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Asymptotic Stability of the Dynamic Solution of an N-Unit Series System with Finite Number of Vacations

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Abstract

We investigate an N-unit series system with finite number of vacations. By analyzing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator we prove that the dynamic solution converges strongly to the steady state solution. Thus we obtain asymptotic stability of the dynamic solution of the system.

Keywords

N-Unit Series System, C_0-Semigroup, Irreducibility, Asymptotic Stability

1. Introduction

The series repairable systems are the classical repairable systems in reliability theory. As a result of the strong practical background of series repairable systems, many researchers have studied them extensively under varying assumptions (see [1] [2] [3] [4]). In [4], the authors studied an N-unit series system with finite number of vacations and obtained some reliability expressions such as the Laplace transform of the reliability, the mean time to the first failure, the availability and the failure frequency of the system by using the supplementary variable method and the generalized Markov progress method as well as the Laplace transform. The authors used the dynamic solution and its asymptotic stability in calculating the availability and the reliability. But they did not prove the existence of the dynamic solution and the asymptotic stability of the dynamic solution. Motivated by this, A. Osman and A. Haji proved in [5] the existence of a unique positive dynamic solution of the system by using C_0-semigroup theory of linear operators. In this paper, we further study this system and prove that the dynamic solution of the system converges strongly to its steady state solution by analyzing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator; thus we obtain the asymptotic stability of the dynamic solution of this system.

The rest of this paper is organized as follows: In Section 2, we present the mathematical model of the system and give some results obtained in [5]. In Section 3, we obtain main result on stability of the system by analyzing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator.

2. Previous Results

According to [4], the N-unit series system with finite number of vacations can be described by the following integro-differential equations:

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t}+\Lambda\right)p_0(t)=\int_0^{+\infty}r(\omega)p_{0M}(t,\omega)\mathrm{d}\omega, \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_k(y)\right)p_k(t,y)=0, k=1,2,\cdots,n, \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \omega}+r(\omega)\right)p_{kj}(t,\omega)=\lambda_i p_{0j}(t,\omega), k=1,2,\cdots,n, j=1,2,\cdots,M, \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \omega}+\Lambda+r(\omega)\right)p_{0j}(t,\omega)=0, j=1,2,\cdots,M, \end{cases} \tag{1}$$

With the boundary conditions

$$\begin{cases} p_k(t,0)=\lambda_k p_0(t)+\sum_{j=1}^{M}\int_0^{\infty}r(\omega)p_{kj}(t,\omega)\mathrm{d}\omega, k=1,2,\cdots,n, \\ p_{kj}(t,0)=0, k=1,2,\cdots,n, j=1,2,\cdots,M, \\ p_{01}(t,0)=\sum_{k=1}^{n}\int_0^{\infty}\mu_k(y)p_k(t,y)\mathrm{d}y, \\ p_{0j}(t,0)=\int_0^{\infty}r(\omega)p_{0,j-1}(t,\omega)\mathrm{d}\omega, j=2,\cdots,M, \end{cases} \tag{2}$$

and the initial conditions

$$\begin{cases} p_0(0,\omega)=1, \\ p_k(0,y)=0, k=1,2,\cdots,n, \\ p_{kj}(0,\omega)=0, k=1,2,\cdots,n, j=1,2,\cdots,M \\ p_{0j}(0,\omega)=0, j=1,2,\cdots,M, \end{cases} \tag{3}$$

where $\Lambda=\lambda_1+\lambda_2+\cdots+\lambda_n$.

Here $(t,y)\in[0,\infty)\times[0,\infty)$; $(t,\omega)\in[0,\infty)\times[0,\infty)$ and the symbols in the equations have the following meaning.

$p_0(t)$: The probability that n units at time t are in working state and the repairman is idle;

$p_k(t,y)\mathrm{d}y$: The probability that at time t the repairman is repairing the failed unit $k(k=1,2,\cdots,n)$, the elapsed repair time lies in $[y,y+\mathrm{d}y)$;

$p_{0j}(t,\omega)\mathrm{d}\omega$: The probability that n units at time t are in working state, the repairman is in $j(j=1,2,\cdots,M)$ vacation and the elapsed repair time lies in $[\omega,\omega+\mathrm{d}\omega)$;

$p_{kj}(t,\omega)\mathrm{d}\omega$: The probability that at time t unit $k(k=1,2,\cdots,n)$ is waiting for repair, the repairman is in $j\left(j=1,2,\cdots,M\right)$ vacation and the elapsed repair time lies in $\left[\omega,\omega+\mathrm{d}\omega\right)$;

$\lambda_k\left(k=1,2,\cdots,n\right)$ is positive constant; $r\left(\omega\right)$ is the vacation rate function of the repairman;

$\mu_k\left(y\right)$ is the repair rate function of unit k $\left(k=1,2,\cdots,n\right)$.

Throughout the paper we require the following assumption for the vacation rate function $r\left(\omega\right)$ and the repair rate functions $\mu_k\left(y\right)\left(k=1,2,\cdots,n\right)$.

