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On Tachibana and Vishnevskii Operators Associated with Certain Structures in the Tangent Bundle

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Abstract

The aim of the present work is to study the complete, vertical and horizontal lifts using Tachibana and Visknnevskii operators along generalized almost r-contact structure in tangent bundle. We also prove certain theorems on Tachibana and Visknnevskii operators with Lie derivative and lifts.

Keywords

Tangent Bundle, Vertical Lift, Complete Lift, Lie Derivative, Tachibana Operator, Vishnevskii Operator

1. Introduction

Let M be an n-dimensional differentiable manifold and let $T(M)=\bigcup_{p\in M}T_p(M)$ be its tangent bundle. Then $T(M)$ is also a differentiable manifold [1]. Let $X=\sum_{i=1}^{n}x^i\left(\frac{\partial}{\partial x^i}\right)$ and $\eta=\sum_{i=1}^{n}\eta^i dx^i$ be the expressions in local coordinates for the vector field X and the 1-form η in M. Let (x^i,y^i) be local coordinates of point in $T(M)$ induced naturally from the coordinate chart (U,x^i) in M.

The complete, vertical and horizontal lifts of tensor field have vital role in differential geometry of tangent bundle. In 2016, [2] studied Tachibana and Vishneveskii operators applied to vertical and horizontal lifts in almost paracontact structure on the tangent bundle T(M). The generalized almost r-contact structure in tangent bundle and integrability of structure is studied by the second author [3].

This paper is organized as follows: Section 2 describes some basic definitions and notations. Section 3 deals with the study of Tachibana and Vishnevskii operators for generalized almost r-contact structure in tangent bundle.

2. Preliminaries

2.1. Vertical Lifts

If *f* is a function in *M*, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$, so that

$$f^V = f \circ \pi \tag{1}$$

where $\circ$ is composition of *f* and *pi*.

Thus, if a point $\tilde{p} \epsilon \pi^{-1}(U)$ has induced coordinates (x^h, y^h) then

$$f^V(\tilde{p}) = f^V(x, y) = f \circ \pi(\tilde{p}) = f(p) = f(x) \tag{2}$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function *f*.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions (Tensor product of *P* and *Q*)

$$(P \otimes Q)^V = P^V \otimes Q^V, (P + R)^V = P^V + R^V \tag{3}$$

P, *Q* and *R* being arbitrary elements of $\Im(M)$.

Furthermore, the vertical lifts of tensor fields obey the general properties [4] [5]:

(a) $(f \cdot g)^V = f^V g^V, (f + g)^V = f^V + g^V$,

(b) $(X + Y)^V = X^V + Y^V, (f \cdot X)^V = f^V X^V, X^V f^V = 0, [X^V, Y^V] = 0$,

(c) $(f \cdot \eta)^V = f^V \eta^V, \eta^V(X^V) = 0, X^V(Y^V) = 0$,

$$\forall f, g \in \Im_0^0(M), X, Y \in \Im_0^1(M), \eta \in \Im_1^0(M).$$

2.2. Complete Lifts

If *f* is a function in *M*, we write f^C for the function in $T(M)$ defined by [1]

$$f^C = i(df)$$

and call f^C the complete lift of the function *f*. The complete lift f^C of a function *f* has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C$$

f being an arbitrary function in *M* and call X^C the complete lift of *X* in $T(M)$.

The complete lift X^C of X with components x^h in M has components

$$X^C:\begin{bmatrix} x^h \\ \partial x^h \end{bmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\eta \in \Im_0^1(M)$ Then a 1-form η^C in $T(M)$ defined by

$$\eta^C\left(X^C\right)=\left(\eta(X)\right)^C$$

X being an arbitrary vector field in M. We call η^C the complete lift of η.

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P\otimes Q)^C = P^C\otimes Q^V + P^V\otimes Q^C, (P+R)^C = P^C + R^C$$

P, Q and R being arbitrary elements of $\Im(M)$.

