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Some Important Properties of Multiple G-Itô Integral in the G-Expectation Space

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Abstract

In the G-expectation space, we propose the multiple Itô integral, which is driven by multi-dimensional G-Brownian motion. We prove the recursive relationship of multiple G-Itô integrals by G-Itô formula and mathematical induction, and we obtain some computational formulas for a kind of multiple G-Itô integrals.

Keywords

G-Brownian Motion, Multiple G-Itô Integrals, G-Itô Formula, Recursive Relationship

1. Introduction

With the rapid development of financial markets, traditional linear expectations cannot explain its uncertainty sometimes. In 2007, Peng [1] introduced a new sublinear expectation–G-expectation, and he introduced G-normal distribution and G-Brownian motion under the G-expectation framework. In 2008, Peng [2] proved the law of large numbers and the central limit theorem under the sublinear expectation, and he defined the Itô integral about G-Brownian motion. Later, Peng [3] obtained the G-Itô formula and proved the existence and uniqueness of solution for the stochastic differential equations driven by G-Brownian motion (G-SDEs for short) and the backward stochastic differential equations driven by G-Brownian motion (G-BSDEs for short).

Since then, G-expectation space and the applications of G-Itô integral have been extensively studied by many researchers. In 2014, Hu, Ji, Peng and Song [4] studied the comparison theorem, nonlinear Feynman-Kac formula and Girsanov transformation of G-BSDE. In 2016, Hu, Wang and Zheng [5] obtained the Ito-Krylov formula under the G-expectation framework. Then they proved the reflection principle of G-Brownian motion, and they got the reflection principle of G-Brownian motion by Krylov's estimate in [6]. [7] studied rough path properties of stochastic integrals of Itô's type and Stratonovich's type with respect to G-Brownian motion. Then, Hu, Ji and Liu [8] studied the strong Markov property for G-SDEs in 2017. Wu [9] introduced the multiple Itô integrals driven by one-dimensional Brownian motion in G-expectation space. He also obtained the relationship between Hermite polynomials and multiple G-Itô integral. In 2012, Yin [10] introduced the Stratonovich integral with respect to G-Brownian motion, and she also researched the properties of G-Stratonovich integrals. In 2014, Sun [11] studied multiple stochastic integrals under one-dimensional G-Brownian motion and developed the $\mathbb{L}^p$ estimation of maximal inequalities for n iterated integrals by the property of Hermite polynomials. The more contents about multiple random integrals can be found in the literature [12].

A nature question is how to define and calculate the multiple G-Itô integral of multi-dimensional G-Brownian motion. This problem will be solved in this paper. We define multiple Itô integrals driven by multi-dimensional G-Brownian motion under G-expectation space. And we prove the recursive relationship between multiple G-Itô integrals strictly by using G-Itô formula and mathematical induction method. Then we obtain some important formulas for calculating multiple G-Itô integrals and make some preparations for further study on scientific calculation of G-SDEs.

The remainder of this paper is organized as follows: In Section 2, we introduce some concepts and lemmas such as G-Brownian motion, G-Itô formula and so on. In Section 3, we define multiple Itô integrals driven by multi-dimensional G-Brownian motion, and prove the recursive relationship between multiple G-Itô integrals. Then we give some important formulas for calculating multiple G-Itô integrals. Finally, several concluding remarks are given in Section 4.

2. Preliminaries and Notation

In this section, we will give some basic theories about G-Brownian motion and multi-indices. Some more details can be found in literatures [1–3] and [12]. Let Ω be a given set, and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. For each c we suppose that $c \in \mathcal{H}$, and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\mathcal{H}$ can be considered as the space of random variables.

2.1. G-Brownian Motion and G-Itô Formula

Firstly, we introduce some notations about G-Brownian motion.

Definition 1. *[3] A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied:*

(i) $B_0(\omega) = 0$;

(ii) For each $t, s \geq 0$, the incremen $B_{t+s} - B_t \sim N(0 \times s\sum)$ is independent from $\{B_{t_1}, B_{t_2}, \ldots, B_{t_n}\}$, for each $n \in N, 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$.

