yields for some $C_1 > 0$ large

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\omega,p}(t) \le \|\sigma_1 - \sigma_2\|^2 + \|f_1 - f_2\|^2 + C_1\left(1 + \|\nabla u_{tt}\| + \|\nabla v_{tt}\|\right)E_{\omega,p}(t). \tag{3.10}$$

Integrating (3.10) from τ to t and using Hölder's inequality, we have

$$E_{\omega,p}(t) \le E_{\omega,p}(\tau) + \int_\tau^t \|\sigma_1(s) - \sigma_2(s)\|^2 \mathrm{d}s + \int_\tau^t \|f_1(s) - f_2(s)\|^2 \mathrm{d}s$$

$$\begin{aligned}&+C_1\int_\tau^t\left(1+\|\nabla u_{tt}\|+\|\nabla v_{tt}\|\right)E_{\omega,p}(s)\,\mathrm{d}s\\&\le E_{\omega,p}(\tau)+\int_\tau^T\|\sigma_1(s)-\sigma_2(s)\|^2\,\mathrm{d}s+\int_\tau^t\|f_1(s)-f_2(s)\|^2\,\mathrm{d}s\\&+C_1\left(\int_\tau^t\left(1+\|\nabla u_{tt}\|+\|\nabla v_{tt}\|\right)^2\mathrm{d}s\right)^{\frac12}\left(\int_\tau^t E^2_{\omega,p}(s)\,\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{3.11}$$

Noting that

$$\int_\tau^T\left(1+\|\nabla u_{tt}\|+\|\nabla v_{tt}\|\right)^2\mathrm{d}s\le C_{RT},$$

then we get for any $t\in[\tau,T]$

$$\begin{aligned}E^2_{\omega,p}(t)&\le 2\left(E_{\omega,p}(\tau)+\int_\tau^T\|\sigma_1(s)-\sigma_2(s)\|^2\,\mathrm{d}s+\int_\tau^T\|f_1(s)-f_2(s)\|^2\,\mathrm{d}s\right)^2\\&+C_{RT}\int_\tau^t E^2_\omega(s)\,\mathrm{d}s.\end{aligned}\tag{3.12}$$

Applying Gronwall's inequality, we see that

$$\begin{aligned}E_{\omega,p}(t)\le\sqrt2\Big(&E_\omega(\tau)+\int_\tau^T\|\sigma_1(s)-\sigma_2(s)\|^2\,\mathrm{d}s\\&+\int_\tau^T\|f_1(s)-f_2(s)\|^2\,\mathrm{d}s\Big)\exp\left(\frac{C_{RT}}{2}T\right),\forall\, t\in[\tau,T].\end{aligned}\tag{3.13}$$

Using $\int_\Omega|u_t|^\rho|\omega_t|^2\,\mathrm{d}x\le\|u_t\|^\rho_{\rho+2}\|\omega_t\|^2_{\rho+2}\le C_{RT}\|\nabla\omega_t\|^2$, we know that $E_{\omega,p}(t)$ is equivalent to the norm of u,θ in $\mathscr{H}$ and we get

$$E_{\omega,p}(\tau)\le C_{RT}\left\|\left(\omega_0^\tau,\omega_1^\tau,p_0^\tau,\zeta^\tau\right)\right\|^2_{\mathscr{H}},$$

which, together with (3.13), gives for all $\tau\le t\le T$

$$\begin{aligned}&\|u(t)-v(t)\|^2_{H_0^1}+\|u_t(t)-v_t(t)\|^2_{H_0^1}+\|\eta^t-\xi^t\|^2_{\mathcal{M}}\\&\le C_{RT}\left(\|u_0^\tau-v_0^\tau\|^2_{H_0^1}+\|u_1^\tau-v_1^\tau\|^2_{H_0^1}+\|\eta^\tau-\xi^\tau\|^2_{\mathcal{M}}+\|\sigma_1-\sigma_2\|^2_{L^2(\tau,T;L^2(\Omega))}\right).\end{aligned}$$

This shows that solutions of (1.17)-(1.21) depend continuously on the initial data. We complete the proof of Theorem 2.1.

4. Uniform Attractors

In this section, we shall establish the existence of uniform attractors for system (1.17)-(1.21). To this end, we shall introduce some basic conceptions and basic lemmas. For more results concerning uniform attractors, we can refer to [31] [36] [37] [43] [44].

Let X be a Banach space, and $\hat\Sigma$ be a parameter set. The operators $\{U_G(t,\tau)\}$ $(t\ge\tau,\tau\in\mathbb{R}^+,G\in\hat\Sigma)$ are said to be a family of processes in X with symbol space $\hat\Sigma$ if for any $G\in\hat\Sigma$,

$$U_G(t,s)U_G(s,\tau)=U_G(t,\tau),\quad\forall\, t\ge s\ge\tau,\tau\in\mathbb{R}^+,\tag{4.1}$$

$$U_G(\tau,\tau)=Id(\text{identity}),\quad\forall\,\tau\in\mathbb{R}^+.\tag{4.2}$$

Let $\{T(s)\}$ be the translation semigroup on $\hat\Sigma$, we say that a family of processes $\{U_G(t,\tau)\}$ $(t\ge\tau,\tau\in\mathbb{R}^+,G\in\hat\Sigma)$ satisfies the translation identity if

$$U_G(t+s,\tau+s)=U_{T(s)G}(t,\tau),\quad \forall\, G\in\hat{\Sigma},t\geq\tau,\tau,s\in\mathbb{R}^+, \tag{4.3}$$

$$T(s)\hat{\Sigma}=\hat{\Sigma},\quad \forall\, s\in\mathbb{R}^+. \tag{4.4}$$

By $B(X)$ we denote the collection of the bounded sets of *X*, and $\mathbb{R}_\tau=[\tau,+\infty),\tau\in\mathbb{R}^+$.

Definition 4.1. A bounded set $B_0\in B(X)$ is said to be a bounded uniformly (w.r.t $G\in\hat{\Sigma}$) absorbing set for $\{U_G(t,\tau)\}(G\in\hat{\Sigma},t\geq\tau,\tau\in\mathbb{R}^+)$ if for any $\tau\in\mathbb{R}^+$ and $B\in B(X)$, there exists a time $T_0=T_0(B,\tau)\geq\tau$ such that

$$\bigcup_{G\in\hat{\Sigma}}U_G(t,\tau)B\subseteq B_0, \tag{4.5}$$

for all $t\geq T_0$.

In the following, as usual, (w.r.t) will represent "with respect to".

Definition 4.2. The family of semi-processes $\{U_\sigma(t,\tau)\}(t\geq\tau,\tau\in\mathbb{R}^+,\sigma\in\hat{\Sigma})$ is said to be asymptotically compact in *X* if $\left\{U_\sigma(t,\tau)\left(u_0^{\tau(n)},u_1^{\tau(n)},\theta_0^{\tau(n)},\eta^{\tau(n)}\right)\right\}$ is precompact in *X*, whenever $\left(u_0^{\tau(n)},u_1^{\tau(n)},\theta_0^{\tau(n)},\eta^{\tau(n)}\right)$ is bounded in *X*, $G^{(n)}\subset\hat{\Sigma}$, and $t_n\in\mathbb{R}_\tau,t_n\to+\infty$ as $n\to+\infty$.