Moreover, the complete lifts of tensor fields obey the general properties [1] [4]:

(a) $(fX)^C = f^C X^V + f^V X^C = (Xf)^C, X^C f^V = (Xf)^V, X^V f^C = (Xf)^V$,

(b) $\phi^V XC = (\phi X)^V, \phi^C X^V = (\phi X)^V, (\phi X)^C = \phi^C X^C$,

(c) $\eta^V X^C = (\eta(X))^C, \eta^C X^V = (\eta(X))^V$,

(d) $\left[X^V, Y^C\right] = [X,Y]^C, I^C = I, I^V I^C = X^V, \left[X^C, Y^C\right] = [X,Y]^C$

$$\forall f, g \in \Im_0^0(M), X, Y \in \Im_0^1(M), \eta \in \Im_1^0(M).$$

2.3. Horizontal Lifts

The horizontal lift f^H of $f \in \Im_0^0(M)$ to the tangent bundle $T(M)$ by

$$(f)^H = f^C - \nabla_\gamma f \tag{4}$$

where

$$\nabla_\gamma f = \gamma(\nabla f),$$

Let $X \in \Im_0^1(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X \tag{5}$$

in $T(M)$, where

$$\nabla_\gamma X = \gamma(\nabla X)$$

The horizontal lift X^H of X has the components

$$\begin{bmatrix} x^h \\ -\Gamma_i^h x^i \end{bmatrix} \tag{6}$$

with respect to the induced coordinates in $T(M)$, where $\Gamma_i^h = y^j \Gamma_{ji}^h$.

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S \tag{7}$$

for all $P,Q \in \Im(M)$. We have

$$\nabla_{\gamma}(P \otimes Q) = (\nabla_{\gamma} P) \otimes Q^{V} + P^{V} \otimes (\nabla_{\gamma} Q)$$

or

$$(P \otimes Q)^{H} = P^{H} \otimes Q^{V} + P^{V} \otimes Q^{H}. \tag{8}$$

In addition, the horizontal lifts of tensor fields obey the general properties [4] [6]:

(a) $X^{H} f^{V} = (Xf)^{V}, \phi^{V} X^{H} = (\phi X)^{V}, \phi^{C} X^{H} = (\phi X)^{H} + (\nabla_{\gamma}\phi) X^{H}$;

(b) $\eta^{V}(X^{H}) = (\eta(X))^{H}, \eta^{C}(X^{H}) = (\eta(X))^{C} - \gamma(\eta \circ (\nabla X))$;

(c) $\eta^{H}(X^{C}) = \eta^{H}(\nabla_{\gamma} X), \eta^{H}(X^{H}) = 0$

$$\forall f,g \in \Im_0^0(M), X,Y \in \Im_0^1(M), \eta \in \Im_1^0(M), \phi \in \Im_1^1(M).$$

Let X be a vector field in an n-dimensional differentiable manifold *M*. The differential transformation L_X is called Lie derivative with respect to X if

(a) $L_X f = Xf, \forall f \in \Im_0^0(M)$,

(b) $L_X Y = [X,Y], \forall X,Y \in \Im_0^1(M)$.

The Lie derivative $L_X F$ of a tensor field F of type (1, 1) with respect to a vector field X is defined by

$$(L_X F) = [X, FY] - F[X,Y] \tag{9}$$

where $[,]$ is Lie bracket [1] page 113.

Let M be an n-dimensional differentiable manifold. Differential transformation of algebra $T(M)$ defined by

$$D = \nabla_X : T(M) \to T(M), X \in \Im_0^1(M), \tag{10}$$

is called as covariant derivation with respect to vector field X if

(a) $\nabla_{fX+gY} t = f \nabla_X t + g \nabla_Y t$,

(b) $\nabla_X f = Xf$,

$$\forall f,g \in \Im_0^0(M), \forall X,Y \in \Im_0^1(M), \forall t \in \Im(M).$$

and a transformation defined by

$$\nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M) \tag{11}$$

is called affine connection [1].