Let $G(\cdot) : \mathbb{S}(d) \to \mathcal{R}$ be a given monotonic and sublinear function. We denote by $\mathbb{S}(d)$ the collection of all $d \times d$ symmetric matrices.There exists a bounded, convex and closed subset $\sum \subset \mathbb{S}_+(d) = \{\theta \in \mathbb{S}(d), \theta \geq 0\}$, such that $G(A) = \frac{1}{2}\sup\limits_{B\in\sum}(A, B), A \in \mathbb{S}(d)$.

In the following sections, we denote by $\Omega = C_0^d(\mathcal{R}^+)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in\mathcal{R}^+}$, with $\omega_0 = 0$, equipped with the distance $\rho(\omega^1,\omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]}|\omega_t^1-\omega_t^2|)\wedge 1]$. For each fixed $T\in[0,\infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega\in\Omega\}$,

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1\wedge T},\ldots,B_{t_n\wedge T}) : n\in\mathbb{N}, t_1,\ldots,t_n\in[0,\infty), \varphi\in C_{l,lip}(\mathbb{R}^n)\},$$

$$L_{ip}(\Omega) := \cup_{n=1}^{\infty} L_{ip}(\Omega_n),$$

where B_t is a canonical process, that is $B_t(\omega)=\omega_t$. For a given $p\geq 1$, we also denote $L_G^p(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

We recall some important notions about G-Itô formula, product rule and so on (see [3]).

Definition 2. *[3] We denote the set of simple process*

$$M_G^{p,0}([0,T]) := \Big\{\eta_t(\omega) := \sum_{j=0}^{N-1}\xi_{t_j}(\omega)I_{[t_j,t_{j+1})}(t); \xi_{t_j}(\omega)\in L_G^p(\Omega_{t_j}),$$
$$\forall N\geq 1, 0=t_0<\ldots<t_N=T, j=0,1,\ldots,N-1\Big\}.$$

And for each $p\geq 1$, we denote by $M_G^p([0,T])$ the completion of $M_G^{p,0}([0,T])$ under the norm

$$\|\eta\|_{M_G^p([0,T])} = \{\mathbb{E}\int_0^T|\eta_t|^p dt\}^{\frac{1}{p}}.$$

Definition 3. *[3] For each $\eta\in M_G^{2,0}([0,T])$, we define the Itô integral of G-Brownian motion is as follows:*

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1}\xi_{t_j}(\omega)(B_{t_{j+1}}-B_{t_j}).$$

Definition 4. *[3] We first consider the quadratic variation process of one-dimensional G-Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \doteq N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$. Let $\pi_t^N, N=1,2,\ldots,$ be a sequence of partitions of $[0,t]$. We consider*

$$B_t^2 = \sum_{j=0}^{N-1}(B_{t_{j+1}^N}^2 - B_{t_j^N}^2)$$
$$= \sum_{j=0}^{N-1}2B_{t_j^N}(B_{t_{j+1}^N}-B_{t_j^N}) + \sum_{j=0}^{N-1}(B_{t_{j+1}^N}-B_{t_j^N})^2.$$

As $\mu(\pi_t^N)\to 0$, the first term of the right side converges to $2\int_0^t B_s dB_s$ in $L_G^2(\Omega)$. The second term must be convergent. We denote its limit by $\langle B\rangle_t$, i.e.,

$$\langle B\rangle_t := \lim_{\mu(\pi_t^N)\to 0}\sum_{j=0}^{N-1}(B_{t_{j+1}^N}-B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

By the above construction, $(\langle B\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. We call it the quadratic variation process of the G-Brownian motion B.

Now let us introduce the following two important lemmas.