Definition 4.3. A set $A\subseteq X$ is said to be uniformly (w.r.t $G\in\hat{\Sigma}$) attracting for the family of semi-processes $\{U_G(t,\tau)\}(t\geq\tau,\tau\in\mathbb{R}^+,G\in\hat{\Sigma})$ if for any fixed $\tau\in\mathbb{R}^+$ and any $B\in B(X)$,

$$\lim_{t\to+\infty}\left(\sup dist\left(U_G(t,\tau)B,A\right)\right)=0, \tag{4.6}$$

here $dist(\cdot,\cdot)$ stands for the usual Hausdorff semidistance between two sets in *X*. In particular, a closed uniformly attracting set $A_{\hat{\Sigma}}$ is said to be the uniform (w.r.t $G\in\hat{\Sigma}$) attractor of the family of the semi-process

$$\{U_G(t,\tau)\}(t\geq\tau,\tau\in\mathbb{R}^+,G\in\hat{\Sigma})$$

if it is contained in any closed uniformly attracting set (minimality property).

Definition 4.4. Let *X* be a Banach space and *B* be a bounded subset of $X,\hat{\Sigma}$ be a symbol (or parameter) space. We call a function $\phi(\cdot,\cdot;\cdot,\cdot)$, defined on $(X\times X)\times(\hat{\Sigma}\times\hat{\Sigma})$ to be a contractive function on $B\times B$ if for any sequence $\{x_n\}_{n=1}^\infty\subseteq B$ and any $\{\mu_n\}\subseteq\hat{\Sigma}$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty\subset\{x_n\}_{n=1}^\infty$ and $\{\mu_{n_k}\}_{k=1}^\infty\subset\{\mu_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi\left(x_{n_k},x_{n_l};\mu_{n_k},\mu_{n_l}\right)=0. \tag{4.7}$$

We denote the set of all contractive functions on $B\times B$ by $Contr(B,\hat{\Sigma})$.

Lemma 4.1. Let $\{U_G(t,\tau)\}(t\geq\tau,\tau\in\mathbb{R}^+,G\in\hat{\Sigma})$ be a family of semi-processes satisfying the translation identities (4.3) and (4.4) on Banach space *X* and has a bounded uniformly (w.r.t $G\in\hat{\Sigma}$) absorbing set $B_0\subseteq X$. Moreover, assuming that for any $\varepsilon>0$, there exist $T=T(B_0,\varepsilon)>0$ and $\phi_T\in Contr(B_0,\hat{\Sigma})$ such that

$$\left\|U_{G_1}(T,0)x-U_{G_2}(T,0)y\right\|\leq\varepsilon+\phi_T(x,y;G_1,G_2),\forall\, G\in\hat{\Sigma},t\geq\tau,\tau\in\mathbb{R}^+. \tag{4.8}$$

Then $\{U_G(t,\tau)\}(t\geq\tau,\tau\in\mathbb{R}^+,G\in\hat{\Sigma})$ is uniformly (w.r.t $G\in\hat{\Sigma}$) asymptotically compact in X.

Proof. This lemma is a version for semi-processes of a result by Khanmamedov [45]. A proof can be found in Sun *et al.* [43], Theorem 4.2.

Next, we will divide into two subsections to prove Theorem 2.3.

4.1. Uniformly (w.r.t. $G \in \Sigma$) Absorbing Set in $\mathscr{H}$

In this subsection we shall establish the family of processes $\{U_G(t,\tau)\}$ has a bounded uniformly absorbing set given in the following theorem.

Theorem 4.1. Assume that $G \in E_1$ and Σ is defined by (2.7), then the family of processes $\{U_G(t,\tau)\}(G \in \Sigma, t \ge \tau, \tau \in \mathbb{R}^+)$ corresponding to (1.17)-(1.21) has a bounded uniformly (w.r.t. $G \in \Sigma$) absorbing set B in $\mathscr{H}$.

Proof. We define

$$F(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{l}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2. \tag{4.9}$$

Using Young's inequality, Poincaré's inequality, we arrive at

$$\begin{aligned} F'(t) &= -\|\nabla u_t\|^2 - (\eta_s^t, \eta^t)_{\mathcal{M}} + (\sigma, u_t) + (f, \theta) \\ &\le -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\|\nabla\theta\|^2 + \frac{1}{2}\int_0^{+\infty} g'(s)\|\nabla\eta^t(s)\|^2 \mathrm{d}s + \frac{1}{2\varepsilon}\left(\|\sigma\|^2 + \|f\|^2\right). \end{aligned} \tag{4.10}$$

Let

$$F_1(t) = F(t) + \frac{1}{2}\int_t^{+\infty}\left(\|\sigma(s)\|^2 + \|f(s)\|^2\right)\mathrm{d}s, \quad \text{for all } t \ge \tau. \tag{4.11}$$

Then (4.11) gives $F_1'(t) \le 0$, whence from (4.9), for $t \ge \tau > 0$

$$\begin{aligned} F(t) \le F_1(t) \le F_1(\tau) &= F(\tau) + \frac{1}{2}\int_\tau^{+\infty}\left(\|\sigma(s)\|^2 + \|f(s)\|^2\right)\mathrm{d}s \\ &= F(\tau) + \frac{1}{2}\left(\|\sigma\|_{L^2(\mathbb{R}_\tau, L^2(\Omega))}^2 + \|f\|_{L^2(\mathbb{R}_\tau, L^2(\Omega))}^2\right), \end{aligned} \tag{4.12}$$

$$\frac{l}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{l}{2}\|\theta\|^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 \le F(t) \le F_1(t) \le F_1(\tau). \tag{4.13}$$

Now we define

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u \mathrm{d}x + \int_\Omega \nabla u_t \cdot \nabla u \mathrm{d}x. \tag{4.14}$$

From (1.17), integration by parts and Young's inequality, we derive for any $\varepsilon \in (0,1)$,

$$\begin{aligned} \Phi'(t) &= \left(l\Delta u + \int_0^{+\infty} g(s)\Delta\eta^t(s)\mathrm{d}s + \Delta u_t - \nabla\theta + \sigma, u\right) \\ &\quad + \frac{1}{\rho+1}\left(|u_t|^\rho u_t, u_t\right) - (\Delta u_t, u_t) \\ &= -l\|\nabla u\|^2 - \left(\nabla u, \int_0^{+\infty} g(s)\nabla\eta^t(s)\mathrm{d}s\right) - (\nabla\theta, u) \\ &\quad + (\Delta u_t, u) + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|^2 + (\sigma, u). \end{aligned} \tag{4.15}$$

Using Young's inequality, Hölder's inequality and Poincaré's inequality, we deduce

$$\left|(\Delta u_t,u)\right| \le \varepsilon\|\nabla u\|^2+\frac{1}{4\varepsilon}\|\nabla u_t\|^2, \tag{4.16}$$

$$\left|-(\nabla\theta,u)\right| \le \varepsilon\|u\|^2+\frac{1}{4\varepsilon}\|\nabla\theta\|^2 \le \varepsilon\lambda^2\|\nabla u\|^2+\frac{1}{4\varepsilon}\|\nabla\theta\|^2, \tag{4.17}$$

$$\begin{aligned}&\left|-\left(\nabla u,\int_0^{+\infty}g(s)\nabla\eta^t(s)\mathrm{d}s\right)\right|\\ &\le \varepsilon\|\nabla u\|^2+\frac{1}{4\varepsilon}\int_\Omega\left(\int_0^{+\infty}g(s)\nabla\eta^t(s)\mathrm{d}s\right)^2\mathrm{d}x\\ &\le \varepsilon\|\nabla u\|^2+\frac{1-l}{4\varepsilon}\int_0^{+\infty}g(s)\left\|\nabla\eta^t(s)\right\|^2\mathrm{d}s,\end{aligned} \tag{4.18}$$

$$\left|(\sigma,u)\right| \le \varepsilon\|u\|^2+\frac{1}{4\varepsilon}\|\sigma\|^2 \le \varepsilon\lambda^2\|\nabla u\|^2+\frac{1}{4\varepsilon}\|\sigma\|^2, \tag{4.19}$$

hereinafter we use λ to represent the Poincaré constant.