General Assumption 2.1: The functions $r\left(\omega\right)$ and $\mu_k\left(y\right)$: $\left[0,+\infty\right)\to\left[0,+\infty\right)$ $\left(k=1,2,\cdots,n\right)$ are measurable and bounded such that

$$r=\lim_{\omega\to+\infty}r\left(\omega\right)>0,\mu_k=\lim_{y\to+\infty}\mu_k\left(y\right)>0,\mu_\infty=\min\left(r,\mu_k\right)\left(k=1,2,\cdots,n\right).\quad(4)$$

In [5], the authors transformed the system (1), (2) and (3) into the following abstract Cauchy problem ([6], Def.II.6.1) on the Banach space $\left(X,\|\cdot\|\right)$.

$$\begin{cases}\dfrac{\mathrm{d}p\left(t\right)}{\mathrm{d}t}=Ap\left(t\right),t\in\left[0,+\infty\right),\\ p\left(0\right)=\left(1,0,0,\cdots,0\right)^{\mathrm{T}}\in X,\end{cases}\quad(5)$$

where

$$X=\mathbb{C}\times\left(L_y^1\left[0,+\infty\right)\right)^n\times\left(L_\omega^1\left[0,+\infty\right)\right)^{(n+1)\times M}\text{ with norm}$$

$$\|p\|=\left|p_0\right|+\sum_{i=1}^n\left\|p_i\right\|_{L_y^1[0,+\infty)}+\sum_{i=1}^n\sum_{1=1}^M\left\|p_{ij}\right\|_{L_\omega^1[0,+\infty)}+\sum_{1=1}^M\left\|p_{0j}\right\|_{L_\omega^1[0,+\infty)}\quad(6)$$

$$\begin{aligned}p=\big(&p_0,p_1\left(y\right),p_2\left(y\right),\cdots,p_n\left(y\right),p_{11}\left(\omega\right),p_{12}\left(\omega\right),\cdots,p_{1M}\left(\omega\right),\\&p_{21}\left(\omega\right),p_{22}\left(\omega\right),\cdots,p_{2M}\left(\omega\right),\cdots,p_{n1}\left(\omega\right),p_{n2}\left(\omega\right),\cdots,p_{nM}\left(\omega\right),\\&p_{01}\left(\omega\right),p_{02}\left(\omega\right),\cdots,p_{0M}\left(\omega\right)\big)^{\mathrm{T}}\in X,\end{aligned}\quad(7)$$

$$Ap=A_mp,D\left(A\right)=\left\{p\in D\left(A_m\right)\mid Lp=\Phi p\right\},$$

$$A_m=\begin{pmatrix}
-\Lambda & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \varphi_M\\
0 & D_1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & D_2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & D_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & D_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & D_{12} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \lambda_1 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_{1M} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda_1\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_{21} & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda_2 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & D_{22} & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \lambda_2 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & D_{2M} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda_2\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & D_{n1} & 0 & \cdots & 0 & \lambda_n & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & D_{n2} & \cdots & 0 & 0 & \lambda_n & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & D_{nM} & 0 & 0 & \cdots & \lambda_n\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & D_{01} & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D_{02} & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_{0M}
\end{pmatrix}\quad(8)$$

$$\varphi_M:f\mapsto\varphi_M\left(f\right)=\int_0^{+\infty}r\left(\omega\right)f\left(\omega\right)\mathrm{d}\omega,\quad(9)$$

$$D_i=-\frac{\mathrm{d}}{\mathrm{d}y}-\mu_i\left(y\right),k=1,2,\cdots,n,\quad(10)$$

$$D_{kj} = -\frac{\mathrm{d}}{\mathrm{d}\omega} - r(\omega), k = 1,2,\cdots,n, j = 1,2,\cdots,M, \quad (11)$$

$$D_{0j} = -\left[\frac{\mathrm{d}}{\mathrm{d}\omega} + \Lambda + r(\omega)\right], j = 1,2,\cdots,M, \quad (12)$$

$$L : D(A_m) \to \partial X, \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix} \to L \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix} = \begin{pmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_n(0) \\ p_{11}(0) \\ p_{12}(0) \\ \vdots \\ p_{1M}(0) \\ p_{21}(0) \\ p_{22}(0) \\ \vdots \\ p_{2M}(0) \\ \vdots \\ p_{n1}(0) \\ p_{n2}(0) \\ \vdots \\ p_{nM}(0) \\ p_{01}(0) \\ p_{02}(0) \\ \vdots \\ p_{0M}(0) \end{pmatrix} \quad (13)$$

$$\Phi LX \to \partial X, \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix} \to \Phi \begin{pmatrix} p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega) \end{pmatrix}$$

$$
= \begin{pmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 & \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1M} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2M} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nM} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \varphi_1 & \varphi_2 & \cdots & \varphi_n & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \varphi_{01} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \varphi_{0M-1} & 0
\end{pmatrix}
\begin{pmatrix}
p_0 \\ p_1(y) \\ p_2(y) \\ \vdots \\ p_n(y) \\ p_{11}(\omega) \\ p_{12}(\omega) \\ \vdots \\ p_{1M}(\omega) \\ p_{21}(\omega) \\ p_{22}(\omega) \\ \vdots \\ p_{2M}(\omega) \\ \vdots \\ p_{n1}(\omega) \\ p_{n2}(\omega) \\ \vdots \\ p_{nM}(\omega) \\ p_{01}(\omega) \\ p_{02}(\omega) \\ \vdots \\ p_{0M}(\omega)
\end{pmatrix}
\tag{14}
$$

$$\varphi_{kj} : f \mapsto \varphi_{kj}(f) = \int_0^{+\infty} r(\omega) f_{kj}(\omega) \mathrm{d}\omega, k = 1,2,\cdots,n, j = 1,2,\cdots,M \tag{15}$$

$$\varphi_k : f \mapsto \varphi_k(f) = \int_0^{+\infty} \mu_k(y) f_k(y) \mathrm{d}y, k = 1,2,\cdots,n, \tag{16}$$

$$\varphi_{0j} : f \mapsto \varphi_{0j}(f) = \int_0^{+\infty} r(\omega) f_{0j-1}(\omega) \mathrm{d}\omega, j = 2,\cdots,n \tag{17}$$

and proved the following results.