Proposition 1. *For any* $X,Y \in \Im_0^1(M)$ [4]

(a) $[X^{V}, Y^{H}] = [X,Y]^{V} - (\nabla_X Y)^{V} = -(\hat{\nabla}_X Y)^{V}$,

(b) $[X^{C}, Y^{H}] = [X,Y]^{H} - \gamma(L_X Y)$,

(c) $[X^{H}, Y^{V}] = [X,Y]^{V} + (\nabla_Y X)^{V}$,

(d) $[X^{C}, Y^{H}] = [X,Y]^{H} - \gamma \hat{R}(X,Y)$

where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

Proposition 2. *For any* $X,Y \in \Im_0^1(M), f \in \Im_0^0(M)$ *and* ∇^{H} *is the* *horizontal lift of the affine connection* ∇ *to* $T(M)$ [1]

(a) $\nabla^H_{X^V} Y^V = 0,$

(b) $\nabla^H_{X^V} Y^H = 0,$

(c) $\nabla^H_{X^H} Y^V = (\nabla_X Y)^V,$

(d) $\nabla^H_{X^H} Y^H = (\nabla_X Y)^H.$

3. Tachibana and Vishnevskii Operators for Generalized Almost R-Contact Structure in Tangent Bundle

Let M be a differentiable manifold of C^∞ class. Suppose that there are given a tensor field ϕ of type (1, 1), a vector field ξ_p and a 1-form $\eta_p, p = 1, 2, \cdots, r$ satisfying [7] [8] [9]

$$\text{(a)}\quad \phi^2 = a^2 I + \epsilon \sum_{p=1}^{r} \xi_p \otimes \eta_p$$

$$\text{(b)}\quad \phi \xi_p = 0$$

$$\text{(c)}\quad \eta_p \circ \phi = 0$$

$$\text{(d)}\quad \eta_p(\xi_q) = -\frac{a^2}{\epsilon}\delta_{pq} \tag{12}$$

where $p = 1, 2, \cdots, r$ and δ_{pq} denote the Kronecker delta while a and ϵ are non-zero complex numbers. The manifold M is called a generalized almost r-contact manifold with a generalized almost r-contact structure or in short with $(\phi, \eta_p, \xi_p, a, \epsilon)$-structure.

Let us suppose that the base space M admits the Lorentzian almost r-para-contact structure. Then there exists a tensor field ϕ of type (1, 1), $r(C^\infty)$ vector fields $\xi_1, \xi_2, \cdots, \xi_p$ and $r(C^\infty)$ 1-forms $\eta_1, \eta_2, \cdots, \eta_p$ such that Equation (12) are satisfied. Taking complete lifts of Equation (12) we obtain the following:

$$\text{(a)}\quad (\phi^H)^2 = a^2 I + \epsilon \sum_{p=1}^{r} \{\xi_p^V \otimes \eta_p^H + \xi_p^H \otimes \eta_p^V\}$$

$$\text{(b)}\quad \phi^H \xi_p^V = 0, \phi^H \xi_p^C = 0$$

$$\text{(c)}\quad \eta_p^V \circ \phi^H = 0, \eta_p^H \circ \phi^V = 0, \eta_p^H \circ \phi^H = 0, \eta_p^V \circ \phi^V = 0$$

$$\text{(d)}\quad \eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0, \eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = -\frac{a^2}{\epsilon}\delta_{pq} \tag{13}$$

Let us define an element $\tilde{J}$ of $J_0^1 T(M)$ by

$$\tilde{J} = \phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right) \tag{14}$$

then in the view of Equation (13), it is easily shown that

$$\tilde{J}^2 X^V = a^2 X^V, \tilde{J}^2 X^H = a^2 X^H$$

which givess that $\tilde{J}$ is GF structure in $T(M)$ [10].