From the expression of $F(t)$, we get

$$\|\nabla u\|^2=\frac{2}{l}F(t)-\frac{2}{l(\rho+2)}\|u_t\|_{\rho+2}^{\rho+2}-\frac{1}{l}\|\nabla u_t\|^2-\frac{1}{l}\|\theta\|^2-\frac{1}{l}\|\eta^t\|_{\mathcal{M}}^2,$$

which, together with (4.15)-(4.19), yields

$$\begin{aligned}\Phi'(t) \le& -\frac{1}{2}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)\|\nabla u\|^2-\frac{1}{2}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)\left(\frac{2}{l}F(t)\right.\\ &\left.-\frac{2}{l(\rho+2)}\|u_t\|_{\rho+2}^{\rho+2}-\frac{1}{l}\|\nabla u_t\|^2-\frac{1}{l}\|\theta\|^2-\frac{1}{l}\|\eta^t\|_{\mathcal{M}}^2\right)\\ &+\frac{1-l}{4\varepsilon}\|\eta^t\|_{\mathcal{M}}^2+\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}+\left(1+\frac{1}{4\varepsilon}\right)\|\nabla u_t\|^2\\ &+\frac{1}{4\varepsilon}\left(\|\sigma\|^2+\|\nabla\theta\|^2\right)\\ \le& -\frac{1}{2}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)\|\nabla u\|^2-\frac{l-2\varepsilon\lambda^2-2\varepsilon}{l}F(t)\\ &+\left(\frac{l-2\varepsilon\lambda^2-2\varepsilon}{l(\rho+2)}+\frac{1}{\rho+1}\right)\|u_t\|_{\rho+2}^{\rho+2}\\ &+\left[1+\frac{1}{4\varepsilon}+\frac{1}{2}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)\right]\|\nabla u_t\|^2\\ &+\left(\frac{1}{2l}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)+\frac{1-l}{4\varepsilon}\right)\|\eta^t\|_{\mathcal{M}}^2\\ &+\frac{1}{4\varepsilon}\|\sigma\|^2+\left(\frac{\lambda}{l}+\frac{1}{4\varepsilon}\right)\|\nabla\theta\|^2\end{aligned} \tag{4.20}$$

Noting that $\|\nabla u_t\|^2 \le 2F(t) \le 2F_1(\tau)$ and the embedding theorem $H^1(\Omega)\rightarrowtail L^{2(\rho+1)}(\Omega)$, we have for any $\varepsilon\in(0,1)$,

$$\begin{aligned}&\left(\frac{l-2\varepsilon\lambda^2-2\varepsilon}{l(\rho+2)}+\frac{1}{\rho+1}\right)\|u_t\|_{\rho+2}^{\rho+2}\\ &\le C_1\|u_t\|\|u_t\|_{2(\rho+1)}^{\rho+1} \le C_1\|\nabla u_t\|^{\rho+1}\|u_t\|\end{aligned}$$

$$\le \varepsilon \|\nabla u_t\|^{2(\rho+1)} + C_\varepsilon \|u_t\|^2$$
$$\le 2^{\rho+1}\varepsilon F_1^\rho(\tau)F(t) + C_\varepsilon \|u_t\|^2,$$

which, together with (4.20) and Poincaré's inequality, gives

$$\begin{aligned}\Phi'(t) \le & -\frac{1}{2}\left(l-2\varepsilon\lambda^2-2\varepsilon\right)\|\nabla u\|^2 - \left(\frac{l-2\varepsilon\lambda^2-\varepsilon}{l} - 2^{\rho+1}\varepsilon F_1^\rho(\tau)\right)F(t) \\ & + \left(1+\frac{1}{4\varepsilon}C_\varepsilon\lambda^2 + \frac{l-2\varepsilon\lambda^2-2\varepsilon}{2l} + \frac{1}{4\varepsilon}\right)\|\nabla u_t\|^2 \\ & + \left(\frac{l-2\varepsilon\lambda^2-2\varepsilon}{2l} + \frac{1-l}{4\varepsilon}\right)\|\eta^t\|_{\mathcal{M}}^2 \\ & + \frac{1}{4\varepsilon}\|\sigma\|^2 + \left(\frac{\lambda}{l} + \frac{1}{4\varepsilon}\right)\|\nabla\theta\|^2.\end{aligned} \tag{4.21}$$

Now we take $\varepsilon \in (0,1)$ so small that

$$l-2\varepsilon\lambda^2-2\varepsilon \ge \frac{l}{2}, \quad \frac{l-2\varepsilon\lambda^2-2\varepsilon}{l} - 2^{\rho+1}\varepsilon F_1^\rho(\tau) \ge \frac{1}{4}. \tag{4.22}$$

Hence from (4.21)-(4.22), it follow

$$\Phi'(t) \le -\frac{l}{4}\|\nabla u\|^2 - \frac{1}{4}F(t) + C_1\|\eta^t\|_{\mathcal{M}}^2 + C_1\left(\|\nabla u_t\|^2 + \|\nabla\theta\|^2\right) + \frac{1}{4\varepsilon}\|\sigma\|^2. \tag{4.23}$$

We define the functional

$$\Psi(t) = \int_\Omega\left(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s\mathrm{d}x. \tag{4.24}$$

It follows from (1.17) that

$$\begin{aligned}\Psi'(t) = & \int_\Omega\left(-l\Delta u - \int_0^{+\infty} g(s)\Delta\eta^t(s)\,\mathrm{d}s + u_t + \nabla\theta - \sigma\right)\int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s \\ & + \int_\Omega\left(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^{+\infty} g(s)\eta_t^t(s)\,\mathrm{d}s \\ := & I_1 + I_2.\end{aligned} \tag{4.25}$$

From Young's inequality, Hölder's inequality and Poincaré's inequality, we derive for any $\delta \in (0,1)$,

$$\left|\int_\Omega(-l\Delta u)\int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s\right| \le \delta\|\nabla u\|^2 + \frac{l^2(1-l)}{4\delta}\|\eta^t\|_{\mathcal{M}}^2, \tag{4.26}$$

$$\left|\int_\Omega -\left(\int_0^{+\infty} g(s)\Delta\eta^t(s)\,\mathrm{d}s\right)^2\mathrm{d}s\right| \le (1-l)\|\eta^t\|_{\mathcal{M}}^2, \tag{4.27}$$

$$\left|\int_\Omega -\Delta u_t \cdot \int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s\right| \le \delta\|\nabla u_t\|^2 + \frac{1-l}{4\delta}\|\eta^t\|_{\mathcal{M}}^2, \tag{4.28}$$

$$\left|\int_\Omega \nabla\theta \cdot \int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s\right| \le \delta\|\nabla\theta\|^2 + \frac{\lambda^2(1-l)}{4\delta}\|\eta^t\|_{\mathcal{M}}^2, \tag{4.29}$$

$$\left|\int_\Omega(-\sigma)\int_0^{+\infty} g(s)\eta^t(s)\,\mathrm{d}s\right| \le \frac{1}{2}\|\sigma\|^2 + \frac{1-l}{2}\|\eta^t\|_{\mathcal{M}}^2,$$

which, together with (4.26)-(4.29), gives

$$I_1 \le \delta\left(\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|\nabla \theta\|^2\right) + \frac{1}{2}\|\sigma\|^2 + \left(\frac{3(1-l)}{2} + \frac{(1+\lambda^2+l^2)(1-l)}{4\delta}\right)\|\eta^t\|^2_{\mathcal{M}}. \tag{4.30}$$