Theorem 2.1: The operator $(A, D(A))$ generates a positive contraction C_0-semigroup $(T(t))_{t\geq 0}$.

Theorem 2.2: The system (1), (2) and (3) has a unique positive dynamic solution

$$
\begin{aligned}
p(t) = \big(&p_0(t), p_1(t,y), p_2(t,y), \cdots, p_n(t,y), p_{11}(t,\omega), p_{12}(t,\omega), \cdots, \\
&p_{1M}(t,\omega), p_{21}(t,\omega), p_{22}(t,\omega), \cdots, p_{2M}(t,\omega), \cdots, p_{n1}(t,\omega), \\
&p_{n2}(t,\omega), \cdots, p_{nM}(t,\omega), p_{01}(t,\omega), p_{02}(t,\omega), \cdots, p_{0M}(t,\omega)\big)^{\mathrm{T}} \in X
\end{aligned}
\tag{18}
$$

which can be expressed as

$$p(t) = T(t) p(0). \tag{19}$$

3. The Main Result

To prove our main result on the asymptotic stability of the dynamic solution of system, we first prove the some lemmas. In [7], A. Haji and A. Radl gave the following result.

Lemma 3.1: Let $\gamma \in \rho(A_0)$, then

1) $\gamma \in \sigma_p(A) \Leftrightarrow 1 \in \sigma_p(\Phi D_\gamma)$.

2) If $\gamma \in \rho(A_0)$ and there exists $\gamma_0 \in \mathbb{C}$ such that $1 \notin \sigma(\Phi D_{\gamma_0})$, then $\gamma \in \sigma(A) \Leftrightarrow 1 \in \sigma(\Phi D_\gamma)$.

In our situation the operator ΦD_γ is $(n+n\times M+M)\times(n+n\times M+M)$-matrix, as follows:

$$
\Phi D_\gamma = \left\{ \begin{array}{ccccccccccccc}
0 & 0 & \cdots & 0 & a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,n+M} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2,n+M+1} & a_{2,n+M+2} & \cdots & a_{2,n+M+M} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
a_{n+nM+1,1} & a_{n+nM+1,2} & \cdots & a_{n+nM+1,n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots
\end{array} \right.
$$

$$
\left. \begin{array}{ccccccccc}
0 & 0 & \cdots & 0 & a_{1,n+nM+1} & a_{1,n+nM+2} & \cdots & a_{1,n+nM+M-1} & a_{1,n+nM+M} \\
0 & 0 & \cdots & 0 & a_{2,n+nM+1} & a_{2,n+nM+2} & \cdots & a_{2,n+nM+M-1} & a_{2,n+nM+M} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n,n+(n-1)M+1} & a_{n,n+(n-1)M+2} & \cdots & a_{n,n+nM} & a_{n,n+nM+1} & a_{n,n+nM+2} & \cdots & a_{n,n+nM+M-1} & a_{n,n+nM+M} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & a_{n+nM+2,n+nM+1} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n+nM+M,n+nM+M} & 0
\end{array} \right\} \tag{20}
$$

where

$$a_{n+nM+1,1}(\gamma)=\int_0^{+\infty}\mu_1(y)\mathrm{e}^{-\gamma y-\int_0^y\mu_1(t)\mathrm{d}t}\mathrm{d}y,\quad a_{n+nM+1,2}(\gamma)=\int_0^{+\infty}\mu_2(y)\mathrm{e}^{-\gamma y-\int_0^y\mu_2(t)\mathrm{d}t}\mathrm{d}y, \tag{21}$$

$$\cdots,$$

$$a_{n+nM+1,n}(\gamma)=\int_0^{+\infty}\mu_n(y)\mathrm{e}^{-\gamma y-\int_0^y\mu_n(t)\mathrm{d}t}\mathrm{d}y, \tag{22}$$

$$a_{1,n+1}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega,\, a_{1,n+2}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{23}$$

$$\cdots,$$

$$a_{1,n+M}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{24}$$

$$a_{2,n+M+1}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega,\, a_{2,n+M+2}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{25}$$

$$\cdots,$$

$$a_{2,n+M+M}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega,\ \cdots, \tag{26}$$

$$a_{n,n+(n-1)M+1}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega,$$

$$a_{n,n+(n-1)M+2}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{27}$$

$$\cdots,$$

$$a_{n,n+nM}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega \tag{28}$$

$$a_{1,n+nM+1}(\gamma)=\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{29}$$

$$a_{2,n+nM+1}(\gamma)=\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{30}$$

$$\cdots,$$

$$a_{n,n+nM+1}(\gamma)=\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{31}$$

$$a_{n+nM+2,n+nM+1}(\gamma)=\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{32}$$

$$a_{1,n+nM+2}(\gamma)=\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{33}$$

$$a_{2,n+nM+2}(\gamma)=\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{34}$$

$$\cdots,$$

$$a_{n,n+nM+2}(\gamma)=\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{35}$$

$$a_{n+nM+3,n+nM+2}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\Lambda\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{36}$$

$$\cdots,$$

$$a_{1,n+nM+M-1}(\gamma)=\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{37}$$

$$a_{2,n+nM+M-1}(\gamma)=\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{38}$$

$$\cdots,$$

$$a_{n,n+nM+M-1}(\gamma)=\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega, \tag{39}$$

$$a_{n+nM+M,n+nM+M-1}(\gamma)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega, \tag{40}$$

$$\begin{aligned}a_{1,n+nM+M}(\gamma)=&\frac{\lambda_1}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&+\frac{\lambda_1}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\end{aligned}, \tag{41}$$

$$\begin{aligned}a_{2,n+nM+M}(\gamma)=&\frac{\lambda_2}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&+\frac{\lambda_2}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\end{aligned}, \tag{42}$$

$$\cdots,$$

$$\begin{aligned}a_{n,n+nM+M}(\gamma)=&\frac{\lambda_n}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&+\frac{\lambda_n}{\Lambda}\int_0^{\infty}r(\omega)\mathrm{e}^{-\gamma\omega-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\end{aligned}. \tag{43}$$