Lemma 1. *[9] We denote B_t be a m-dimensional G-Brownina motion. Let $\Phi \in C^2(\mathbb{R}^n)$ be bounded with bounded derivatives and $\partial^2_{x^i x^j}\Phi$ are uniformly Lipschitz. Let $s \in [0,T]$ be fixed and let X_t^i be the i $(i = 1,\ldots,d)$-th component of $X_t = (X_t^1,\ldots,X_t^d)^\top$ satisfying*

$$X_t^i = X_0^i + \int_0^t a_s^i ds + \sum_{j=1}^m \int_0^t \eta_s^{i,j} d\langle B^j\rangle_s + \sum_{j=1}^m \int_0^t \sigma_s^{i,j} dB_s^j,$$

where a^i be the i-th of $a = (a^1,\ldots,a^d)^\top$, $\eta^{i,j}$ and $\sigma^{i,j}$ is the lines i-th and j-th of $\eta = (\eta^{i,j})_{d\times m}$ and $\sigma = (\sigma^{i,j})_{d\times m}$, and they are bounded process on $M_G^2(0,T)$. For $t,s \geq 0$, then we have

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) = &\sum_{i=1}^d [\int_s^t \partial_{x^i}\Phi(X_u)a_u^i du + \sum_{j=1}^m \int_s^t \partial_{x^i}\Phi(X_u)\sigma_u^{i,j} dB_u^j] \\ &+ \int_s^t [\sum_{i=1}^d \sum_{j=1}^m \partial_{x^i}\Phi(X_u)\eta_u^{i,j} \\ &+ \frac{1}{2}\sum_{i,l=1}^d \sum_{j=1}^m \partial^2_{x^i x^j}\Phi(X_u)\sigma_u^{i,j}\sigma_u^{l,j}] d\langle B^j\rangle_u.\end{aligned}$$

Lemma 2. *[1, 3] In G-expectation space, the following product rule is established:*

$$dB_t^i dB_t^j = \delta_{ij} = \begin{cases} d\langle B^i\rangle_t, & i = j \\ 0, & i \neq j \end{cases},$$

$$dt\,dt = 0,\ \ dt\,d\langle B\rangle_t = 0,\ \ dt\,dB_t = 0,\ \ d\langle B\rangle_t\,dt = 0,\ \ d\langle B\rangle_t\,d\langle B\rangle_t = 0,$$
$$d\langle B\rangle_t\,dB_t = 0,\ \ dB_t\,dt = 0,\ \ dB_t\,d\langle B\rangle_t = 0,\ \ dB_t\,dB_t = d\langle B\rangle_t.$$

2.2. Multi-Indices

Let us introduce some notations about multi-indices for simplify statements and proof. We shall call a row vector $\alpha = (j_1, j_2, \ldots, j_l)$, where $j_i \in \{-m, -(m-1), \ldots, -1, 0, 1, 2, \ldots, m\}, i \in \{1, 2, \ldots, l\}$ and $m, l = 1, 2, 3, \ldots.$ a multi-index of length $l := l(\alpha) \in \{1, 2, \ldots\}$.

Definition 5. *[12] We denote the set of all multi-indices by $\mathcal{M}$, so*

$$\mathcal{M} = \{(j_1, j_2, \ldots, j_l) : j_i \in \{-m, -(m-1), \ldots, -1, 0, 1, 2, \ldots, m\}, \\ i \in \{1, 2, \ldots, l\}, l \in \{1, 2, 3, \ldots\}\} \cup \{v\},$$

where v is the multi-index of length zero.

We write $n(\alpha)$ for the number of components of a multi-index α that are equal to 0 and $s(\alpha)$ for the number of components of a multi-index α that are equal to -1. Moreover, we write $\alpha-$ for the multi-index obtained by deleting the first component of α and $-\alpha$ for the multi-index obtained by deleting the first component of α. $\alpha - (j)$ for the multi-index obtained by deleting the last component of $\alpha = (j_1, j_2, \ldots, j_k, j)$ so we can get the multi-index $(j_1, j_2, \ldots, j_k)$. Additionally, given two multi-indices $\alpha_1 = (j_1, j_2, \ldots, j_k)$ and $\alpha_2 = (i_1, i_2, \ldots, i_l)$, we introduce the concatenation operator $*$ on $\mathcal{M}$ defined by

$$\alpha_1 * \alpha_2 = (j_1, \ldots, j_k, i_1, \ldots, i_l),$$

where $\alpha_1, \alpha_2 \in \mathcal{M}$. The operator allows us to combine two multi-indices. For instance, assuming $m = 2$ one obtains

$$l((0,-1,1)) = 3,\ n((0,1,-1,2,0)) = 2,\ s((0,1,-1,2,0)) = 1,$$
$$(0,-1,1)- = (0,-1),\ (0,1,-1) * (0,2) = (0,1,-1,0,2).$$