Noting that

$$\int_0^{+\infty} g(s)\eta_t^t(s)\,\mathrm{d}s = -\int_0^{+\infty} g(s)\eta_s^t(s)\,\mathrm{d}s + \int_0^{+\infty} g(s)u_t(t)\,\mathrm{d}s = \int_0^{+\infty} g'(s)\eta^t(s)\,\mathrm{d}s + (1-l)u_t,$$

then we have

$$I_2 = -(1-l)\|\nabla u_t\|^2 - \frac{1-l}{\rho+1}\|u_t\|^{\rho+2}_{\rho+2} + \int_0^{+\infty} g'(s)\int_\Omega \Delta u_t(t)\eta^t(s)\,\mathrm{d}x\mathrm{d}s + \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)\eta^t(s)\,\mathrm{d}s\mathrm{d}x. \tag{4.31}$$

By Young's inequality, we derive

$$\begin{aligned}&\int_0^{+\infty} g'(s)\int_\Omega \Delta u_t(t)\eta^t(s)\,\mathrm{d}x\mathrm{d}s\\ &\le -\int_0^{+\infty} g'(s)\|\nabla u_t(t)\|\|\nabla \eta^t(s)\|\,\mathrm{d}s\\ &\le \frac{1-l}{4}\|\nabla u_t(t)\|^2 - \frac{1}{1-l}\int_0^{+\infty} g'(s)\|\nabla \eta^t(s)\|^2\,\mathrm{d}s,\end{aligned} \tag{4.32}$$

and for any $\varepsilon > 0$

$$\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)\eta^t(s)\,\mathrm{d}s\mathrm{d}x \le \varepsilon\|\nabla u_t\|^2 - C_\varepsilon\int_0^{+\infty} g'(s)\|\nabla \eta^t(s)\|^2\,\mathrm{d}s,$$

which, together with (4.30)-(4.32) and taking $\varepsilon > 0$ small enough, yields

$$I_2 \le -\frac{1-l}{2}\|\nabla u_t\|^2 - \frac{1-l}{\rho+1}\|u_t\|^{\rho+2}_{\rho+2} - C_1\int_0^{+\infty} g'(s)\|\nabla \eta^t(s)\|^2\,\mathrm{d}s. \tag{4.33}$$

Inserting (4.30) and (4.33) into (4.25), we arrive at

$$\Psi'(t) \le -\frac{1-l}{4}\|\nabla u_t\|^2 + \delta\left(\|\nabla u\|^2 + \|\nabla \theta\|^2\right) + C_1\|\eta^t\|^2_{\mathcal{M}} + \frac{1}{2}\|\sigma\|^2 - C_1\int_0^{+\infty} g'(s)\|\nabla \eta^t(s)\|^2\,\mathrm{d}s - \frac{1-l}{\rho+1}\|u_t\|^{\rho+2}_{\rho+2}. \tag{4.34}$$

Set

$$H(t) = MF(t) + \varepsilon\Phi(t) + \Psi(t), \tag{4.35}$$

where M and ε are positive constants.

Then it follows from (4.10), (4.23), (4.34) and (2.2) that

$$\begin{aligned}H'(t) \le{}& \frac{M}{2}\|u_t\|^2 - \left(\frac{l\varepsilon}{4} - \delta\right)\|\nabla u\|^2 - \frac{\varepsilon}{4}F(t) - \left(\frac{1-l}{4} - C_1\varepsilon + \frac{M}{2}\right)\|\nabla u_t\|^2\\ &- \left(\frac{M}{2} - \delta - C_1\varepsilon\right)\|\nabla \theta\|^2 + \left(\frac{M}{2} - C_1 - C_1\xi_2\right)\int_0^{+\infty} g'(s)\|\nabla \eta^t(s)\|^2\,\mathrm{d}s\\ &+ \left(\frac{M}{2\varepsilon} + \frac{\varepsilon}{4\varepsilon} + \frac{1}{2}\right)\|\sigma\|^2 + \frac{M}{2\varepsilon}\|f\|^2 - \frac{1-l}{\rho+1}\|u_t\|^{\rho+2}_{\rho+2}.\end{aligned} \tag{4.36}$$

Now we claim that there exist two constants $\beta_1,\beta_2>0$ such that

$$\beta_1 F(t)\le H(t)\le \beta_2 F(t),\ t\ge 0. \tag{4.37}$$

For any $t\ge\tau$, we take ε so small that

$$\frac{M}{2}+\frac{1-l}{4}-C_1\varepsilon>0. \tag{4.38}$$

For fixed ε, we choose δ small enough and M so large that

$$\frac{M}{2}+\delta-C_1\varepsilon>0,\frac{l\varepsilon}{4}-\delta>0,\ \frac{M}{2}-C_1-C_1\xi_2>0.$$

Then there exist a constant $\gamma>0$ such that

$$H'(t)\le -\gamma F(t)+C_1\left(\|\sigma\|^2+\|f\|^2\right), \tag{4.39}$$

which, together with (4.37), gives

$$H'(t)\le -\frac{\gamma}{\beta_2}H(t)+C_1\left(\|\sigma\|^2+\|f\|^2\right). \tag{4.40}$$

Integrating (4.40) over $[\tau,t]$ with respect to t and using Lemmas 2.2-2.3, we obtain

$$\begin{aligned}H(t)&\le H(\tau)\mathrm{e}^{-\frac{\gamma}{\beta_2}(t-\tau)}+C_1\int_\tau^t \mathrm{e}^{-\frac{\gamma}{\beta_2}(t-s)}\left(\|\sigma(s)\|^2+\|f(s)\|^2\right)\mathrm{d}s\\ &\le C_{\mathcal{B}_0}\mathrm{e}^{-\frac{\gamma}{\beta_2}(t-\tau)}+C_1\frac{1}{1-\mathrm{e}^{-\frac{\gamma}{\beta_2}}}\sup_{t\ge\tau}\int_t^{t+1}\left(\|\sigma(s)\|^2+\|f(s)\|^2\right)\mathrm{d}s\\ &\le C_{\mathcal{B}_0}\mathrm{e}^{-\frac{\gamma}{\beta_2}(t-\tau)}+C_1\frac{1}{1-\mathrm{e}^{-\frac{\gamma}{\beta_2}}}\eta_G(1).\end{aligned} \tag{4.41}$$

Now for any bounded set $\mathcal{B}_0\subseteq\mathscr{H}$, for any $(u_0^\tau,u_1^\tau,\theta_0^\tau,\eta^\tau)\in\mathcal{B}_0$, there exists a constant $C_{\mathcal{B}_0}>0$ such that $F(\tau)\le C_{\mathcal{B}_0}\le C_1$. Taking

$$R_0^2=2\left(2C_1\frac{\eta_{G_0}(1)}{1-\mathrm{e}^{-\frac{\gamma}{\beta_2}}}+1\right),$$

$$t_0=\tau-\left(\frac{\gamma}{\beta_2}\right)^{-1}\log\left(\frac{C_1\eta_{G_0}(1)+1}{C_{\mathcal{B}_0}\left(1-\mathrm{e}^{-\frac{\gamma}{\beta_2}}\right)}\right),$$

then for any $t\ge t_0\ge 1$, we have

$$H(t)\le C_{\mathcal{B}_0}\mathrm{e}^{-\frac{\gamma}{\beta_2}(t-\tau)}+C_1\frac{1}{1-\mathrm{e}^{-\frac{\gamma}{\beta_2}}}\eta_{G_0}(1)\le\frac{R_0^2}{2},$$

which gives

$$\left\|(u,u_t,\theta,\eta^t)\right\|_{\mathscr{H}}\le 2H(t)=R_0^2,$$

i.e.,

$$B=B(0,R_0)=\left\{(u,u_t,\theta,\eta^t)\in\mathscr{H}:\left\|(u,u_t,\theta,\eta^t)\right\|_{\mathscr{H}}^2\le R_0^2\right\}\subseteq\mathscr{H}$$

is a uniform absorbing ball for any $G\in E_1$. The proof is now complete.