Lemma 3.2: For the operator $(A,D(A))$ we have $0\in\sigma_p(A)$.

Proof: All the entries of ΦD_0 are positive and one can compute each column sum of the $(n+n\times M+M)\times(n+n\times M+M)$-matrix ΦD_0 as follows.

$$\begin{aligned}&a_{1,1}(0)+a_{2,1}(0)+\cdots+a_{n+nM+1,1}(0)+\cdots+a_{n+nM+M,1}(0)\\&=a_{n+nM+1,1}(0)=\int_0^{+\infty}\mu_1(y)\mathrm{e}^{-\int_0^{y}\mu_1(t)\mathrm{d}t}\mathrm{d}y=1\end{aligned}, \tag{44}$$

$$\begin{aligned}&a_{1,2}(0)+a_{2,2}(0)+\cdots+a_{n+nM+1,2}(0)+\cdots+a_{n+nM+M,2}(0)\\&=a_{n+nM+1,2}(0)=\int_0^{+\infty}\mu_2(y)\mathrm{e}^{-\int_0^{y}\mu_2(t)\mathrm{d}t}\mathrm{d}y=1\end{aligned}, \tag{45}$$

$$\cdots$$

$$\begin{aligned}&a_{1,n}(0)+a_{2,n}(0)+\cdots+a_{n+nM+1,n}(0)+\cdots+a_{n+nM+M,n}(0)\\&=a_{n+nM+1,n}(0)=\int_0^{+\infty}\mu_n(y)\mathrm{e}^{-\int_0^{y}\mu_n(t)\mathrm{d}t}\mathrm{d}y=1\end{aligned}, \tag{46}$$

$$\begin{aligned}&a_{1,n+1}(0)+a_{2,n+1}(0)+\cdots+a_{n+nM+M,n+1}(0)\\&=a_{1,n+1}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned}, \tag{47}$$

$$\begin{aligned}&a_{1,n+2}(0)+a_{2,n+2}(0)+\cdots+a_{n+nM+M,n+2}(0)\\&=a_{1,n+2}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned}, \tag{48}$$

$$\cdots$$

$$\begin{aligned}&a_{1,n+M}(0)+a_{2,n+M}(0)+\cdots+a_{n+nM+M,n+M}(0)\\&=a_{1,n+M}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned}, \tag{49}$$

$$\begin{aligned}&a_{1,n+M+1}(0)+a_{2,n+M+1}(0)+\cdots+a_{n+nM+M,n+M+1}(0)\\&=a_{2,n+M+1}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned}, \tag{50}$$

$$\begin{aligned}&a_{1,n+M+2}(0)+a_{2,n+M+2}(0)+\cdots+a_{n+nM+M,n+M+2}(0)\\&=a_{2,n+M+2}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{51}$$

$$\cdots$$

$$\begin{aligned}&a_{1,n+M+M}(0)+a_{2,n+M+M}(0)+\cdots+a_{n+nM+M,n+M+M}(0)\\&=a_{2,n+M+M}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{52}$$

$$\cdots$$

$$\begin{aligned}&a_{1,n+(n-1)M+1}(0)+a_{2,n+(n-1)M+1}(0)+\cdots+a_{n,n+(n-1)M+1}(0)+\cdots+a_{n+nM+M,n+(n-1)M+1}(0)\\&=a_{n,n+(n-1)M+1}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{53}$$

$$\begin{aligned}&a_{1,n+(n-1)M+2}(0)+a_{2,n+(n-1)M+2}(0)+\cdots+a_{n,n+(n-1)M+2}(0)+\cdots+a_{n+nM+M,n+(n-1)M+2}(0)\\&=a_{n,n+(n-1)M+2}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{54}$$

$$\cdots$$

$$\begin{aligned}&a_{1,n+nM}(0)+a_{2,n+nM}(0)+\ldots+a_{n,n+nM}(0)+\ldots+a_{n+nM+M,n+nM}(0)\\&=a_{n,n+nM}(0)=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{55}$$

$$\begin{aligned}&a_{1,n+nM+1}(0)+a_{2,n+nM+1}(0)+\cdots+a_{n,n+nM+1}(0)+\cdots+a_{n+nM+2,n+nM+1}(0)+\cdots\\&+a_{n+nM+M,n+nM+1}(0)\\&=a_{1,n+nM+1}(0)+a_{2,n+nM+1}(0)+\cdots+a_{n,n+nM+1}(0)+\cdots+a_{n+nM+2,n+nM+1}(0)\\&=\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\cdots\\&\quad+\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{56}$$