Now in view of the Equation (15), we have

$$\text{(a)}\quad \tilde{J}X^H = (\phi X)^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left\{(\eta_p(X))^V \xi_p^V\right\}$$

$$\text{(b)}\quad \tilde{J}X^V = (\phi X)^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\left\{(\eta_p(X))^V \xi_p^H\right\} \qquad (15)$$

for all $X \in \Im_0^1(M)$.

3.1. Tachibana Operator

Let ϕ be a tensor fieldof type (1, 1) *i.e.* $\phi \in \Im_1^1(M)$ and $\phi \in \Im(M) = \sum_{r,s=0}^{\infty}\Im_r^s(M)$ be a tensor algebra over *R*. A map $\Phi_\phi\big|_{r+s>0}$ is called a Tachibana operator or Φ_ϕ operator on *M* if [2] [11]

(a) Φ_ϕ is linear with respect to constant coefficient,

(b) $\Phi_\phi : \Im^*(M) \to \Im_{s+1}^r(M)$ for all *r* and *s*,

(c) $\Phi_\phi\left(K \otimes^C L\right) = \left(\Phi_\phi K\right)\otimes L + K \otimes \Phi_\phi L$ for all $K, L \in \Im^*(M)$,

(d) $\Phi_{\phi X} Y = -\left(L_Y \phi\right) X$ for all $X, Y \in \Im_0^1(M)$

where L_Y is Lie derivation with respect to *Y*,

$$\text{(e)}\quad \begin{aligned}\left(\Phi_{\phi\eta}\right)Y &= (d(\Im_Y \eta(\Phi X) - (d(\Im_Y(\eta \circ \Phi)X + \eta((L_Y\phi)X) \\ &= (\Phi X(\Im_Y \eta))(\Phi X) - X(\Im_{\phi X}\eta) + \eta((L_Y\phi)X)\end{aligned} \qquad (16)$$

for all $\eta \in \Im_1^0(M)$ and $X, Y \in \Im_0^1(M)$, where $\Im_Y \eta = \eta(X) = \eta \otimes^C Y$, $\Im_r^{*s}(M)$ the module of pure tensor fields of type (r,s) on *M* with respect to the affinor field φ [12] [13].

Theorem 3. *For Tachibana operator on* M, L_X *the operator Lie derivation with respect to* $X, \tilde{J} \in \Im_1^1(T(M))$ *defined by*

$\tilde{J} = \phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)$ *and* $\eta(Y) = 0$, *we have*

$$\text{(a)}\quad \Phi_{\tilde{J}Y^V} X^H = -\left(\left(\hat{\nabla}_X \phi\right)Y\right)^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\hat{\nabla}_X \eta_p\right)Y\right)^V \xi_p^H$$

$$\text{(b)}\quad \Phi_{\tilde{J}Y^H} X^H = -\left(\left(L_X \phi\right)Y\right)^H + \gamma\hat{R}(X,\phi Y) + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_X \eta_p\right)Y\right)^V \xi_p^V - \tilde{J}\gamma\hat{R}(X,Y)$$

$$\text{(c)}\quad \Phi_{\tilde{J}Y^V} X^V = 0$$

$$\text{(d)}\quad \begin{aligned}\Phi_{\tilde{J}Y^H} X^V &= -\left(\left(L_X \phi\right)Y\right)^V + \left(\left(\nabla_X \phi\right)Y\right)^V - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_X \eta_p\right)Y\right)^V \xi_p^H \\ &+ \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\nabla_X \eta_p\right)Y\right)^V \xi_p^H\end{aligned} \qquad (17)$$

where $X, Y \in \Im_0^1(M)$, a tensor field $\phi \in \Im_1^1(M)$, a vector field ξ and a 1-form $\eta \in \Im_1^0(M)$.