4.2. Uniformly (w.r.t. $\sigma\in\Sigma$) Asymptotic Compactness in $\mathscr{H}$

In this subsection, we will prove the uniformly (w.r.t. $G\in\Sigma$) asymptotic compactness in $\mathscr{H}$, which is given in the following theorem.

Theorem 4.2. Assume that $G\in E_1$ and Σ is defined by (2.8), then the family of processes $\{U_G(t,\tau)\}(G\in\Sigma,t\ge\tau,\tau\in\mathbb{R}^+)$ corresponding to (1.17)-(1.21) is uniformly (w.r.t. $G\in\Sigma$) asymptotically compact in $\mathscr{H}$.

Proof. For any $(u_{0i}^\tau,u_{1i}^\tau,\theta_{0i}^\tau,\eta_i^\tau)\in B,\ i=1,2$. We consider two symbols σ_1,f_1 and σ_2,f_2 and the corresponding solutions u_1,θ_1 and u_2,θ_2 of problem (1.17)-(1.21) with initial data $(u_{0i}^\tau,u_{1i}^\tau,\theta_{0i}^\tau,\eta_i^\tau)$, $i=1,2$, respectively. Let $\omega(t)=u_1(t)-u_2(t)$, $p(t)=\theta_1(t)-\theta_2(t)$, $\zeta^t(x,s)=\eta_1^t(x,s)-\eta_2^t(x,s)$.

Then (ω,ζ^t) verifies

$$\begin{aligned}&|u_{1t}|^\rho\omega_{tt}+u_{2tt}\left(|u_{1t}|^\rho-|u_{2t}|^\rho\right)-l\Delta\omega-\Delta\omega_{tt}\\&-\int_0^{+\infty}g(s)\Delta\zeta(s)\mathrm{d}s-\Delta\omega_t+\nabla p=\sigma_1-\sigma_2,\ x\in\Omega,\ t>\tau,\end{aligned}\tag{4.42}$$

$$p_t-\Delta p+\mathrm{div}\omega=f_1-f_2\tag{4.43}$$

$$\zeta_t^t+\zeta_s^t=\omega_t,\tag{4.44}$$

with Dirichlet boundary conditions and initial conditions

$$\omega(x,\tau)=\omega_0^\tau,\ \omega_t(x,\tau)=\omega_1^\tau,\ p(x,\tau)=p_0^\tau,\ \zeta^\tau=\eta_1^\tau-\eta_2^\tau.\tag{4.45}$$

The corresponding energy for (4.42)-(4.45) is defined

$$E_{\omega,p}(t)=\frac{1}{2}\int_\Omega|u_{1t}|^\rho\omega_t^2\mathrm{d}x+\frac{l}{2}\|\nabla\omega\|^2+\frac{1}{2}\|\nabla\omega_t\|^2+\frac{1}{2}\|\theta\|^2+\frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2.\tag{4.46}$$

Clearly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\omega,p}(t)=&-\|\nabla\omega_t\|^2+\frac{1}{2}\int_0^{+\infty}g'(s)\|\nabla\zeta^t(s)\|^2\mathrm{d}s\\&+\int_\Omega(\sigma_1-\sigma_2)\omega_t\mathrm{d}x+\int_\Omega(f_1-f_2)p\mathrm{d}x\\&+\frac{\rho}{2}\int_\Omega|u_{1t}|^{\rho-1}u_{1tt}\omega_t^2\mathrm{d}x-\int_\Omega u_{2tt}\omega_t\left(|u_{1t}|^\rho-|u_{2t}|^\rho\right)\mathrm{d}x.\end{aligned}\tag{4.47}$$

Using Hölder's inequality, Poincaré's inequality and Theorem 4.1, we derive

$$\left|\int_\Omega(\sigma_1-\sigma_2)\omega_t\mathrm{d}x\right|\le\|\sigma_1-\sigma_2\|\|\omega_t\|,\tag{4.48}$$

$$\left|\int_\Omega(f_1-f_2)p\mathrm{d}x\right|\le\|f_1-f_2\|\|p\|,\tag{4.49}$$

$$\begin{aligned}\frac{\rho}{2}\left|\int_\Omega|u_{1t}|^{\rho-1}u_{1tt}\omega_t^2\mathrm{d}x\right|&\le\frac{\rho}{2}\|u_{1t}\|_{2(\rho+1)}^{\rho-1}\|u_{1tt}\|\|\omega_t\|_{2(\rho+1)}\|\omega_t\|\\&\le C_B\|\nabla u_{1t}\|^{\rho-1}\|\nabla\omega_t\|\|\nabla u_{1tt}\|\|\omega_t\|\\&\le C_B\|\omega_t\|\|\nabla u_{1tt}\|,\end{aligned}\tag{4.50}$$

$$\begin{aligned}&\left|-\int_\Omega u_{2tt}\left(|u_{1t}|^\rho-|u_{2t}|^\rho\right)\omega_t\mathrm{d}x\right|\\&\le C_1\|u_{2tt}\|_{2(\rho+1)}\left(\|u_{1t}\|_{2(\rho+1)}^\rho+\|u_{2t}\|_{2(\rho+1)}^\rho\right)\|\omega_t\|\\&\le C_1\|\nabla u_{2tt}\|\left(\|\nabla u_{1t}\|^\rho+\|\nabla u_{2t}\|^\rho\right)\|\omega_t\|\\&\le C_B\|\nabla u_{2tt}\|\|\omega_t\|,\end{aligned}$$

which, combined with (4.47)-(4.50), yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\omega,p}(t)\le&-\|\omega_t\|^2-\|\nabla p\|^2+\frac{1}{2}\int_0^{+\infty}g'(s)\|\nabla\zeta^t(s)\|^2\mathrm{d}s+\|\omega_t\|\|\sigma_1-\sigma_2\|\\&+\|f_1-f_2\|\|p\|+C_{B_0}\left(\|\nabla u_{1tt}\|+\|\nabla u_{2tt}\|\right)\|\omega_t\|.\end{aligned}\tag{4.51}$$

We define

$$\Phi_{\omega,p}(t)=\int_\Omega|u_{1t}|^\rho\omega_t\omega\mathrm{d}x+\int_\Omega\nabla\omega_t\cdot\nabla\omega\mathrm{d}x+\frac{1}{2}\int_\Omega(\nabla\omega)^2\mathrm{d}x.\tag{4.52}$$

It is very easy to verify

$$\begin{aligned}\left|\Phi_{\omega,p}(t)\right|&\le\frac{1}{2}\left(\|\nabla\omega\|^2+\|\nabla\omega_t\|^2\right)+\|u_{1t}\|_{2(\rho+1)}^\rho\|\omega_t\|_{2(\rho+1)}\|\omega\|+\frac{1}{2}\|\omega\|^2\\&\le C_B\left(\|\nabla\omega\|^2+\|\nabla\omega_t\|^2\right)\le C_BE_{\omega,p}(t).\end{aligned}\tag{4.53}$$