$$\begin{aligned}&a_{1,n+nM+2}(0)+a_{2,n+nM+2}(0)+\cdots+a_{n,n+nM+2}(0)+\cdots+a_{n+nM+3,n+nM+2}(0)+\cdots\\&+a_{n+nM+M,n+nM+2}(0)\\&=a_{1,n+nM+2}(0)+a_{2,n+nM+2}(0)+\cdots+a_{n,n+nM+2}(0)+\cdots+a_{n+nM+3,n+nM+2}(0)\\&=\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\cdots\\&\quad+\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^{\omega}r(s)\mathrm{d}s}\mathrm{d}\omega=1\end{aligned},\tag{57}$$

$$\cdots,$$

$$
\begin{aligned}
& a_{1,n+nM+M-1}(0)+a_{2,n+nM+M-1}(0)+\cdots+a_{n,n+nM+M-1}(0)+\cdots \\
& +a_{n+nM+M,n+nM+M-1}(0)+\cdots+a_{n+nM+M,n+nM+M-1}(0) \\
& =a_{1,n+nM+M-1}(0)+a_{2,n+nM+M-1}(0)+\cdots+a_{n,n+nM+M-1}(0)+\cdots+a_{n+nM+M,n+nM+M-1}(0) \\
& =\frac{\lambda_1}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_2}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\cdots \\
& +\frac{\lambda_n}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty} r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega \\
& =\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty} r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega \\
& =\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega=1
\end{aligned}
, \quad (58)
$$

$$
\begin{aligned}
& a_{1,n+nM+M}(0)+a_{2,n+nM+M}(0)+\cdots+a_{n,n+nM+M}(0)+\cdots+a_{n+nM+M,n+nM+M}(0) \\
& =a_{1,n+nM+M}(0)+a_{2,n+nM+M}(0)+\cdots+a_{n,n+nM+M}(0) \\
& =\frac{\lambda_1}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega+\frac{\lambda_1}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega \\
& +\frac{\lambda_2}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega+\frac{\lambda_2}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\cdots \\
& +\frac{\lambda_n}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega+\frac{\lambda_n}{\Lambda}\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega \\
& =\int_0^{+\infty} r(\omega)\mathrm{e}^{-\int_0^{\omega} r(s)\mathrm{d}s}\mathrm{d}\omega=1
\end{aligned}
. \quad (59)
$$

From these results we know that the matrix ΦD_0 is column stochastic and thus $1\in\sigma_p(\Phi D_0)$. Applying Lemma 3.1 1), we immediately obtain $0\in\sigma_p(A)$.

Using Lemma 3.1 2) we can show that 0 is the only spectral point of A on the imaginary axis.

Lemma 3.3: The spectrum $\sigma(A)$ of A satisfies

$$\sigma(A)\cap iR=\{0\}. \quad (60)$$

Proof: By Lemma 3.1 it suffices to prove that $ai\notin\sigma(\Phi D_{ai})$ for all $a\in R$, $a\neq 0$.

Since the General Assumption 2.1 implies that there exists $r\in\mathbb{R}_+$ such that $\mu_k(y)>0, k=1,2,\cdots,n$ and $r(x)>0$ for all $y,\omega\in\left[r,r+\frac{2\pi}{a}\right]$. Using the abbreviation $g(y):=\mu_k(y)\mathrm{e}^{-\int_0^y\mu_k(\tau)\mathrm{d}\tau}, k=1,2,\cdots,n$ we can estimate

$$
\begin{aligned}
& \left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|=\left|\int_0^{+\infty}\mathrm{e}^{-aiy}\mu_k(y)\mathrm{e}^{-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|=\left|\int_0^{+\infty}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right| \\
& \leq\left|\int_r^{r+\frac{2\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right|+\left|\int_0^r\mathrm{e}^{-aiy}g(y)\mathrm{d}y+\int_{r+\frac{2\pi}{a}}^{+\infty}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right| \\
& \leq\left|\int_r^{r+\frac{2\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right|+\int_0^r g(y)\mathrm{d}y+\int_{r+\frac{2\pi}{a}}^{+\infty}g(y)\mathrm{d}y,\ k=1,2,\cdots,n
\end{aligned}
\quad (61)
$$

The first term on the right hand side of the inequality (61) can be estimated as

$$\begin{aligned}&\left|\int_r^{r+\frac{2\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right|=\left|\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y+\int_{r+\frac{\pi}{a}}^{r+\frac{2\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y\right|\\&=\left|\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y+\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-ai\left(y+\frac{\pi}{a}\right)}g\left(y+\frac{\pi}{a}\right)\mathrm{d}y\right|\\&=\left|\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-aiy}g(y)\mathrm{d}y+\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-\pi i}\mathrm{e}^{-aiy}g\left(y+\frac{\pi}{a}\right)\mathrm{d}y\right|\\&=\left|\int_r^{r+\frac{\pi}{a}}\mathrm{e}^{-aiy}\left[g(y)-g\left(y+\frac{\pi}{a}\right)\right]\mathrm{d}y\right|\le\int_r^{r+\frac{\pi}{a}}\left|g(y)-g\left(y+\frac{\pi}{a}\right)\right|\mathrm{d}y\\&<\int_r^{r+\frac{\pi}{a}}\left[g(y)+g\left(y+\frac{\pi}{a}\right)\right]\mathrm{d}y=\int_r^{r+\frac{\pi}{a}}g(y)\mathrm{d}y+\int_{r+\frac{\pi}{a}}^{r+\frac{2\pi}{a}}g(y)\mathrm{d}y=\int_r^{r+\frac{2\pi}{a}}g(y)\mathrm{d}y,\end{aligned}\tag{62}$$

where we used the strict positivity of $\mu_k(y)$ on $\left[r,r+\frac{2\pi}{a}\right]$ in the last inequality. By inserting (62) into (61) we have

$$\begin{aligned}&\left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|<\int_r^{r+\frac{2\pi}{a}}g(y)\mathrm{d}y+\int_0^r g(y)\mathrm{d}y+\int_{r+\frac{2\pi}{a}}^{+\infty}g(y)\mathrm{d}y\\&=\int_0^{+\infty}g(y)\mathrm{d}y=1,\ k=1,2,\cdots,n\end{aligned}\tag{63}$$