Proof. For $\tilde{J} = \phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)$ and $\eta(Y) = 0$, we get

$$\Phi_{\tilde{J}Y^V} X^H = -\left(L_{X^H}\tilde{J}\right)Y^V = -\left(L_{X^H}\tilde{J}Y^V - \tilde{J}L_{X^H}Y^V\right), \text{ since } L_X Y = [X,Y]$$

$$= -\left[X^H, \tilde{J}Y^V\right] + \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\left[X^H, Y^V\right]$$

$$\text{(a)} \quad = -\left[X^H, (\phi Y)^V\right] + \phi^H\left([X,Y]^V + (\nabla_X Y)^V\right)$$

$$+\frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^V\left([X,Y]^V + (\nabla_Y X)^V\right)\xi_p^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^H\left([X,Y]^V + (\nabla_Y X)^V\right)\xi_p^H$$

$$= -\left[X^H, (\phi Y)^V\right](\nabla_{\phi Y} X)^V + \phi^H\left([X,Y]^V + (\nabla_Y X)^V\right)$$

$$+\frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^V\left([X,Y]^V + (\nabla_Y X)^V\right)\xi_p^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^H\left([X,Y]^V + (\nabla_Y X)^V\right)\xi_p^H$$

$$= -\left(\left(\hat{\nabla}_X \phi\right)Y\right)^V - \left(\phi\hat{\nabla}_X Y\right)^V + \left(\phi\left(\hat{\nabla}_X Y\right)\right)^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_X \eta_p\right)Y\right)^V \xi_p^H \tag{18}$$

$$-\frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\hat{\nabla}_X \eta_p\right)Y\right)^V \xi_p^H - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\eta_p\left(L_X Y\right)\right)^V \xi_p^H$$

$$\text{as } \eta\left(L_X Y\right) = -\left(L_X \eta_p\right)Y = -\left(\left(\hat{\nabla}_X \phi\right)Y\right)^V - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\hat{\nabla}_X \eta_p\right)Y\right)^V \xi_p^H.$$

$$\Phi_{\tilde{J}Y^H} X^H = -\left(L_{X^H}\tilde{J}\right)Y^H = -\left(L_{X^H}\tilde{J}Y^H - \tilde{J}L_{X^H}Y^H\right) \text{ since } L_X Y = [X,Y]$$

$$= -\left[X^H, \tilde{J}Y^H\right] + \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\left[X^H, Y^H\right]$$

$$\text{(b)} \quad = -\left[X^H, (\phi Y)^H\right] + \phi^H\left[X^H, Y^H\right] + \frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^V\left[X^H, Y^H\right]\xi_p^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^H\left[X^H, Y^H\right]\xi_p^H \tag{19}$$

$$\text{since } \left[X^H, Y^H\right] = [X,Y]^H - \gamma\hat{R}(X,Y),$$

$$= -\left(\left(L_X \phi\right)Y\right)^H + \gamma\hat{R}(X, \phi Y) - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_X \eta_p\right)Y\right)^V \xi_p^V - \tilde{J}\gamma\hat{R}(X,Y).$$

$$\Phi_{\tilde{J}Y^V} X^V = -\left(L_{X^V}\tilde{J}\right)Y^V = -\left(L_{X^V}\tilde{J}Y^V - \tilde{J}L_{X^V}Y^V\right) \text{ since } L_X Y = [X,Y]$$

$$= -\left[X^V, \tilde{J}Y^V\right] + \tilde{J}\left[X^V, Y^V\right], \quad \left[X^V, Y^V\right] = 0$$

$$\text{(c)} \quad = -\left[X^V, \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)Y^V\right] \tag{20}$$

$$\text{as } \left(\eta_p(Y)\xi_p\right)^H = 0 = -\left[X^V, (\phi Y)^V\right] - \frac{\epsilon}{a}\sum_{p=1}^{r}\left[X^V, \left(\eta_p(Y)\xi_p\right)^H\right] = 0.$$

$$\Phi_{\tilde{J}Y^H} X^V = -\left(L_{X^V}\tilde{J}\right)Y^H = -L_{X^V}\tilde{J}Y^H + \tilde{J}L_{X^V}Y^H, \text{ since } L_X Y = [X,Y]$$