Taking the derivative of $\Phi_\omega(t)$, it follows from (4.42)-(4.43) that

$$\begin{aligned}\Phi'_{\omega,p}(t)=&-\int_\Omega u_{2tt}\left(|u_{1t}|^\rho-|u_{2t}|^\rho\right)\omega\mathrm{d}x-l\|\nabla\omega\|^2\\&-\int_\Omega\nabla\omega\int_0^{+\infty}g(s)\nabla\zeta^t(s)\mathrm{d}s\mathrm{d}x+\int_\Omega\nabla p\omega\mathrm{d}x+\int_\Omega(\sigma_1-\sigma_2)\omega\mathrm{d}x\\&+\int_\Omega\left(\rho|u_{1t}|^{\rho-1}u_{1tt}\omega_t\omega+|u_{1t}|^\rho\omega_t^2\right)\mathrm{d}x+\|\nabla\omega_t\|^2\\=&\sum_{i=1}^5A_i-l\|\nabla\omega\|^2+\|\nabla\omega_t\|^2.\end{aligned}\tag{4.54}$$

Applying Hölder's inequality, Young's inequality, Poinceré's inequality and Theorem 4.1, we get

$$\begin{aligned}|A_1|&\le C_1\|u_{2tt}\|_{2(\rho+1)}\left(\|u_{1t}\|_{2(\rho+1)}^{\rho-1}+\|u_{2t}\|_{2(\rho+1)}^{\rho-1}\right)\|\omega_t\|_{2(\rho+1)}\|\omega\|\\&\le C_1\|\nabla u_{2tt}\|\left(\|\nabla u_{1t}\|^{\rho-1}+\|\nabla u_{2t}\|^{\rho-1}\right)\|\nabla\omega_t\|\|\omega\|\\&\le C_B\|\nabla u_{2tt}\|\|\omega\|,\end{aligned}\tag{4.55}$$

$$|A_2|\le\|\nabla\omega\|\left\|\int_0^{+\infty}g(s)\nabla\zeta^t(s)\mathrm{d}s\right\|\le\varepsilon\|\nabla\omega\|^2+\frac{1-l}{4\varepsilon}\|\zeta^t\|^2,\quad\forall\varepsilon\in(0,1),\tag{4.56}$$

$$|A_3|\le\|\sigma_1-\sigma_2\|\|\omega\|,\tag{4.57}$$

$$|A_4|\le\|\nabla p\|\|\omega\|,\tag{4.58}$$

$$\begin{aligned}|A_5|&\le C_1\|u_{1tt}\|_{2(\rho+1)}\|u_{1t}\|_{2(\rho+1)}^{\rho-1}\|\omega_t\|_{2(\rho+1)}\|\omega\|+C_1\|u_{1t}\|_{\rho+2}^\rho\|\omega_t\|_{\rho+2}^2\\&\le C_B\|\nabla u_{1tt}\|\|\omega\|+C_B\|\nabla\omega_t\|^2.\end{aligned}\tag{4.59}$$

By virtue of (4.46), we have

$$\|\nabla\omega\|^2=\frac{2}{l}E_{\omega,p}(t)-\frac{1}{l}\int_\Omega|u_{1t}|^\rho\omega_t^2\mathrm{d}x-\frac{1}{l}\|\nabla\omega_t\|^2-\frac{1}{l}\|\nabla p\|^2-\frac{1}{l}\|\zeta^t\|_{\mathcal{M}}^2.\tag{4.60}$$

Then from (4.54)-(4.59), we can conclude

$$
\begin{aligned}
\Phi'_{\omega,p}(t) \le & -\frac{l-\varepsilon}{2}\|\nabla\omega\|^2 - \frac{l-\varepsilon}{2}\left(\frac{2}{l}E_{\omega,p}(t) - \frac{1}{l}\int_\Omega |u_{1t}|^\rho \omega_t^2 \mathrm{d}x - \frac{1}{l}\|\nabla\omega_t^2\| \right.\\
& \left. -\frac{1}{l}\|\nabla p\|^2 - \frac{1}{l}\|\zeta^t\|_{\mathcal{M}}^2\right) + C_B\|\nabla\omega_t\|^2 + C_B\|\omega\|\left(\|\nabla u_{1tt}\|\right.\\
& \left. + \|\nabla u_{2tt}\| + \|\nabla p\|\right) + C_1\|\sigma_1-\sigma_2\|\|\omega\| + \frac{1-l}{4\varepsilon}\|\zeta^t\|_{\mathcal{M}}^2 \\
\le & -\frac{l-\varepsilon}{2}\|\nabla\omega\|^2 - (l-\varepsilon)E_\omega(t) + C_\varepsilon\|\nabla\omega_t\|^2 + C_\varepsilon\|\nabla p\|^2 + C_\varepsilon\|\zeta^t\|_{\mathcal{M}}^2 \\
& + C_1\|\sigma_1-\sigma_2\|\|\omega\| + C_B\|\omega\|\left(\|\nabla u_{1tt}\| + \|\nabla u_{2tt}\| + \|\nabla p\|\right).
\end{aligned}
\tag{4.61}
$$

Now we define

$$
\Psi_{\omega,p}(t) = \int_\Omega \left(\Delta\omega_t - |u_{1t}|^\rho \omega_t\right)\left(\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\right)\mathrm{d}x. \tag{4.62}
$$

From (4.42)-(4.43) and integration by parts, we derive

$$
\begin{aligned}
\Psi'_{\omega,p}(t) = & \int_\Omega u_{2tt}\left(|u_{1t}|^\rho - |u_{2t}|^\rho\right)\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x + l\int_\Omega \nabla\omega\int_0^{+\infty} g(s)\nabla\zeta^t(s)\mathrm{d}s\mathrm{d}x \\
& + \int_\Omega\left(\int_0^{+\infty} g(s)\nabla\zeta^t(s)\mathrm{d}s\right)^2\mathrm{d}x - \int_\Omega \Delta\omega_t\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x \\
& + \int_\Omega \nabla p\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x - \int_\Omega(\sigma_1-\sigma_2)\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x \\
& - \rho\int_\Omega |u_{1t}|^{\rho-1} u_{1tt}\omega_t\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x + \int_\Omega \Delta\omega_t\int_0^{+\infty} g(s)\zeta_t^t(s)\mathrm{d}s\mathrm{d}x \\
& - \int_\Omega |u_{1t}|^\rho \omega_t\int_0^{+\infty} g(s)\zeta_t^t(s)\mathrm{d}s\mathrm{d}x = \sum_{i=1}^{9} B_i.
\end{aligned}
\tag{63}
$$

Using Hölder's inequality, Poincaré's inequality and Theorem 4.1, we derive for any $\delta\in(0,1)$,

$$
\begin{aligned}
B_1 & \le \|u_{2tt}\|_{2(\rho+1)}\left(\|u_{1t}\|_{2(\rho+1)}^{\rho-1} + \|u_{2t}\|_{2(\rho+1)}^{\rho-1}\right)\|\omega_t\|\left\|\int_0^{+\infty} g(s)\zeta^t(s)\mathrm{d}s\right\|_{2(\rho+1)} \\
& \le C_B\|\nabla u_{2tt}\|\|\omega_t\|\left\|\int_0^{+\infty} g(s)\nabla\zeta^t(s)\mathrm{d}s\right\| \\
& \le C_B(1-l)^{\frac{1}{2}}\|\nabla u_{2tt}\|\|\omega_t\|\|\zeta^t\|_{\mathcal{M}} \\
& \le C_B(1-l)^{\frac{1}{2}}\|\nabla u_{2tt}\|\|\omega_t\|\left(\|\eta_1^t\|_{\mathcal{M}} + \|\eta_2^t\|_{\mathcal{M}}\right) \\
& \le C'_B\|\nabla u_{2tt}\|\|\omega_t\|,
\end{aligned}
\tag{4.64}
$$

$$
B_2 \le \delta\|\nabla\omega\|^2 + \frac{(1-l)l^2}{4\delta}\|\zeta^t\|_{\mathcal{M}}^2 \tag{4.65}
$$