Using the same way we can also estimate

$$\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega\right|<1,\tag{64}$$

$$\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\right|<\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(s)\mathrm{d}s}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega,\tag{65}$$

$$\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\Lambda\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega\right|<\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^\omega r(s)\mathrm{d}s}\mathrm{d}\omega.\tag{66}$$

Using (63)-(66) we can estimate each column sum of absolute entries of ΦD_{ai} as follows.

$$\begin{aligned}&\left|a_{1,k}(ai)\right|+\left|a_{2,k}(ai)\right|+\cdots+\left|a_{n+nM+1,k}(ai)\right|+\cdots+\left|a_{n+nM+M,k}(ai)\right|\\&=\left|a_{an+nM+1,k}(ai)\right|=\left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|<1,\ k=1,2,\cdots,n,\end{aligned}\tag{67}$$

$$\begin{aligned}&\left|a_{1,n+j}(ai)\right|+\left|a_{2,n+j}(ai)\right|+\cdots+\left|a_{n+nM+1,n+j}(ai)\right|+\cdots+\left|a_{n+nM+M,n+j}(ai)\right|\\&=\left|a_{1,n+j}(ai)\right|=\left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|<1,\ j=1,2,\cdots,M\end{aligned}\tag{68}$$

$$\begin{aligned}&\left|a_{1,n+M+j}(ai)\right|+\left|a_{2,n+M+j}(ai)\right|+\cdots+\left|a_{n+nM+M,n+M+j}(ai)\right|\\&=\left|a_{2,n+M+j}(ai)\right|=\left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|<1,\ j=1,2,\cdots,M,\end{aligned}\tag{69}$$

$$\cdots,$$

$$\begin{aligned}&\left|a_{1,n+(n-1)M+j}(ai)\right|+\left|a_{2,n+(n-1)M+j}(ai)\right|+\cdots+\left|a_{n,n+(n-1)M+j}(ai)\right|\\&+\cdots+\left|a_{n+nM+M,n+(n-1)M+j}(ai)\right|\\&=\left|a_{n,n+(n-1)M+j}(ai)\right|=\left|\int_0^{+\infty}\mu_k(y)\mathrm{e}^{-aiy-\int_0^y\mu_k(\tau)\mathrm{d}\tau}\mathrm{d}y\right|<1,\ j=1,2,\cdots,M,\end{aligned}\tag{70}$$

$$\begin{aligned}&\left|a_{1,n+nM+j}(ai)\right|+\left|a_{2,n+nM+j}(ai)\right|+\cdots+\left|a_{n,n+nM+j}(ai)\right|+\cdots\\&+\left|a_{n+nM+2,n+nM+j}(ai)\right|+\cdots+\left|a_{n+nM+M,n+nM+j}(ai)\right|\\&=\left|a_{1,n+nM+j}(ai)\right|+\left|a_{2,n+nM+j}(ai)\right|+\cdots+\left|a_{n,n+nM+j}(ai)\right|+\cdots+\left|a_{n+nM+2,n+nM+j}(ai)\right|\\&=\left|\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\right|+\left|\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\right|\\&\quad+\cdots+\left|\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\right|+\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\Lambda\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\right|\\&=\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega\right|+\left|\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\Lambda\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\right|\\&<\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega=1,\ j=1,2,\cdots,M-1,\end{aligned}\tag{71}$$

$$\begin{aligned}&\left|a_{1,n+nM+M}(ai)\right|+\left|a_{2,n+nM+M}(ai)\right|+\cdots+\left|a_{n,n+nM+M}(ai)\right|+\cdots+\left|a_{n+nM+M,n+nM+M}(ai)\right|\\&=\left|a_{1,n+nM+M}(ai)\right|+\left|a_{2,n+nM+M}(ai)\right|+\cdots+\left|a_{n,n+nM+M}(ai)\right|\\&=\left|\frac{\lambda_1}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_1}{\Lambda+ai}\int_0^{+\infty}r(\omega)\mathrm{e}^{-(ai+\Lambda)\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\right|\\&\quad+\left|\frac{\lambda_2}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_2}{\Lambda+ai}\int_0^{+\infty}r(\omega)\mathrm{e}^{-(ai+\Lambda)\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\right|\\&\quad+\cdots+\left|\frac{\lambda_n}{\Lambda}\int_0^{+\infty}r(\omega)\mathrm{e}^{-ai\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\lambda_n}{\Lambda+ai}\int_0^{+\infty}r(\omega)\mathrm{e}^{-(ai+\Lambda)\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\right|\\&\le\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\frac{\Lambda}{\Lambda^2+a^2}\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\\&<\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(\tau)\mathrm{d}\tau}\left(1-\mathrm{e}^{-\Lambda\omega}\right)\mathrm{d}\omega+\int_0^{+\infty}r(\omega)\mathrm{e}^{-\Lambda\omega-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega\\&=\int_0^{+\infty}r(\omega)\mathrm{e}^{-\int_0^\omega r(\tau)\mathrm{d}\tau}\mathrm{d}\omega=1\end{aligned}\tag{72}$$

From (67)-(72) we deduce $\|\Phi D_{ai}\|<1$, thus the spectral radius fulfills $r(\Phi D_{ai})\le\|\Phi D_{ai}\|<1$. This implies $1\in\rho(\Phi D_{ai})$ for all $a\in R,a\neq 0$, *i.e.* $ai\notin\sigma(\Phi D_{ai})$ for all $a\in R,a\neq 0$. By Lemma 3.1 2) we obtain that $ai\notin\sigma(A)$ for all $a\in R,a\neq 0$, *i.e.*, $\sigma(A)\cap iR=\{0\}$.