3. Main Results

In this section, by a component $j \in \{1, 2, \ldots, m\}$ of a multi-index we will denote in a multiple stochastic integral the integration with respect to the j-th Wiener process. A component $j = 0$ will denote integration with respect to time. Lastly, a component $j \in \{-m, -(m-1), \ldots, -1\}$ refer to an integration with respect to quadratic variation process. We shall define three sets of adapted right continuous stochastic processes $g = \{g(t, \omega), t \in [0, T]\}$ with left hand limits.

$$\mathcal{H}_v = \{g : \sup_{t \in [0,T]} \mathbb{E}(|g(t, \omega)|) < \infty\};$$

$$\mathcal{H}_{(0)} = \{g : \mathbb{E}(\int_0^T |g(t, \omega)| ds) < \infty\};$$

$$\mathcal{H}_{(j)} = \{g : \mathbb{E}(\int_0^T |g(s, \omega)|^2 ds) < \infty\},$$

where $j \in \{1, 2, \ldots, m\}$.

Definition 6. *Let ϱ and τ be two stopping times with $0 \le \varrho \le \tau \le T$ a.s.. Then for a multi-index $\alpha \in \mathcal{M}$ and a process $g(\cdot) \in \mathcal{H}_\alpha$, we define the multiple $G-$Itô integral $I_\alpha[g(\cdot)]_{\varrho,\tau}$ recursively by*

$$I_\alpha[g(\cdot)]_{\varrho,\tau} = \begin{cases} g(\tau), & l = 0, \\ \int_\varrho^\tau I_{\alpha-}[g(\cdot)]_{\varrho,\tau} dz, & l \ge 1 j_l = 0, \\ \int_\varrho^\tau I_{\alpha-}[g(\cdot)]_{\varrho,\tau} dB_z^{j_l}, & l \ge 1 j_l \in \{1, 2, \ldots, m\}, \\ \int_\varrho^\tau I_{\alpha-}[g(\cdot)]_{\varrho,z} d\langle B^{-j_l}\rangle_z, & l \ge 1 j_l \in \{-m, -(m-1), \ldots, -1\}, \end{cases} \tag{1}$$

where $g(\cdot) = g(\cdot, v_1, \ldots, v_{s(\alpha)})$.

We use the following example to illustrate Definition 6: $I_v[g(\cdot)]_{0,t} = g(t), I_{(0)}[g(\cdot)]_{0,t} = \int_0^t g(z)dz$,

$$I_{(1)}[g(\cdot)]_{\rho,\tau} = \int_\rho^\tau g(z) dB_z,$$

$$I_{(2,0)}[g(\cdot)]_{\rho,\tau} = \int_\rho^\tau \int_\rho^{z_2} g(z_1) dB_{z_1}^2 dz_2,$$

$$I_{(0,-1)}[g(\cdot)]_{\rho,\tau} = \int_\rho^\tau \int_\rho^{z_2} g(z_1) dz_1 d\langle B^1\rangle_{z_2}.$$

For a multi-index $\alpha = (j_1, j_2, \ldots, j_l) \in \mathcal{M}$ and $l(\alpha) > 1$, we define the set $\mathcal{H}_\alpha$ to be the totality of adapted right continuous processes $g = \{g(t), t \ge 0\}$ with left hand limits such that the integral process $\{I_\alpha[g(\cdot)]_{\rho,t}, t \in [0, T]\}$ considered as a function of t satisfies $I_\alpha[g(\cdot)]_{\rho,\cdot} \in \mathcal{H}_{(j_l)}$. For convenience we write $I_{\alpha,t} = I_\alpha[1]_{0,t}$ and $B_t^0 = t$ for $\alpha \in \mathcal{M}, t \ge 0$.