$$
B_3 \le (1-l)\|\zeta^t\|_{\mathcal{M}}^2, \tag{4.66}
$$

$$
B_4 \le \delta\|\nabla\omega_t\|^2 + \frac{\lambda^2}{4\delta}(1-l)\|\zeta^t\|_{\mathcal{M}}^2, \tag{4.67}
$$

$$
B_5 \le \delta\|\nabla p\|^2 + \frac{(1-l)l^2}{4\delta}\|\zeta^t\|_{\mathcal{M}}^2 \tag{4.68}
$$

$$
B_6 \le \lambda(1-l)^{\frac{1}{2}}\|\sigma_1-\sigma_2\|\|\zeta^t\|_{\mathcal{M}} \le C_1\|\sigma_1-\sigma_2\|^2 + C_1\|\zeta^t\|_{\mathcal{M}}^2 \tag{4.69}
$$

$$\begin{aligned} B_7 &\le \|u_{1t}\|_{2(\rho+1)}^{\rho-1}\|u_{1tt}\|_{2(\rho+1)}\|\omega_t\|\left\|\int_0^{+\infty} g(s)\zeta^t(s)\,\mathrm{d}s\right\|_{2(\rho+1)} \\ &\le C_B\|\nabla u_{1tt}\|\|\omega_t\|\left\|\int_0^{+\infty} g(s)\nabla\zeta^t(s)\,\mathrm{d}s\right\| \\ &\le C_B(1-l)^{\frac{1}{2}}\|\nabla u_{1tt}\|\|\omega_t\|\|\zeta^t\|_{\mathcal{M}} \le C_B'\|\nabla u_{1tt}\|\|\omega_t\|. \end{aligned} \tag{4.70}$$

Noting that

$$\begin{aligned} \int_0^{+\infty} g(s)\zeta_t^t(s)\,\mathrm{d}s &= \int_0^{+\infty} g(s)\,\mathrm{d}s\cdot\omega_t - \int_0^{+\infty} g(s)\zeta_s^t(s)\,\mathrm{d}s \\ &= (1-l)\omega_t + \int_0^{+\infty} g'(s)\zeta^t(s)\,\mathrm{d}s, \end{aligned}$$

then we see that

$$\begin{aligned} B_8 &\le -(1-l)\|\nabla\omega_t\|^2 - \int_0^{+\infty} g'(s)\|\nabla\omega_t(t)\|\|\nabla\zeta^t(s)\|\,\mathrm{d}s \\ &\le -\frac{1-l}{2}\|\nabla\omega_t\|^2 - \frac{1}{2(1-l)}\int_0^{+\infty} g'(s)\|\nabla\zeta^t(s)\|^2\,\mathrm{d}s, \end{aligned} \tag{4.71}$$

$$\begin{aligned} B_9 &\le -(1-l)\int_\Omega |u_t|^\rho\omega_t^2\,\mathrm{d}x + \|u_{1t}\|_{2(\rho+1)}^\rho\|\omega_t\|_{2(\rho+1)}\left\|\int_0^{+\infty} g'(s)\zeta^t(s)\,\mathrm{d}s\right\| \\ &\le -(1-l)\int_\Omega |u_t|^\rho\omega_t^2\,\mathrm{d}x + \delta\|\nabla\omega_t\|^2 - \frac{C_1\lambda^2 g(0)}{4\delta}\int_0^{+\infty} g'(s)\|\nabla\zeta^t(s)\|^2\,\mathrm{d}s. \end{aligned} \tag{4.72}$$

Plugging (4.64)-(4.72) into (4.63), we get

$$\begin{aligned} \Psi'_{\omega,p}(t) &\le C_B'\left(\|\nabla u_{1tt}\| + \|\nabla u_{2tt}\|\right)\|\omega_t\| + C_1\|\zeta^t\|_{\mathcal{M}}^2 + \delta\left(\|\omega_t\|^2 + \|\nabla p\|^2\right) \\ &\quad + C_1\|\sigma_1-\sigma_2\|^2 - \left(\frac{1-l}{2} - 2\delta\right)\|\nabla\omega_t\|^2 \\ &\quad - C_1\int_0^{+\infty} g'(s)\|\nabla\zeta^t(s)\|^2\,\mathrm{d}s - (1-l)\int_\Omega |u_{1t}|^\rho\omega_t^2\,\mathrm{d}x. \end{aligned} \tag{4.73}$$

On the other hand, we can get

$$\begin{aligned} \left|\Psi_{\omega,p}(t)\right| &\le (1-l)^{\frac{1}{2}}\|\nabla\omega_t\|\|\zeta^t\|_{\mathcal{M}} + \|u_{1t}\|_{\rho+2}^\rho\|\omega_t\|_{\rho+2}\left\|\int_0^{+\infty} g(s)\zeta^t(s)\,\mathrm{d}s\right\| \\ &\le (1-l)^{\frac{1}{2}}\|\nabla\omega_t\|\|\zeta^t\|_{\mathcal{M}} + C_1(1-l)^{\frac{1}{2}}\lambda\|\nabla u_{1t}\|^\rho\|\nabla\omega_t\|\|\zeta^t\|_{\mathcal{M}} \\ &\le C_B\left(\|\nabla\omega_t\|^2 + \|\zeta^t\|_{\mathcal{M}}^2\right) \le C_B E_{\omega,p}(t). \end{aligned} \tag{4.74}$$

Define

$$G_{\omega,p}(t) = ME_{\omega,p}(t) + \varepsilon\Phi_{\omega,p}(t) + \Psi_{\omega,p}(t), \tag{4.75}$$

which, together with (4.53) and (4.74), yields

$$(M - C_B\varepsilon - C_B)E_{\omega,p}(t) \le G_{\omega,p}(t) \le (M + C_B\varepsilon + C_B)E_{\omega,p}(t). \tag{4.76}$$

Now we take $\varepsilon > 0$ so small and M so large that

$$\frac{M}{2}E_\omega(t) \le G_\omega(t) \le 2ME_\omega(t). \tag{4.77}$$

Then for any $t \ge \tau$, we have

$$G'_{\omega,p}(t) \le -(l-\varepsilon)\varepsilon E_{\omega,p}(t) - \left(\frac{M}{2}\xi_2 - C_1\xi_2 - C_1\right)\|\zeta^t\|_{\mathcal{M}}^2$$

$$
\begin{aligned}
&-\left(\frac{M}{2}-C_{\varepsilon}-\delta\right)\|\nabla p\|^2-\left(\frac{l-\varepsilon}{2}-\delta\right)\|\nabla \omega\|^2 \\
&-\left(M+\frac{1-l}{2}-2\delta-C_{\varepsilon}\varepsilon\right)\|\nabla \omega_t\|^2 \\
&+C(B,M,\varepsilon)\left(\|\nabla u_{1tt}\|+\|\nabla u_{2tt}\|+\|\nabla p\|\right)\left(\|\omega_t\|+\|\omega\|\right) \\
&+C(B,M,\varepsilon)\left(\|\omega_t\|+\|\omega\|\right)\|\sigma_1-\sigma_2\|+C_1\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right).
\end{aligned} \tag{4.78}
$$

Now we take $\delta>0$ and $\varepsilon>0$ so small that

$$
M+\frac{1-l}{2}-2\delta-C_{\varepsilon}\varepsilon>0,\frac{l-\varepsilon}{2}-\delta>0,\frac{M}{2}-C_{\varepsilon}-\delta>0.
$$