Lemma 3.4: If the operator $(A_0,D(A_0))$ is defined as

$$A_0p=A_mp,D(A_0)=\{p\in D(A_m)\mid Lp=0\},\tag{73}$$

then for the set $S=\{\gamma\in\mathbb{C}\mid\operatorname{Re}\gamma>-\mu_\infty\}$ we have $S\subseteq\rho(A_0)$. Moreover, if $\gamma\in S$, then

$$
R(\gamma, A_0) = \left(\begin{array}{ccccccccccc}
s_{11} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & s_{22} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & s_{33} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & s_{n+1,n+1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & s_{n+2,n+2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & s_{n+3,n+3} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & s_{n+M+1,n+M+1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & s_{n+M+2,n+M+2} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & s_{n+M+3,n+M+3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}\right.
$$

$$
\begin{array}{ccccccc}
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & s_{n+2M+1,n+2M+1} & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & s_{n+(n-1)M+2,n+(n-1)M+2} & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & s_{n+(n-1)M+3,n+(n-1)M+3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & s_{n+nM+1,n+nM+1} \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}
$$

$$\left.\begin{matrix}
0 & 0 & \cdots & s_{1,n+nM+M+1} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 \\
s_{n+2,n+nM+2} & 0 & \cdots & 0 \\
0 & s_{n+3,n+nM+3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & s_{n+M+1,n+nM+M+1} \\
s_{n+M+2,n+nM+2} & 0 & \cdots & 0 \\
0 & s_{n+M+3,n+nM+3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & s_{n+2M+1,n+nM+M+1} \\
\cdots & \cdots & \cdots & \cdots \\
s_{n+(n-1)M+2,n+nM+2} & 0 & \cdots & 0 \\
0 & s_{n+(n-1)M+3,n+nM+3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & s_{n+nM+1,n+nM+M+1} \\
s_{n+nM+2,n+nM+2} & 0 & \cdots & 0 \\
0 & s_{n+nM+3,n+nM+3} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & s_{n+nM+M+1,n+nM+M+1}
\end{matrix}\right), \tag{74}$$

where

$$s_{11} = \frac{1}{\gamma + \Lambda}, s_{22} = R(\gamma, D_1), s_{33} = R(\gamma, D_2), \cdots, s_{n+1,n+1} = R(\gamma, D_n), \tag{75}$$

$$s_{n+2,n+2} = R(\gamma, D_{11}), s_{n+3,n+3} = R(\gamma, D_{12}), \ \cdots, \ s_{n+M+1,n+M+1} = R(\gamma, D_{1M}) \tag{76}$$

$$s_{n+M+2,n+M+2} = R(\gamma, D_{21}), s_{n+M+3,n+M+3} = R(\gamma, D_{22}), \ \cdots, \ s_{n+2M+1,n+2M+1} = R(\gamma, D_{2M}), \tag{77}$$

$$\cdots,$$

$$s_{n+(n-1)M+2,n+(n-1)M+2} = R(\gamma, D_{n1}), \ s_{n+(n-1)M+3,n+(n-1)M+3} = R(\gamma, D_{n2}) \tag{78}$$

$$\cdots,$$

$$s_{n+nM+1,n+nM+1} = R(\gamma, D_{nM}), \ s_{n+2,n+nM+2} = \lambda_1 R(\gamma, D_{11}) R(\gamma, D_{01}), \tag{79}$$

$$s_{n+M+2,n+nM+2} = \lambda_2 R(\gamma, D_{21}) R(\gamma, D_{01}), \tag{80}$$

$$\cdots,$$

$$s_{n+(n-1)M+2,n+nM+2} = \lambda_n R(\gamma, D_{n1}) R(\gamma, D_{01}), \tag{81}$$

$$s_{n+nM+2,n+nM+2} = R(\gamma, D_{01}), \ s_{n+3,n+nM+3} = \lambda_1 R(\gamma, D_{12}) R(\gamma, D_{02}), \tag{82}$$

$$s_{n+M+3,n+nM+3} = \lambda_2 R(\gamma, D_{22}) R(\gamma, D_{02}), \tag{83}$$

$$\cdots,$$

$$s_{n+(n-1)M+3,n+nM+3} = \lambda_n R(\gamma, D_{n2}) R(\gamma, D_{02}), s_{n+nM+3,n+nM+3} = R(\gamma, D_{02}) \tag{84}$$

$$s_{1,n+nM+M+1} = \frac{1}{\gamma+\Lambda}\varphi_M R(\gamma, D_{0M}), \tag{85}$$

$$s_{n+M+1,n+nM+M+1} = \lambda_1 R(\gamma, D_{1M}) R(\gamma, D_{0M}),$$
$$s_{n+2M+1,n+nM+M+1} = \lambda_2 R(\gamma, D_{2M}) R(\gamma, D_{0M}), \tag{86}$$
$$\cdots,$$

$$s_{n+nM+1,n+nM+M+1} = \lambda_n R(\gamma, D_{nM}) R(\gamma, D_{0M}),\quad s_{n+nM+M+1,n+nM+M+1} = R(\gamma, D_{0M}) \tag{87}$$