Now, we will give our main theorems.

Theorem 1. *For multi-index $\alpha^n = (j_1, j_2, \ldots, j_n), j_i \in \{1, 2, \ldots, m\}$, where $j_1, j_2, \ldots, j_n$ are not equal with each other. The set $C(\alpha^n)$ be the all of the n level arrangement of α^n. We define*

$$C(\alpha^n) = \{(a_1, a_2, \ldots, a_n) | a_i \in \{j_1, j_2, \ldots, j_n\}, i = 1, \ldots, n, 2 \le n \le m\},$$

such that

$$H_{C(\alpha^n)} = \sum_{\alpha \in C(\alpha^n)} I_{\alpha,t} = \prod_{i=1}^n B_t^{j_i}.$$

Proof. For $n = 2$, we have $I_{(i,j),t} + I_{(j,i),t} = \int_0^t \int_0^s dB_r^i dB_s^j + \int_0^t \int_0^s dB_r^j dB_s^i = B_t^i B_t^j$;

For $n = k$ we have $H_{C(\alpha^k)} = \sum_{\alpha \in C(\alpha^k)} I_{\alpha,t} = \prod_{i=1}^{k} B_t^{j_i}$. We need to prove that

$$H_{C(\alpha^{k+1})} = \sum_{\alpha \in C(\alpha^{k+1})} I_{\alpha,t} = \prod_{i=1}^{k+1} B_t^{j_i}.$$

Actually, we only need to prove that

$$\sum_{l=1}^{k+1} \int_0^t H_{C(\alpha^{k+1}-(j_l)),t} dB_t^{j_l} = \sum_{l=1}^{k+1} \int_0^t \prod_{i=1,i\neq l}^{k} B_t^{j_i} dB_t^{j_l} = \prod_{i=1}^{k+1} B_t^{j_i}. \quad (2)$$

where $\alpha = (j_1, j_2, \ldots, j_k, j)$ and $\alpha^{k+1} - (j_l)$ for the k-index obtained by deleting the last component j_l of α^{k+1}. Applying G-Itô formula and independence of Brown motion, one has

$$d \prod_{i=1}^{k+1} B_t^{j_i} = \sum_{l=1}^{k+1} \prod_{i=1,i\neq l}^{k} B_t^{j_i} dB_t^{j_l}. \quad (3)$$

Taking integral on Equation (3) and combined with Equation (2), the proof is completed.

Example 1. *For $i, j, k \in \{1, 2, 3, \ldots, m\}$, and i, j, k are different with each other. Using G-Itô formula and Theorem 1, we can get*

$$\begin{aligned} &I_{(i,j,k),t} + I_{(j,i,k),t} + I_{(k,i,j),t} + I_{(i,k,j),t} + I_{(k,j,i),t} + I_{(j,k,i),t} \\ =&\int_0^t B_z^i B_z^j dB_z^k + \int_0^t B_z^k B_z^i dB_z^j + \int_0^t B_z^k B_z^j dB_z^i \\ =&B_t^i B_t^j B_t^k. \end{aligned}$$

Now we shall prove the recursive relationship between multiple G-Itô integrals.

Theorem 2. *Let $j_1, \ldots, j_l \in \{0, 1, \ldots, m\}$ and $\alpha = (j_1, \ldots, j_l) \in \mathcal{M}$, where $l = 1, 2, 3, \ldots$ Then for $t \geq 0$,*

$$B_t^j I_{\alpha,t} = \sum_{i=0}^{l} I_{(j_1,\ldots,j_i,j,j_{i+1},\ldots,j_l),t} + \sum_{i=1}^{l} I_{\{j_i=j\neq 0\}} I_{(j_1,\ldots,j_{i-1},-j_i,j_{i+1},\ldots,j_l),t}. \quad (4)$$