$$= -\left[X^V, \tilde{J}Y^H\right] + \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\left[X^V, Y^H\right]$$

$$= -[X, \phi Y]^V + (\nabla_X \phi Y)^V + \phi^H\left([X,Y]^V - (\nabla_X Y)^V\right)$$

$$\text{(d)} \quad +\frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^V\left([X,Y]^V - (\nabla_X Y)^V\right)\xi_p^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\eta_p^H\left([X,Y]^V - (\nabla_X Y)^V\right)\xi_p^H \tag{21}$$

$$\text{since } \eta_p L_X Y = L_X \eta_p(Y) - \left(L_X \eta_p\right)Y, \ \eta_p \nabla_X Y = \nabla_X \eta_p(Y) - \left(\nabla_X \eta_p\right)Y$$

$$= -\left(\left(L_X \phi\right)Y\right)^V + \left(\left(\nabla_X \phi\right)Y\right)^V - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_X \eta_p\right)Y\right)^V \xi_p^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\nabla_X \eta_p\right)Y\right)^V \xi_p^H.$$

□

Corollary 1. *If we put* $Y=\xi_p$ *i.e.* $\eta_p^H\left(\xi_p^H\right)=\eta_p^V\left(\xi_p^V\right)=0$, $\eta_p^H\left(\xi_p^V\right)=\eta_p^V\left(\xi_p^H\right)=-\frac{a^2}{\epsilon}$, *then we have*

(a) $\Phi_{\tilde{J}\xi_p^V}X^H=a\sum_{p=1}^r\left(L_{\xi_p}X\right)^H-a\gamma\hat{R}\left(X,\xi_p\right)-\left(\left(\hat{\nabla}_X\phi\right)\xi_p\right)^V+\left(\left(\hat{\nabla}_X\eta_p\right)\xi_p\right)^V\xi_p^H$

(b)

$$\Phi_{\tilde{J}\xi_p^H}X^H=a\left(\hat{\nabla}_X\xi_p\right)^V-\left(\left(L_X\phi\right)\xi_p\right)^H-\phi^H\gamma\hat{R}\left(X,\xi_p\right)-\frac{\epsilon}{a}\sum_{p=1}^r\left(\left(L_X\eta_p\right)\xi_p\right)^V\xi_p^V$$
$$-\frac{\epsilon}{a}\sum_{p=1}^r\eta_p^V\gamma\hat{R}\left(X,\xi_p\right)\xi_p^V-\frac{\epsilon}{a}\sum_{p=1}^r\eta_p^H\gamma\hat{R}\left(X,\xi_p\right)\xi_p^H.$$

(c) $\Phi_{\tilde{J}\xi_p^V}X^V=-a\left(\hat{\nabla}_{\xi_p}X\right)^V$

(d)
$$\Phi_{\tilde{J}\xi_p^H}X^V=-\left(\left(L_X\phi\right)\xi_p\right)^V+\left(\left(\nabla_X\phi\right)\xi_p\right)^V-\frac{\epsilon}{a}\sum_{p=1}^r\left(\left(L_X\eta_p\right)\xi_p\right)^V\xi_p^H$$
$$+\frac{\epsilon}{a}\sum_{p=1}^r\left(\left(\nabla_X\eta_p\right)\xi_p\right)^V\xi_p^H. \tag{22}$$

3.2. Vishnevskii Operator

Let ∇ is a linear connection and ϕ be a tensor field of type (1, 1) on *M*. If the condition (d) of Tachibana operator replace by

$$\text{(d')}\quad \Psi_{\phi X}Y=\nabla_{\phi X}Y-\phi\nabla_XY, \tag{23}$$

$\forall X,Y\in\Im_0^1(M)$, is a mapping wih linear connection ∇. A map $\Psi_\phi:\Im^*(M)\to\Im(M)$, which satisfies conditions (a), (b), (c), (e) of Tachibana operator and the condition (d'), is called Vishnevskii operator on *M* [2] [14].