The resolvent operator of the differential operators $D_{i,j,0}$ where $D_{i,j,0} = D_{i,j}$ with domain $D(D_{i,j,0}) = \{g(x) \in W^{1,1}(0,+\infty) : g(0) = 0\}$, ($i = 1,2,\cdots,n+nM+M+1$, $j = 1,2,\cdots,n+nM+M+1$) are given by

$$\left(R(\gamma, D_{0M})g\right)(x) = \mathrm{e}^{-(\gamma+\Lambda)\omega - \int_0^\omega r(\tau)\mathrm{d}\tau} \int_0^\omega \mathrm{e}^{(\gamma+\Lambda)s + \int_0^s r(\tau)\mathrm{d}\tau} g(s)\mathrm{d}s, \tag{88}$$

$$\left(R(\gamma, D_i)g\right)(x) = \mathrm{e}^{-\gamma y - \int_0^y \mu_i(\tau)\mathrm{d}\tau} \int_0^y \mathrm{e}^{\gamma s + \int_0^s \mu_i(\tau)\mathrm{d}\tau} g(s)\mathrm{d}s, i = 1,2,\cdots,n \tag{89}$$

$$\left(R(\gamma, D_{ij})g\right)(x) = \mathrm{e}^{-\gamma\omega - \int_0^\omega r(\tau)\mathrm{d}\tau} \int_0^\omega \mathrm{e}^{\gamma s + \int_0^s r(\tau)\mathrm{d}\tau} g(s)\mathrm{d}s, i = 1,2,\cdots,n, j = 1,2,\cdots,M, \tag{90}$$

$$\left(R(\gamma, D_{0j})g\right)(x) = \mathrm{e}^{-(\gamma+\Lambda)\omega - \int_0^\omega r(\tau)\mathrm{d}\tau} \int_0^\omega \mathrm{e}^{(\gamma+\Lambda)s + \int_0^s r(\tau)\mathrm{d}\tau} g(s)\mathrm{d}s, j = 1,2,\cdots,M. \tag{91}$$

Applying the same method as in [7] we can express the resolvent of A in terms of the resolvent of A_0, the Dirichlet operator D_γ and the boundary operator as follows.

Lemma 3.5: If $\gamma \in \rho(A_0) \cap \rho(A)$, then

$$R(\gamma, A) = R(\gamma, A_0) + D_\gamma \left(Id - FD_\gamma\right)^{-1} \Phi R(\gamma, A_0). \tag{92}$$

The following property of C_0-semigroup $(T(t))_{t\ge 0}$ generated by the system operator $(A, D(A))$ is useful to prove the asymptotic stability of the dynamic solution of the system.

Theorem 3.6: The semigroup $(T(t))_{t\ge 0}$ generated by $(A, D(A))$ is irreducible.

Proof: The representation (4) for the resolvent of A_0 shows that it is a positive operator for $\gamma > 0$. We know from the proof of Lemma 3.3 that $\|\Phi D_\gamma\| < 1$, Hence the inverse of $Id_{\partial x} - \Phi D_\gamma$ can be computed via the Neumann series

$$\left(Id_{\partial x} - \Phi D_\gamma\right)^{-1} = \sum_{n=0}^{\infty} \left(\Phi D_\gamma\right)^n. \tag{93}$$

Using Lemma 3.5 we can now prove as in ([7] Lemma 3.9) that $R(\gamma, A)$ transforms any positive vector $p \in X$ into a strictly positive vector:

$$p \in X, p > 0 \Rightarrow R(\gamma, A)p \gg 0. \tag{94}$$

By ([8] Def. C-III 3.1) this is equivalent to the irreducibility of the semigroup $(T(t))_{t\ge 0}$ generated by $(A, D(A))$.

Using Lemma 3.2, Lemma 3.3 and Theorem 3.6 we obtain the following result.

Theorem 3.7: The space X can be decomposed into the direct sum

$$X = X_1 \oplus X_2, \tag{95}$$

where $X_1 = fix(T(t))_{t\ge 0} = \ker A$ is one-dimensional and spanned by a strictly positive eigenvector $\hat{p}$ of A. In addition, the restriction $\left(T(t)\big|_{X_2}\right)_{t\ge 0}$ is strongly stable.

Corollary 3.8: For all $p \in X$, there exists $\alpha > 0$, such that

$$\lim_{t\to+\infty} T(t)p = \alpha\hat{p}, \tag{96}$$

where $\ker A = \langle \hat{p} \rangle, \hat{p} \gg 0$.

From Corollary 3.8 together with Theorem 2.2 we obtain our main result as follows.

Corollary 3.9: The dynamic solution of the system (1), (2) and (3) converges strongly to the steady state solution as time tends to infinity, that is, there exists $\alpha' > 0$, such that

$$\lim_{t\to+\infty} p(t) = \alpha'\hat{p}, \tag{97}$$

where $\hat{p}$ as in Corollary 3.8.

4. Conclusion

In this paper, we investigated an N-unit series system with finite number of vacations. The study of the dynamic solution as well as its stability is in demand in terms of theory and practice. We discussed the asymptotic stability of the dynamic solution and proved that the dynamic solution converges strongly to the steady state solution by analyzing the spectral distribution of the system operator and taking into account the irreducibility of the semigroup generated by the system operator.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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