Proof. We consider multi linear G-Itô process $X = \{X_t, t \geq 0\}$, which defined as follows

$$\begin{aligned} X_t =&(X_t^{(0)}, \ldots, X_t^{(m)}, X_t^{(j_1,j_2)}, \ldots, X_t^{(j_1,j_2,j_3)}, \ldots, X_t^{(j_1,\ldots,j_l)})^\top \\ =&(I_{(0),t}, \ldots, I_{(m),t}, I_{(j_1,j_2),t}, \ldots, I_{(j_1,j_2,j_3),t}, \ldots, I_{(j_1,j_2,\ldots,j_l),t})^\top, \end{aligned} \quad (5)$$

where each component of X_t is a multi G-Itô integral. For β-th component $\beta(j_1^{'}, \ldots, j_r^{'})$ the coefficients are

$$a^\beta = \begin{cases} x^{\beta-}, & j_r^{'} = 0 \\ 0, & otherwise \end{cases}, b^{\beta,j} = \begin{cases} x^{\beta-}, & j = j_r^{'} \in \{1, \ldots, m\} \\ 0, & otherwise \end{cases},$$

$$c^\beta = \begin{cases} x^{\beta-}, & j = j_r^{'} \in \{-m, -(m-1) \ldots, -1\} \\ 0, & otherwise \end{cases}.$$

By Definition 4 and G-Itô formula, one have

$$B_t^j I_{\alpha,t} = I_{(j),t} I_{\alpha,t} = \int_0^t I_{\alpha,s} dI_{(j),s} + \int_0^t I_{(j),s} I_{\alpha-,s} dB_s^{j_l} + I_{\{j_l = j \neq 0\}} \int_0^t I_{\alpha-,s} d\langle B^{j_l} \rangle_s. \tag{6}$$

For $l = 1$, Equation (6) is established obviously. For $l \geq 2$ we have

$$B_t^j I_{\alpha,t} = I_{(j_1,\ldots,j_l,j),t} + \int_0^t I_{(j),s} I_{\alpha-,s} dB_s^{j_l} + I_{\{j_l = j \neq 0\}} I_{(j_1,\ldots,j_{l-1},-j_l),t}.$$

The proof is completed.

Example 2. *Particularly, for $j = 1, \alpha = (0,1)$, from Theorem 2 it follows that*

$$\begin{aligned} B_t^1 I_{(0,1),t} &= I_{(j,j_1,j_2),t} + I_{(j_1,j,j_2),t} + I_{(j_1,j_2,j,),t} + I_{\{j_2 = j = 1 \neq 0\}} \cdot I_{(j_1,-j_2),t} \\ &= I_{(1,0,1),t} + 2I_{(0,1,1),t} + I_{(0,-1),t}; \end{aligned}$$

For $j = 2, \alpha = (0,1,3)$, applying the Theorem 2 we can get

$$\begin{aligned} B_t^2 I_{(0,1,3),t} &= I_{(j,j_1,j_2,j_3),t} + I_{(j_1,j,j_2,j_3),t} + I_{(j_1,j_2,j,j_3),t} + I_{(j_1,j_2,j_3,j),t} \\ &\quad + I_{\{j_3 \neq j\}} \cdot I_{(j_1,j_2,-j_3),t} \\ &= I_{(2,0,1,3),t} + I_{(0,2,1,3),t} + I_{(0,1,2,3),t} + I_{(0,1,3,2),t} + 0 \\ &= I_{(2,0,1,3),t} + I_{(0,2,1,3),t} + I_{(0,1,2,3),t} + I_{(0,1,3,2),t}. \end{aligned}$$

4. Concluding Remarks and Future Work

In this work, we define G-Itô integral driven by multi-dimensional G-Brownian motion in G-expectation space. And we use G-Itô formula and mathematical induction to obtain a kind of multiple G-Itô integrals. As discussed in Section 1, this effort focuses on multiple G-Itô integrals driven by multi-dimensional G-Brownian motion rather than one-dimensional G-Brownian motion. Our future efforts will focus on introducing the properties of Stratonovich integral driven by multi-dimensional G-Brownian motion, and exploring the relationship between Stratonovich integral and G-Itô integral under the G-expectation framework.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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