Theorem 4. *For* Ψ_ϕ *Vishnevskii operator on M and* ∇^H *the horizontal lift of an affine connection* ∇ *in M to* $T(M)$, $\tilde{J}\in\Im_1^1(T(M))$ *defined by* (14), *we have*

(a) $\Psi_{\tilde{J}X^V}Y^H=\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p(X)\nabla_{\xi_p}Y\right)^H$

(b)
$$\Psi_{\tilde{J}X^H}Y^V=\left(\left(\hat{\nabla}_Y\phi\right)X\right)^V-\left(\left(L_X\phi\right)X\right)^V-\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p\hat{\nabla}_YX\right)^V\xi_p^H$$
$$+\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_pL_YX\right)^V\xi_p^H$$

(c) $\Psi_{\tilde{J}X^V}Y^V=\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p(X)\right)^V\nabla_{\xi_p^H}^HY^V$

(d)
$$\Psi_{\tilde{J}X^H}Y^H=\left(\left(\hat{\nabla}_Y\phi\right)X\right)^H-\left(\left(L_X\phi\right)X\right)^H-\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p\hat{\nabla}_YX\right)^V\xi_p^V$$
$$+\frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_pL_YX\right)^V\xi_p^V \tag{24}$$

where $X,Y\in\Im_0^1(M)$, a tensor field $\phi\in\Im_1^1(M)$, vector fields ξ_p and a 1-form $\eta_p\in\Im_1^0(M), p=1,\cdots,r$.

Proof.

$$\begin{aligned}
\Psi_{\tilde{J}X^V} Y^H &= \nabla^H_{\tilde{J}X^V} Y^H - \tilde{J}\nabla^H_{X^V} Y^H \\
&= \nabla^H_{\left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)X^V} Y^H - \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^V} Y^H \\
\text{(a)}\quad &= \nabla_{(\phi X)^V + \frac{\epsilon}{a}\sum_{p=1}^r(\eta_p X)^V \xi_p^H} Y^H \quad \text{as } \nabla^H_{X^V} Y^H = 0 \qquad (25) \\
&= \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p X\right)^V\left(\nabla_{\xi_p} Y\right)^H \quad \text{as } \nabla^H_{(\phi X)^V} Y^H = 0 \\
&= \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p(X)\nabla_{\xi_p} Y\right)^H .
\end{aligned}$$

$$\begin{aligned}
\Psi_{\tilde{J}X^H} Y^V &= \nabla^H_{\tilde{J}X^H} Y^H - \tilde{J}\nabla^H_{X^H} Y^V \\
&= \nabla^H_{\left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)X^H} Y^V - \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^H} Y^V \\
&= \nabla^H_{(\phi X)^H} Y^V - \phi^H\left(\nabla_X Y\right)^V - \frac{\epsilon}{a}\sum_{p=1}^r \eta_p^H\left(\nabla_X Y\right)^V \xi_p^H \\
\text{(b)}\quad &= \left(\hat{\nabla}_Y \phi X\right)^V + [\phi X, Y]^V - \phi^H\left(\left(\hat{\nabla}_Y X\right)^V + [X,Y]^V\right) \qquad (26) \\
&\quad - \frac{\epsilon}{a}\sum_{p=1}^r \eta_p^H\left(\left(\hat{\nabla}_Y X\right)^V + [X,Y]^V\right)\xi_p^H \\
&= \left(\left(\hat{\nabla}_Y \phi\right)X\right)^V - \left(\left(L_Y\phi\right)X\right)^V - \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p \hat{\nabla}_Y X\right)^V \xi_p^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p L_Y X\right)^V \xi_p^H
\end{aligned}$$