For fixed ε and δ, we choose M so large that

$$
\frac{M}{2}\xi_2-C_1\xi_2-C_1>0.
$$

Then there exist some constant $\beta>0$ such that

$$
\begin{aligned}
G'_{\omega,p}(t)&\le-\beta E_{\omega,p}(t)+C_1\left(\|\nabla u_{1tt}\|+\|\nabla u_{2tt}\|+\|\nabla p\|\right)\left(\|\omega_t\|+\|\omega\|\right) \\
&\quad+C_1\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right)+C_1\|\sigma_1-\sigma_2\|\left(\|\omega_t\|+\|\omega\|\right) \\
&\le-\frac{\beta}{2M}G_{\omega,p}(t)+C_1\left(\|\nabla u_{1tt}\|+\|\nabla u_{2tt}\|\right)\left(\|\omega_t\|+\|\omega\|\right) \\
&\quad+C_1\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right)+C_1\|\sigma_1-\sigma_2\|\left(\|\omega_t\|+\|\omega\|\right).
\end{aligned} \tag{4.79}
$$

Integrating (4.79) over (τ,t) with respect to t, we derive

$$
\begin{aligned}
G_{\omega,p}(t)&\le G_{\omega,p}(\tau)\mathrm{e}^{-\frac{\beta}{2M}(t-\tau)}+C_1\int_\tau^t\mathrm{e}^{-\frac{\beta}{2M}(t-s)}\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right)\mathrm{d}s \\
&\quad+C_1\int_\tau^t\mathrm{e}^{-\frac{\beta}{2M}(t-s)}\left(\|\nabla u_{1tt}\|+\|\nabla u_{2tt}\|+\|p\|\right)\left(\|\omega_t\|+\|\omega\|\right)\mathrm{d}s \\
&\quad+C_1\int_\tau^t\mathrm{e}^{-\frac{\beta}{2M}(t-s)}\|\sigma_1-\sigma_2\|\left(\|\omega_t\|+\|\omega\|\right)\mathrm{d}s \\
&\le G_{\omega,p}(\tau)\mathrm{e}^{-\frac{\beta}{2M}(t-\tau)}+C_1\left(\int_\tau^t\left(\|\omega_t\|^2+\|\omega\|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}+C_1\int_\tau^t\|\sigma_1-\sigma_2\|^2\,\mathrm{d}s \\
&\quad+C_1\left(\int_\tau^t\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_\tau^t\left(\|\omega_t\|^2+\|\omega\|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.80}
$$

For any fixed $\varepsilon\in(0,1)$, we choose $T>\tau$ so large that

$$
G_{\omega,p}(\tau)\mathrm{e}^{-\frac{\beta}{2M}(T-\tau)}\le\varepsilon,
$$

which, together with (4.77) and (4.80), gives

$$
\begin{aligned}
E_{\omega,p}(t)&\le\varepsilon+C_1\left(\int_\tau^t\left(\|\omega_t\|^2+\|\omega\|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}+C_1\int_\tau^t\left(\|\sigma_1-\sigma_2\|^2+\|f_1-f_2\|^2\right)\mathrm{d}s \\
&\quad+C_1\left(\int_\tau^t\|\sigma_1-\sigma_2\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_\tau^t\left(\|\omega_t\|^2+\|\omega\|^2\right)\mathrm{d}s\right)^{\frac{1}{2}}
\end{aligned} \tag{4.81}
$$

Let

$$
\begin{aligned}
&\phi_T\left(\left(u_{01}^\tau,u_{11}^\tau,\theta_{01}^\tau,\eta_1^\tau\right),\left(u_{02}^\tau,u_{12}^\tau,\theta_{02}^\tau,\eta_2^\tau\right);G_1,G_2\right) \\
&=\int_\tau^t\int_\Omega(\sigma_1-\sigma_2)\,\omega_t\mathrm{d}x\mathrm{d}s+\int_\tau^t\int_\Omega(f_1-f_2)\,p\mathrm{d}x\mathrm{d}s
\end{aligned} \tag{4.82}
$$

Then

$$E_{\omega,p}(t)\le\varepsilon+\phi_T\left(\left(u_{01}^\tau,u_{11}^\tau,\theta_{01}^\tau,\eta_1^\tau\right),\left(u_{02}^\tau,u_{12}^\tau,\theta_{02}^\tau,\eta_2^\tau\right);G_1,G_2\right). \tag{4.83}$$

It suffices to show $\phi_T(\cdot,\cdot,\cdot,\cdot)\in Contr(B,\Sigma)$ for each fixed $T>\tau$. From the proof of existence theorem, we can deduce that for any fixed $T>\tau$, and the bound B depends on T,

$$\bigcup_{G\in\Sigma}\bigcup_{t\in[\tau,T]}U_G(t,\tau)B \tag{4.84}$$

is bounded in $\mathscr{H}$.

Let $(u_n,u_{nt},\theta_n,\eta_n^t)$ be the solutions corresponding to initial data $\left(u_{0n}^\tau,u_{1n}^\tau,\theta_{0n}^\tau,\eta_n^\tau\right)\in B$ with respect to symbol $G_n\in\Sigma, n=1,2,\cdots$. Then from (4.84), we get

$$u_n\to u \quad \star\text{-weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right), \tag{4.85}$$

$$u_{nt}\to u_t \quad \star\text{-weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right), \tag{4.86}$$

$$\theta_n\to \theta \quad \star\text{-weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right). \tag{4.87}$$

Taking $u_1=u_n$, $u_2=u_m$, $\theta_1=\theta_n$, $\theta_2=\theta_m$, $\sigma_1=\sigma_n$, $f_1=f_n$, $f_2=f_m$, $\sigma_2=\sigma_m$, noting that compact embedding $H_0^1(\Omega)\rightarrowtail L^2(\Omega)$, passing to a subsequence if necessary, we have

$$u_n \text{ and } u_{nt} \text{ converge strongly in } C\left([\tau,T];L^2(\Omega)\right).$$

Therefore we get

$$\int_\tau^T\left\|u_{nt}-u_{mt}\right\|^2\mathrm{d}s\to 0,\quad \text{as } m,n\to+\infty, \tag{4.88}$$

$$\int_\tau^T\left\|u_n-u_m\right\|^2\mathrm{d}s\to 0,\quad \text{as } m,n\to+\infty, \tag{4.89}$$

$$\int_\tau^T\left\|\theta_n-\theta_m\right\|^2\mathrm{d}s\to 0,\quad \text{as } m,n\to+\infty. \tag{4.90}$$

On the other hand, by $\sigma_m,\sigma_n,f_m,f_n\in\Sigma$, we see that

$$\int_\tau^T\left\|\sigma_n-\sigma_m\right\|^2\mathrm{d}s\to 0,\quad \text{as } m,n\to+\infty, \tag{4.91}$$

$$\int_\tau^T\left\|f_n-f_m\right\|^2\mathrm{d}s\to 0,\quad \text{as } m,n\to+\infty. \tag{4.92}$$

Hence it follows from (4.88)-(4.92)

$$\phi_T\left(\left(u_n,u_{nt},\theta_n,\eta_n^t\right),\left(u_m,u_{mt},\theta_m,\eta_m^t\right);G_n,G_m\right)\to 0 \quad \text{as } m,n\to+\infty, \tag{4.93}$$

that is, $\phi_T\in Contr(B,\Sigma)$.

Therefore by Lemma 3.1, the semigroup $\{U_G(t,\tau)\}(t\ge\tau>0,G\in\Sigma)$ is uniformly asymptotically compact and the proof is now complete.

Proof of Theorem 2.3. Combining Theorems 4.1-4.2, we can complete the proof of Theorem 2.3.

Acknowledgements

Shanghai Polytechnical University and the key discipline Applied Mathematics of Shanghai Polytechnic University with contract number XXKPY1604.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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