$$\begin{aligned}
\Psi_{\tilde{J}X^V} Y^V &= \nabla^H_{\tilde{J}X^V} Y^V - \tilde{J}\nabla^H_{X^V} Y^V \\
&= \nabla^H_{\left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)X^V} Y^V - \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^V} Y^V \\
\text{(c)}\quad &= \nabla^H_{(\phi X)^V} Y^V + \frac{\varepsilon}{a}\sum_{p=1}^r\left(\eta_p(X)\right)^V \nabla^H_{\xi_p^H} Y^V \qquad (27) \\
&= \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p(X)\right)^V \nabla^H_{\xi_p^H} Y^V \quad \text{as } \nabla^H_{(\phi X)^V} Y^V = 0.
\end{aligned}$$

$$\begin{aligned}
\Psi_{\tilde{J}X^H} Y^H &= \nabla^H_{\tilde{J}X^H} Y^H - \tilde{J}\nabla^H_{X^H} Y^H \\
&= \nabla^H_{\left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)X^H} Y^H - \left(\phi^H + \frac{\epsilon}{a}\sum_{p=1}^r\left(\xi_p^V \otimes \eta_p^V + \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^H} Y^H \\
&= \nabla^H_{(\phi X)^H} Y^H - \phi^H\left(\nabla_X Y\right)^H - \frac{\epsilon}{a}\sum_{p=1}^r \eta_p^V\left(\nabla_X Y\right)^H \xi_p^V \\
\text{(d)}\quad &= \left(\hat{\nabla}_Y \phi X\right)^H + [\phi X, Y]^H - \phi^H\left(\left(\hat{\nabla}_Y X\right)^H + [X,Y]^H\right) \qquad (28) \\
&\quad - \frac{\epsilon}{a}\sum_{p=1}^r \eta_p^V\left(\left(\hat{\nabla}_Y X\right)^H + [X,Y]^H\right)\xi_p^H \\
&= \left(\left(\hat{\nabla}_Y \phi\right)X\right)^H - \left(\left(L_Y\phi\right)X\right)^H - \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p \hat{\nabla}_Y X\right)^V \xi_p^V + \frac{\epsilon}{a}\sum_{p=1}^r\left(\eta_p L_Y X\right)^V \xi_p^V
\end{aligned}$$

Corollary 2. *If we put* $Y = \xi_p$ *i.e.* $\eta_p^H\left(\xi_p^H\right) = \eta_p^V\left(\xi_p^V\right) = 0$,

$\eta_p^H\left(\xi_p^V\right) = \eta_p^V\left(\xi_p^H\right) = -\frac{a^2}{\epsilon}\delta_{pq}$, *then we have*

(a) $\Psi_{\tilde{J}\xi_p^V} Y^H = -a\left(\nabla_{\xi_p} Y\right)^H$

(b) $$\Psi_{\tilde{J}\xi_p^H} Y^V = -\phi^H\left(\hat{\nabla}_Y \xi_p\right)^V - \left(\left(L_Y\phi\right)\xi_p\right)^V + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\eta_p\left(\hat{\nabla}_Y\xi_p\right)^V\right)\xi_p^H - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\eta_p\left(L_Y\xi_p\right)^V\right)\xi_p^H$$

(c) $\Psi_{\tilde{J}\xi_p^V} Y^V = -a\left(\nabla_{\xi_p} Y\right)^V$

(d) $$\Psi_{\tilde{J}\xi_p^H} Y^H = \left(\left(\hat{\nabla}_Y\phi\right)\xi_p\right)^H - \left(\left(L_Y\phi\right)\xi_p\right)^H + \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(\hat{\nabla}_Y\eta_p\right)\xi_p\right)^V \xi_p^V - \frac{\epsilon}{a}\sum_{p=1}^{r}\left(\left(L_Y\eta_p\right)\xi_p\right)^V \xi_p^V. \quad (29)$$

4. Conclusion

The generalized almost r-contact structure on Tachibana and Visknnevskii operators in tangent bundle are introduced. We deduced the theorems on Tachibana and Visknnevskii operators with respect to Lie derivative and lifting theory.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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